

INDUCED DECOMPOSITIONS OF GRAPHS

ADRIAN BONDY

LYON 1 AND PARIS 6

(WITH JAYME SZWARCFITER, UFRJ)

BORDEAUX GRAPH WORKSHOP

UNIVERSITÉ BORDEAUX 1

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DECOMPOSITIONS

F, G : two graphs

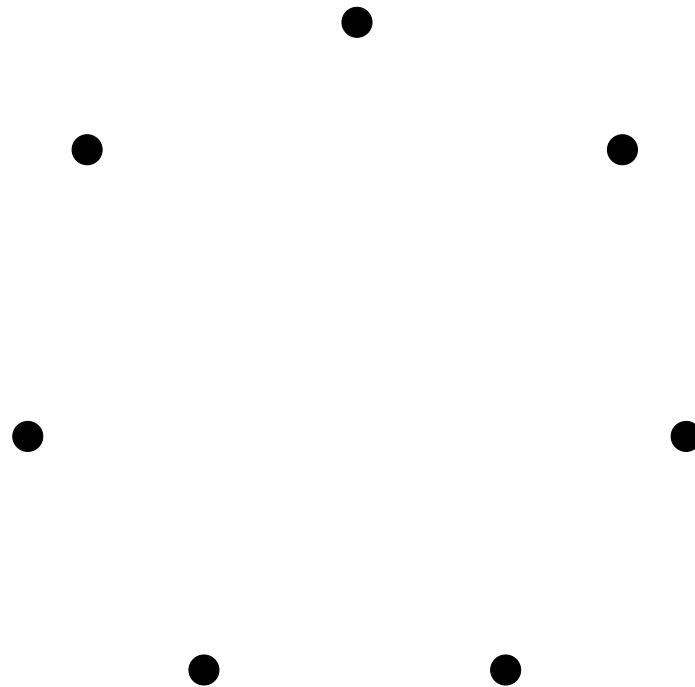
F -decomposition of G :

set $\mathcal{F} = \{F_1, F_2, \dots, F_p\}$ of edge-disjoint subgraphs of G such that

$$F_i \cong F, \quad 1 \leq i \leq p, \quad \text{and} \quad \bigcup_{i=1}^p F_i = G$$

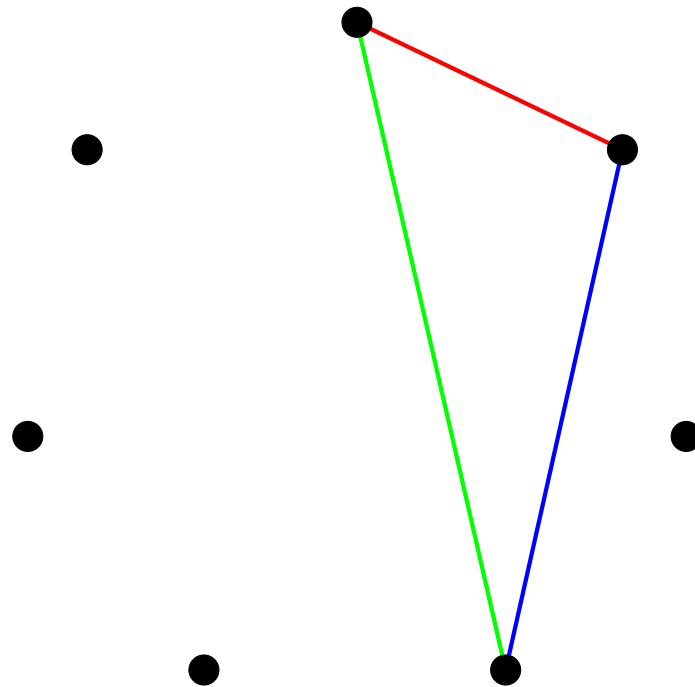
EXAMPLE

Decomposition of K_7 into triangles:



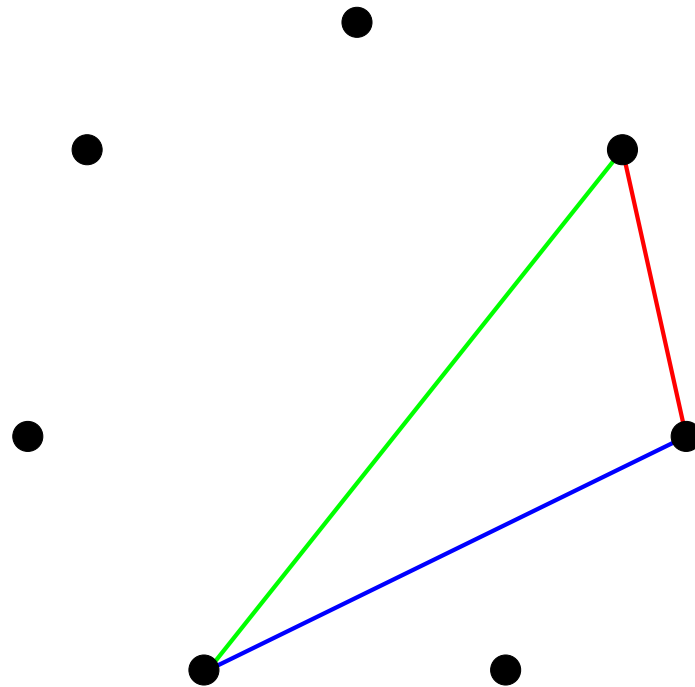
EXAMPLE

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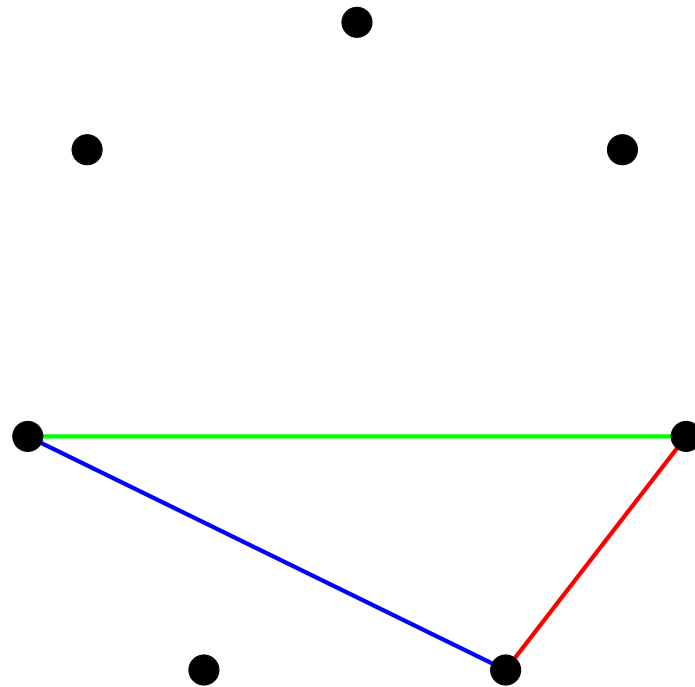
EXAMPLE

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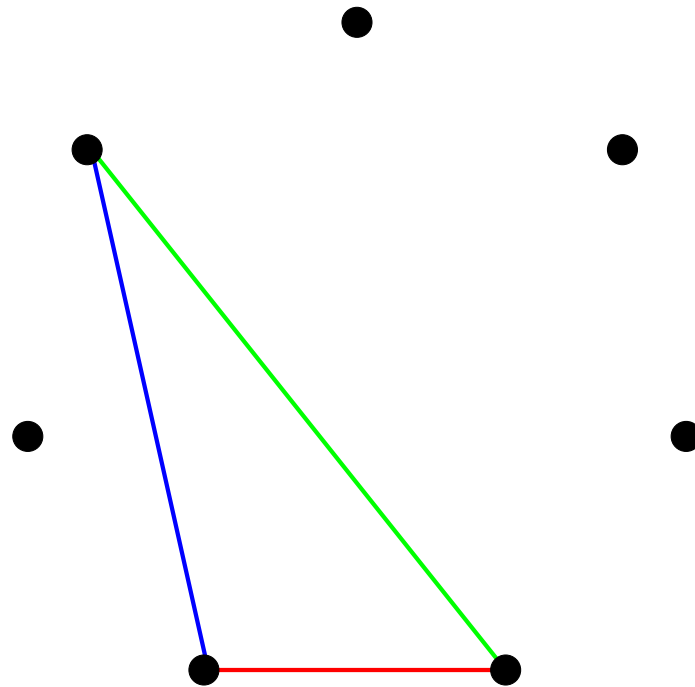
EXAMPLE

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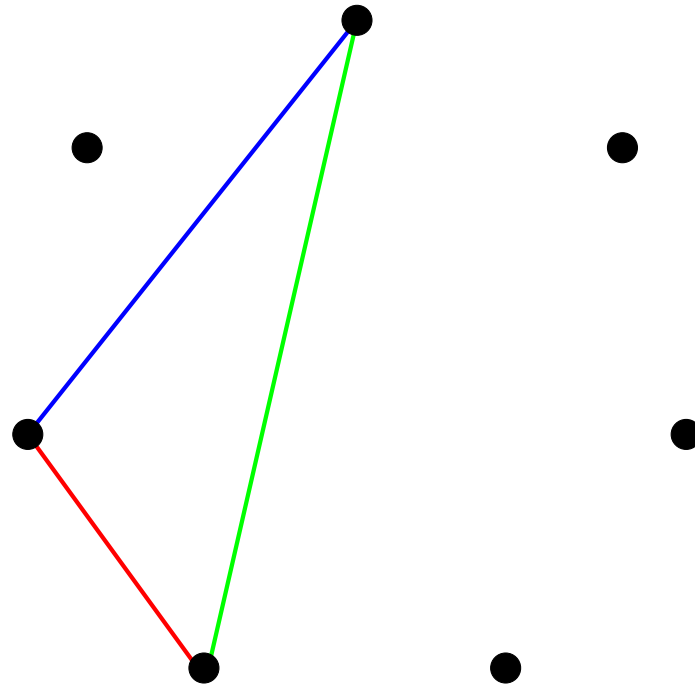
EXAMPLE

Decomposition of K_7 into triangles:



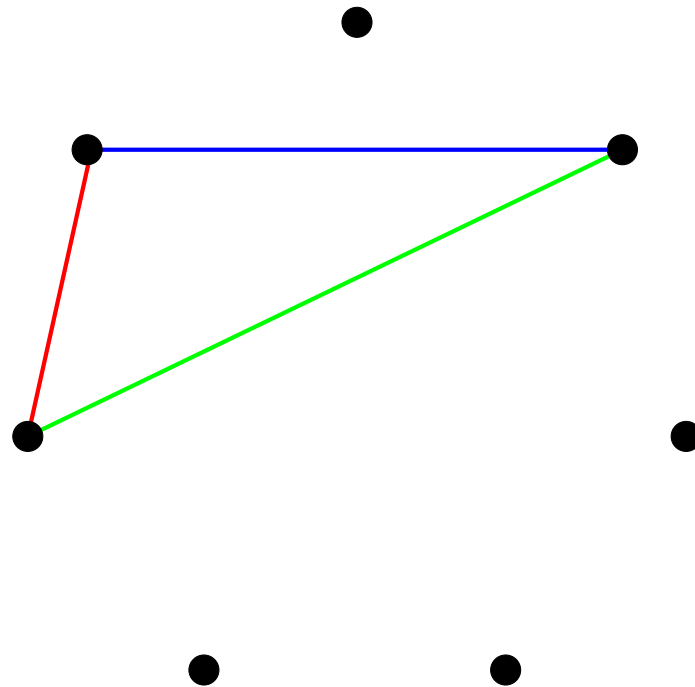
EXAMPLE

Decomposition of K_7 into triangles:



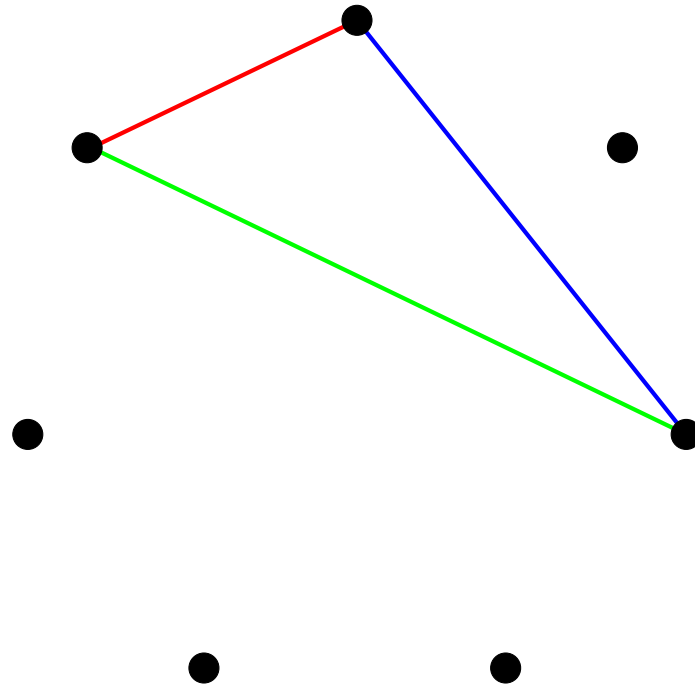
EXAMPLE

Decomposition of K_7 into triangles:



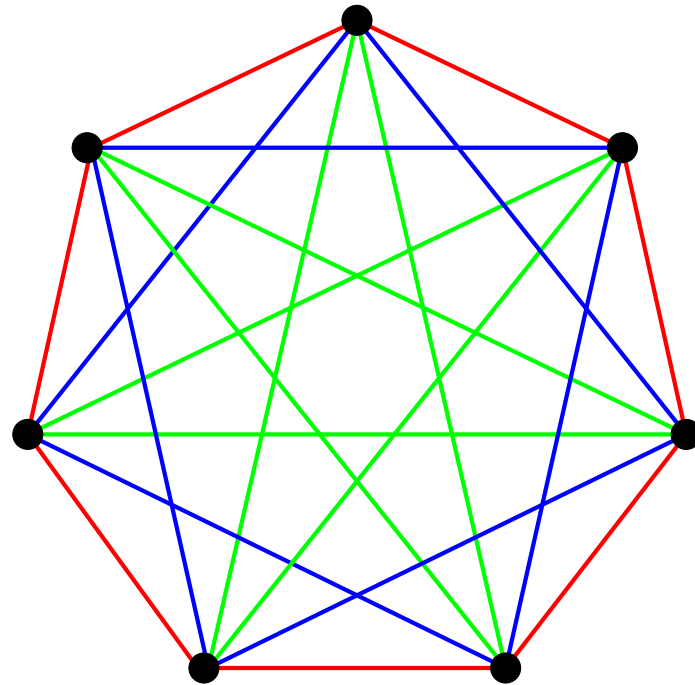
EXAMPLE

Decomposition of K_7 into triangles:



EXAMPLE

Decomposition of K_7 into triangles:



DECOMPOSITIONS

Steiner triple system: decomposition of K_n into triangles

Necessary conditions:

$$n - 1 \equiv 0 \pmod{2}, \quad \binom{n}{2} \equiv 0 \pmod{3}$$

That is:

$$n \equiv 1, 3 \pmod{6}$$

Kirkman 1847: *A Steiner triple system on n elements exists if and only if $n \equiv 1, 3 \pmod{6}$.*

DECOMPOSITIONS

Other values of n ? How many edges of K_n can be decomposed into triangles?

Spencer 1968 answered this question.

Other graphs F ?

Wilson 1976 gave necessary and sufficient conditions for the existence of an F decomposition of K_n for **any** graph F , provided that n is sufficiently large. In particular, there is a K_r decomposition of K_n if n is sufficiently large, and

$$n - 1 \equiv 0 \pmod{r - 1}, \quad \binom{n}{2} \equiv 0 \pmod{\binom{r}{2}}$$

INDUCED DECOMPOSITIONS

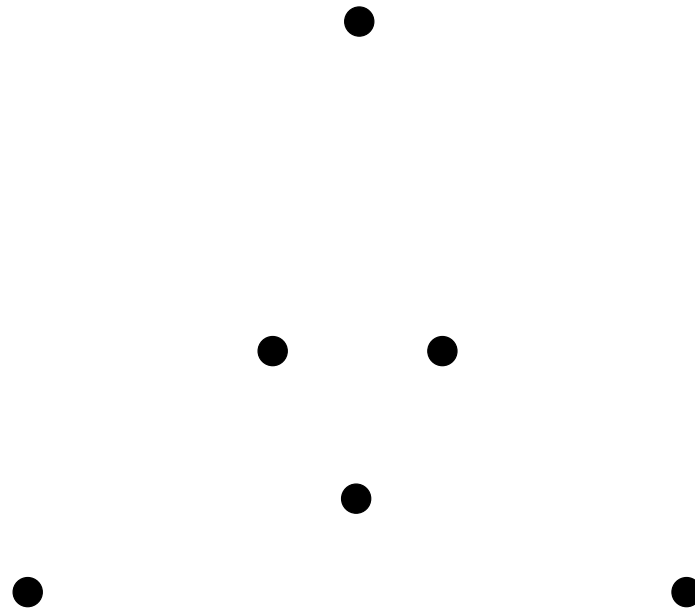
induced F-decomposition of G :

set $\mathcal{F} = \{F_1, F_2, \dots, F_p\}$ of edge-disjoint *induced* subgraphs of G such that

$$F_i \cong F, \quad 1 \leq i \leq p, \quad \text{and} \quad \bigcup_{i=1}^p F_i = G$$

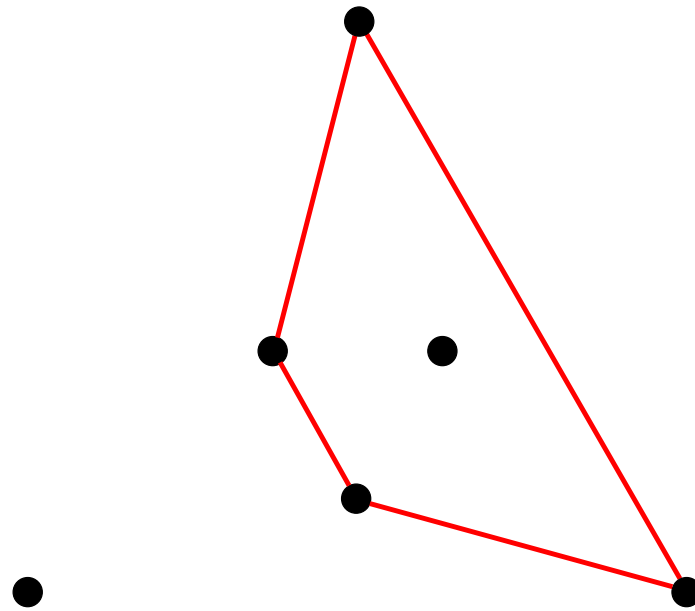
EXAMPLE

Induced decomposition of the octahedron into 4-cycles:



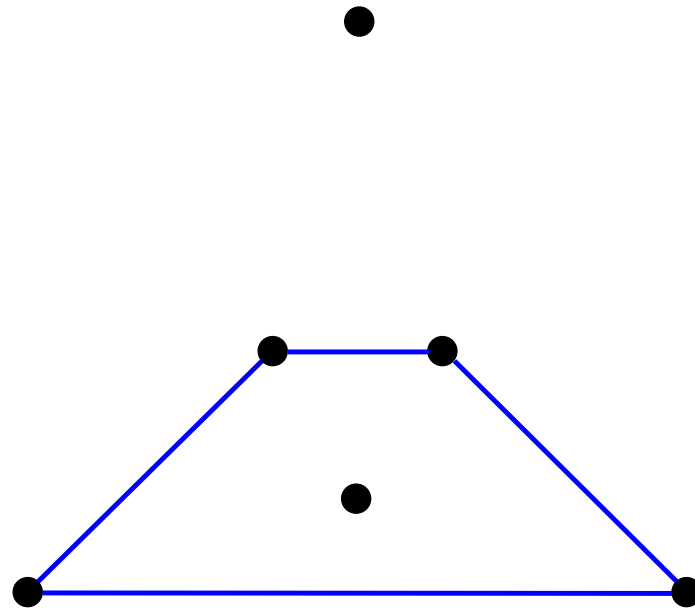
EXAMPLE

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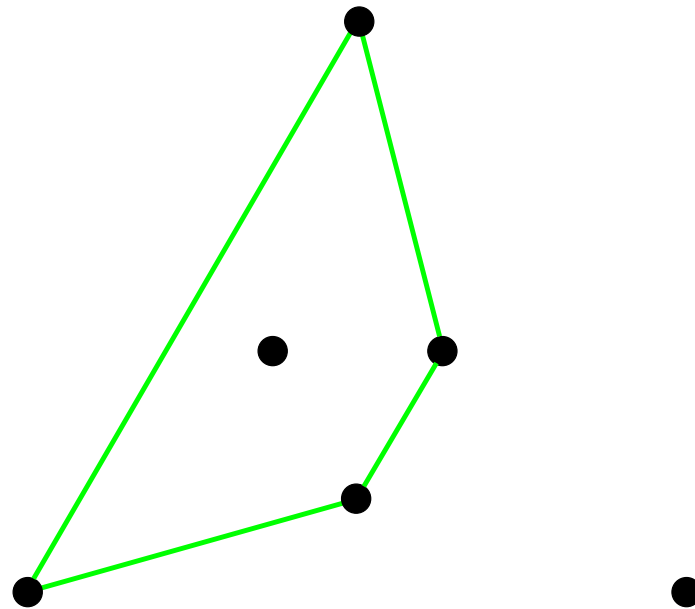
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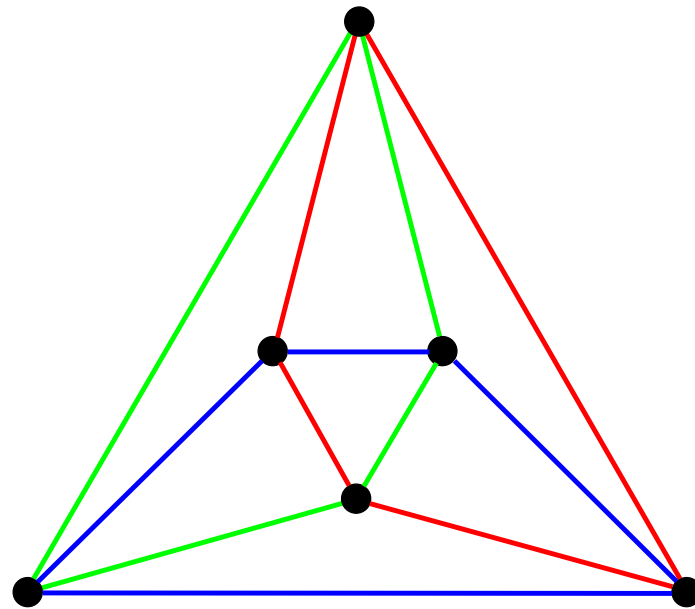
EXAMPLE

Induced decomposition of the octahedron into 4-cycles:



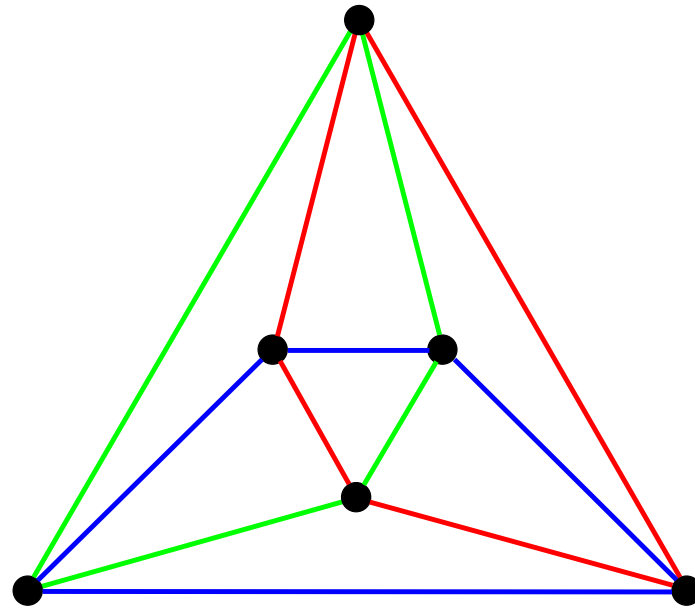
EXAMPLE

Induced decomposition of the octahedron into 4-cycles:



EXAMPLE

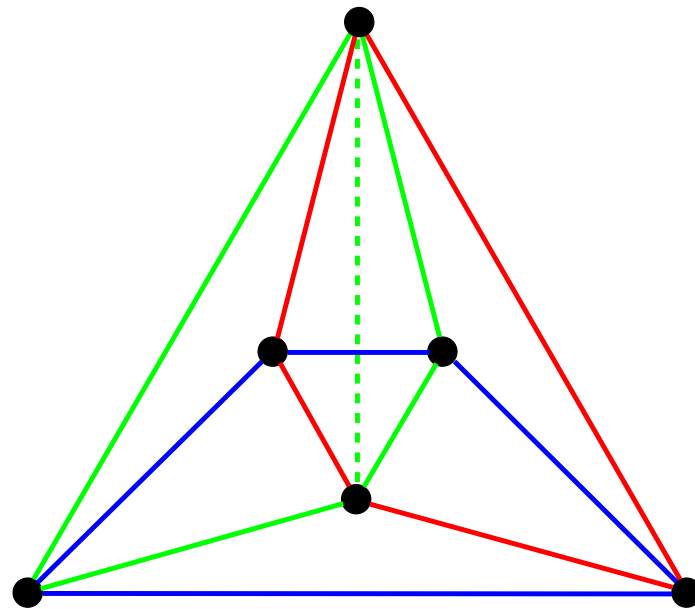
Induced decomposition of the octahedron into 4-cycles:



No two of the 4-cycles share an edge.

EXAMPLE

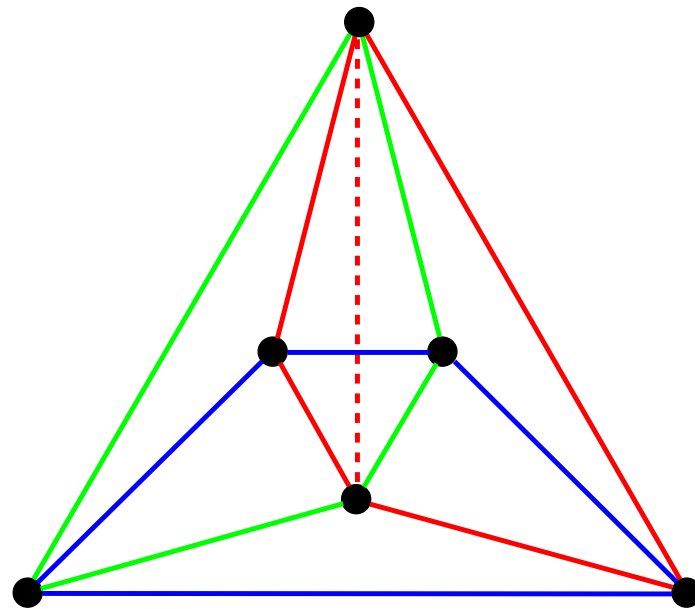
Induced decomposition of the octahedron into 4-cycles:



No two of the 4-cycles share an edge. But they *do* share **nonedges**.

EXAMPLE

Induced decomposition of the octahedron into 4-cycles:



No two of the 4-cycles share an edge. But they *do* share **nonedges**.

INDUCED DECOMPOSITIONS

$ex[n, F]$: maximum number of edges in a graph on n vertices which admits an induced F -decomposition.

extremal graph for F : graph G which has $ex[n, F]$ edges, where $n = v(G)$, and which admits an induced F -decomposition.

EXAMPLES

- $ex[7, K_3] = 21$ K_7 is an extremal graph for K_3
- $ex[6, C_4] = 12$ the octahedron is an extremal graph for C_4

LEXICOGRAPHIC PRODUCTS

lexicographic product $G[H]$ of graphs G and H :

- a copy H_v of H for every vertex v of G
- the copies are pairwise disjoint
- the vertices of H_u are adjacent to the vertices of H_v whenever u and v are adjacent in G

If H is an **empty graph** on t vertices, $G[H]$ is denoted $G[t]$.

EXAMPLE

$K_r[t]$ is the complete r -partite graph with t vertices in each part.

COMPLETE r -PARTITE GRAPHS

THEOREM If K_k admits a K_r decomposition, then:

- $K_k[t]$ admits an induced $K_r[t]$ decomposition
- $ex[tk, K_r[t]] = t^2 \binom{k}{2}$
- the unique extremal graph is $K_k[t]$

Proof

- Since $K_k[t]$ admits an induced $K_r[t]$ decomposition, and $v(K_k[t]) = tk$

$$ex[tk, K_r[t]] \geq e(K_k[t]) = t^2 \binom{k}{2}$$

- Let G be an extremal graph for $K_r[t]$, with $v(G) = tk$.
- Each vertex of G which lies in a copy of $K_r[t]$ is nonadjacent to $t - 1$ vertices of this copy, so has degree at most $t(k - 1)$ in G .
- Each vertex which lies in no copy of $K_r[t]$ has degree zero.
- Therefore

$$ex[tk, K_r[t]] = e(G) \leq \frac{1}{2}tn \times t(k - 1) = t^2 \binom{k}{2}$$

FOUR-CYCLES

K_n clearly admits a K_2 decomposition.

Moreover the complete bipartite graph $K_2[2]$ is the four-cycle C_4 .

Setting $r = 2$ in the theorem:

COROLLARY For all $k \geq 1$,

$$ex[2k, C_4] = 2k(k - 1)$$

and the unique extremal graph is $K_k[2]$.

EXAMPLE When $k = 3$, the extremal graph is $K_3[2]$, the octahedron.

This solves the extremal problem for four-cycles when $v(G)$ is even.

FOUR-CYCLES

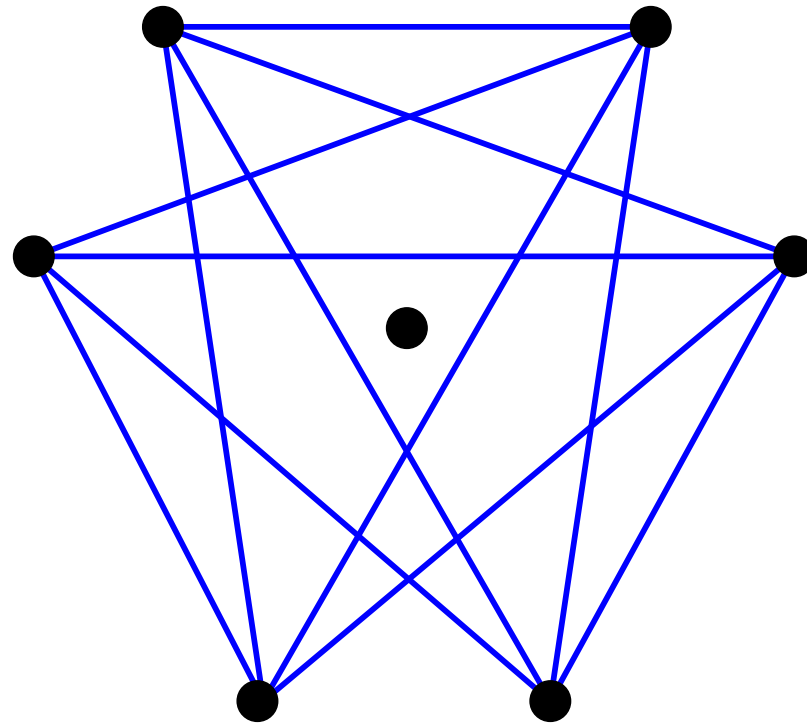
What happens when $v(G)$ is odd?

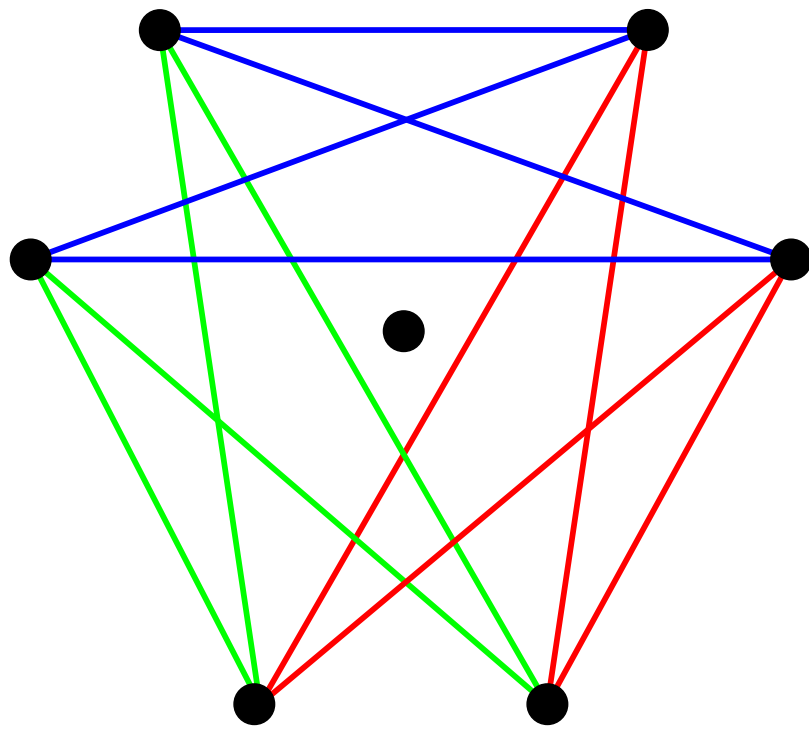
THEOREM For all $k \geq 1$,

$$ex[2k + 1, C_4] = 2k(k - 1) = ex[2k, C_4]$$

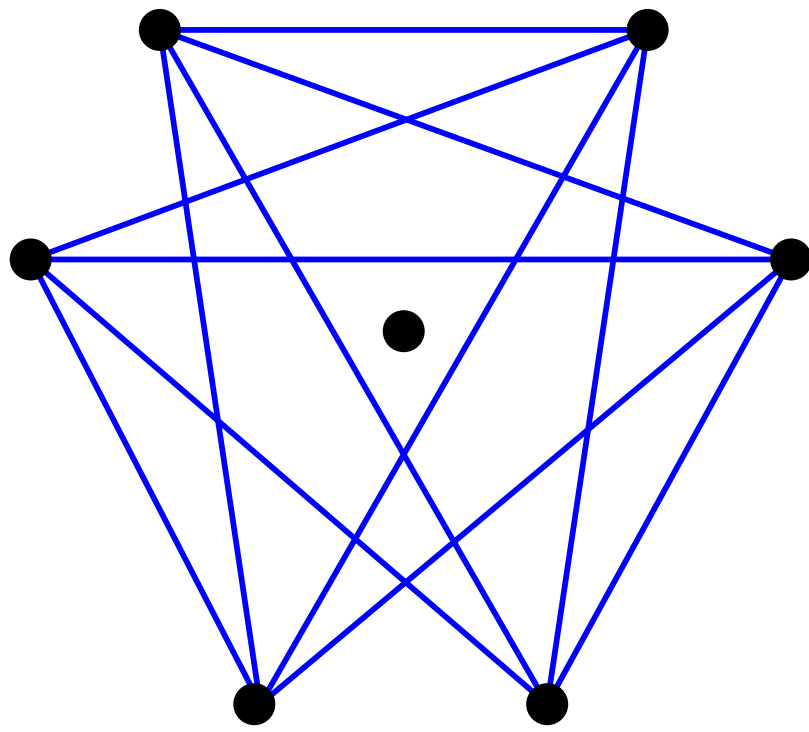
One extremal graph is $K_1 + K_k[2]$. But there are others.

EXAMPLE: $k = 3, n = 7$

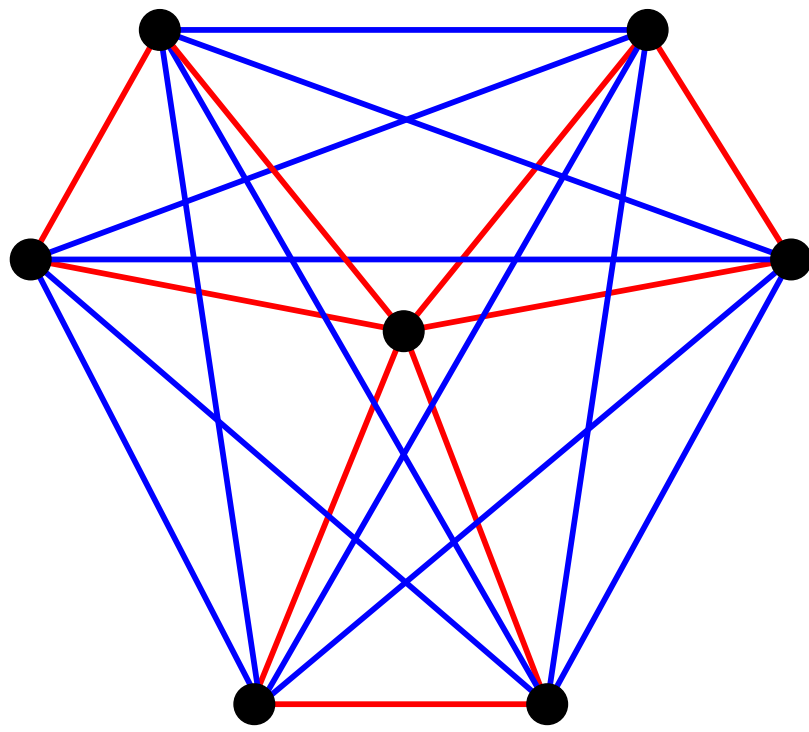




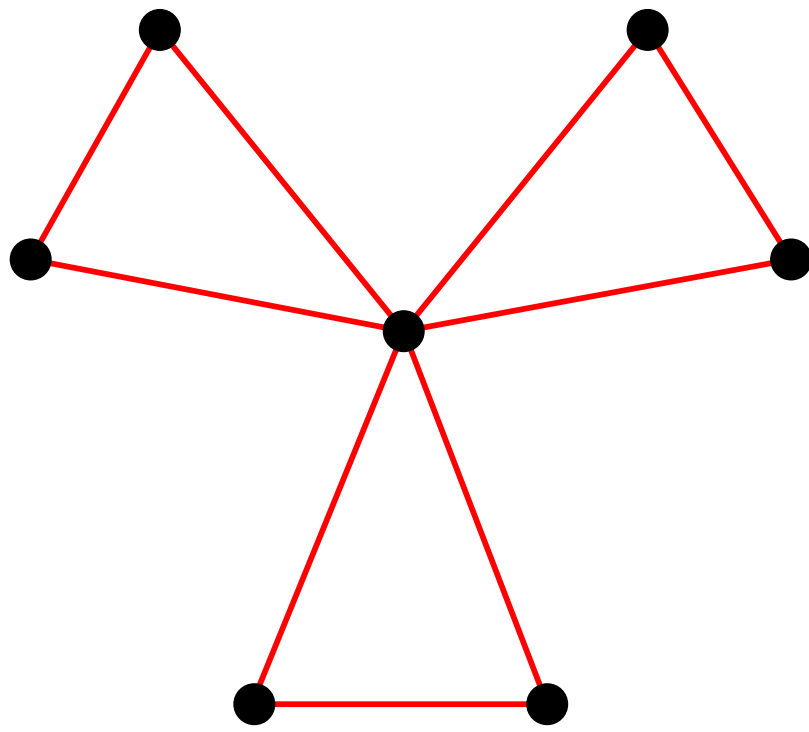
INDUCED DECOMPOSITION



WHAT IS THIS GRAPH?

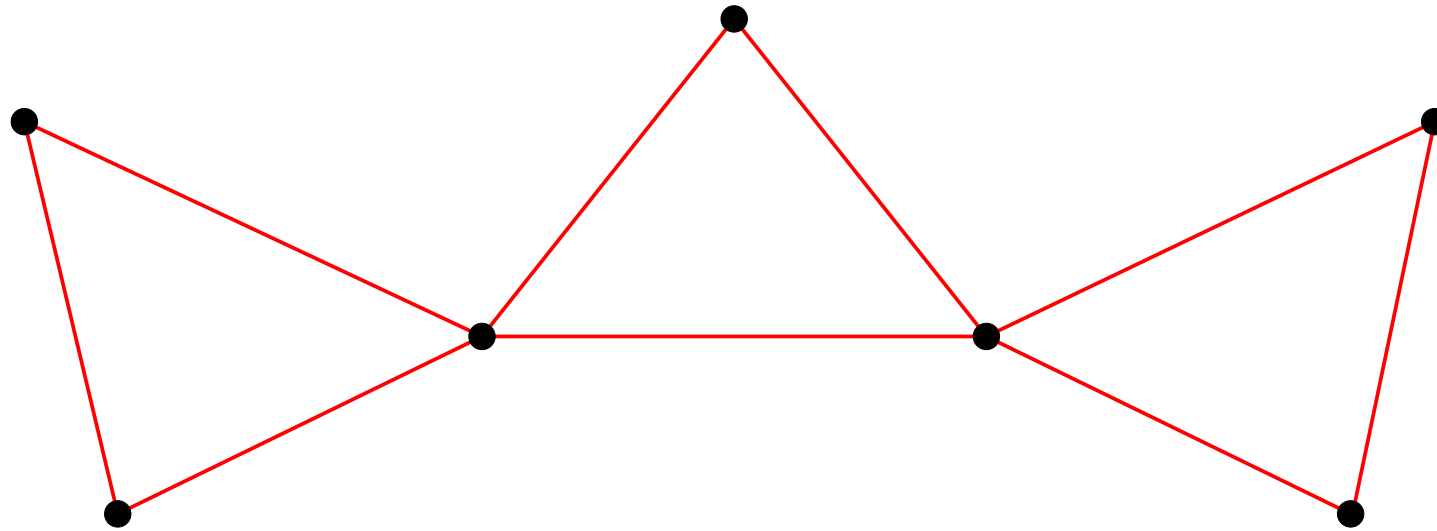


COMPLEMENT

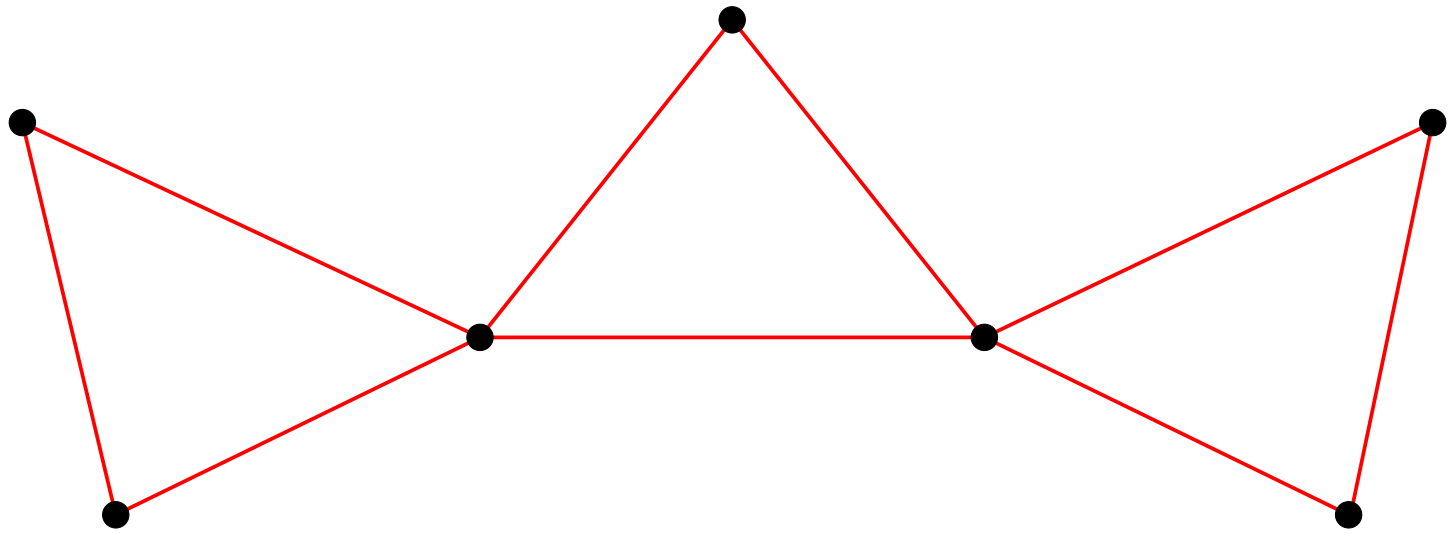


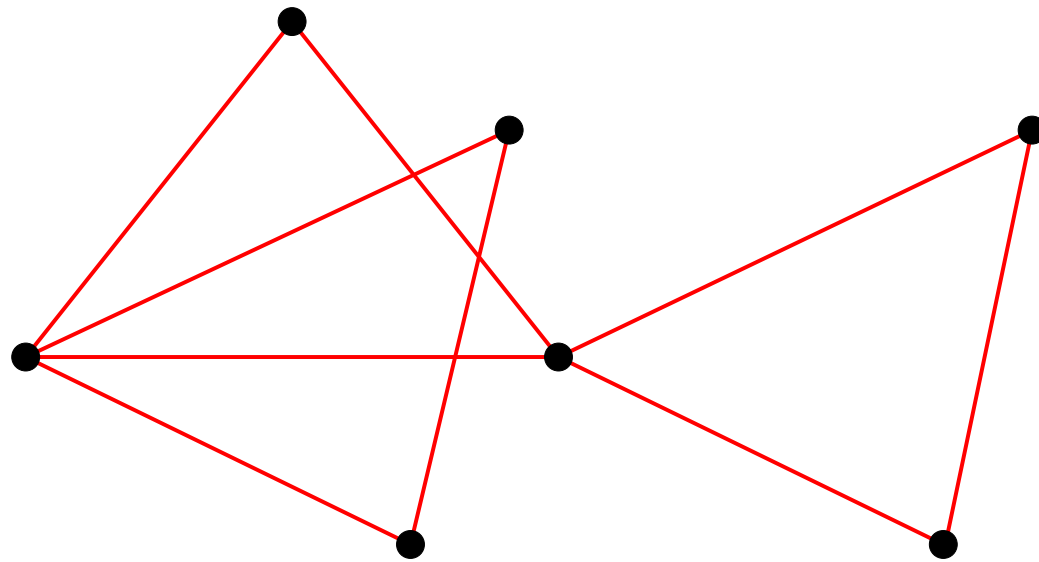
TRIANGULAR CACTUS

triangular cactus: connected graph all of whose blocks are triangles

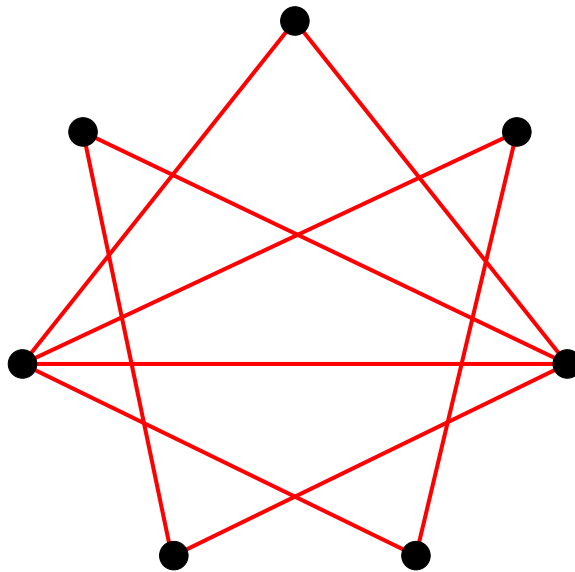


THEOREM For n odd, the extremal graphs for four-cycles are the complements of triangular cacti.

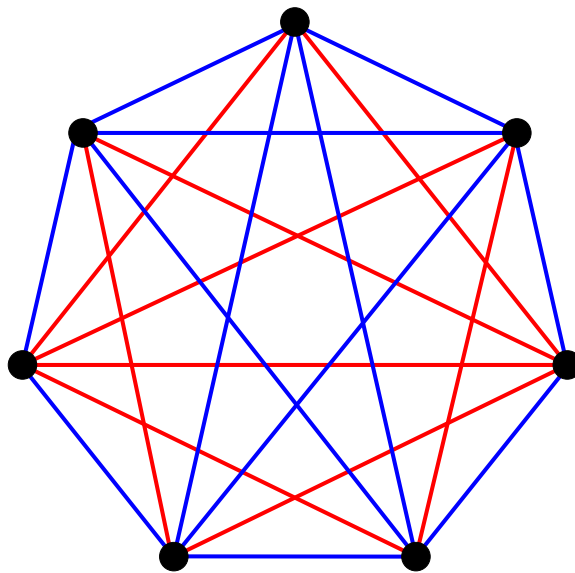




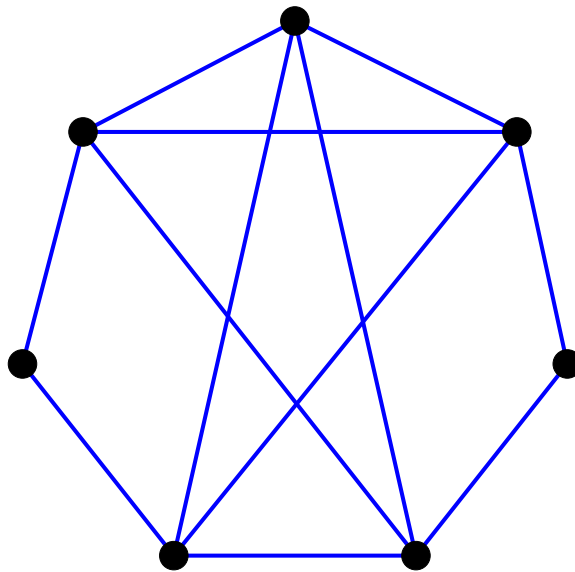
TRIANGULAR CACTUS REDRAWN



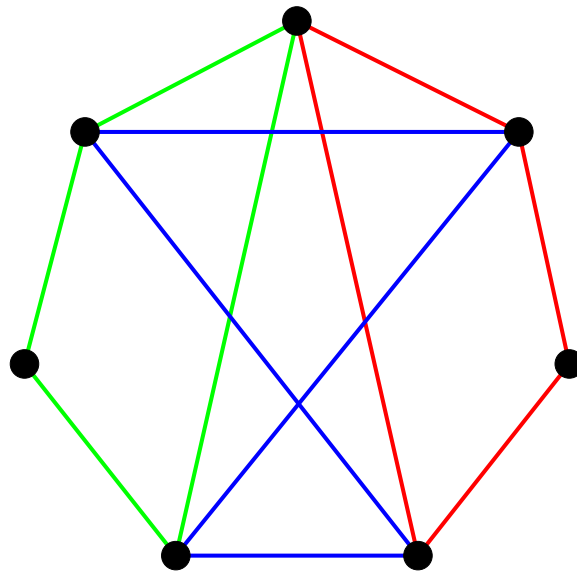
TRIANGULAR CACTUS REDRAWN



COMPLEMENT



COMPLEMENT



INDUCED DECOMPOSITION

STARS

THEOREM

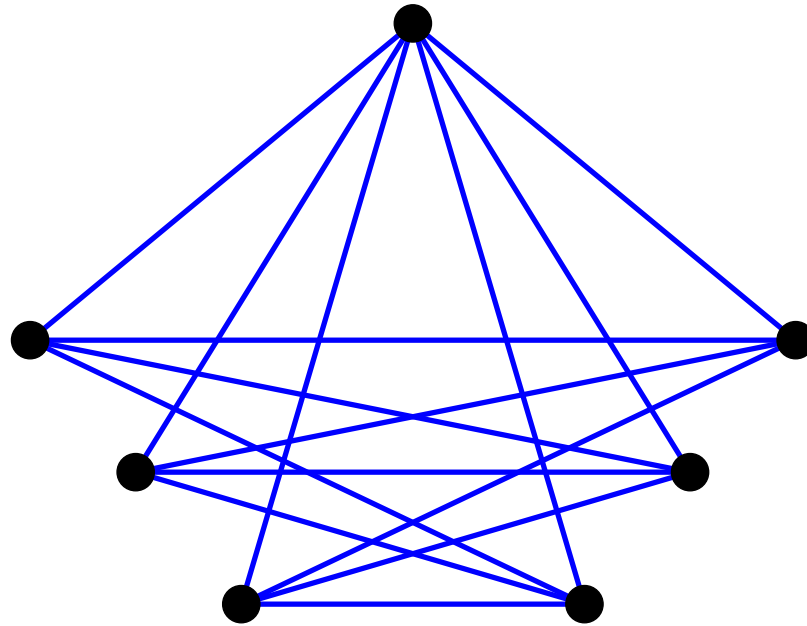
Let $n \equiv r \pmod{k}$, where $0 \leq r \leq k - 1$. Then

$$ex[n, K_{1,k}] = \frac{1}{2}(n - r)(n - k + r)$$

and the unique extremal graph is the complete $\lceil n/k \rceil$ -partite graph in which each part except possibly one has k vertices.

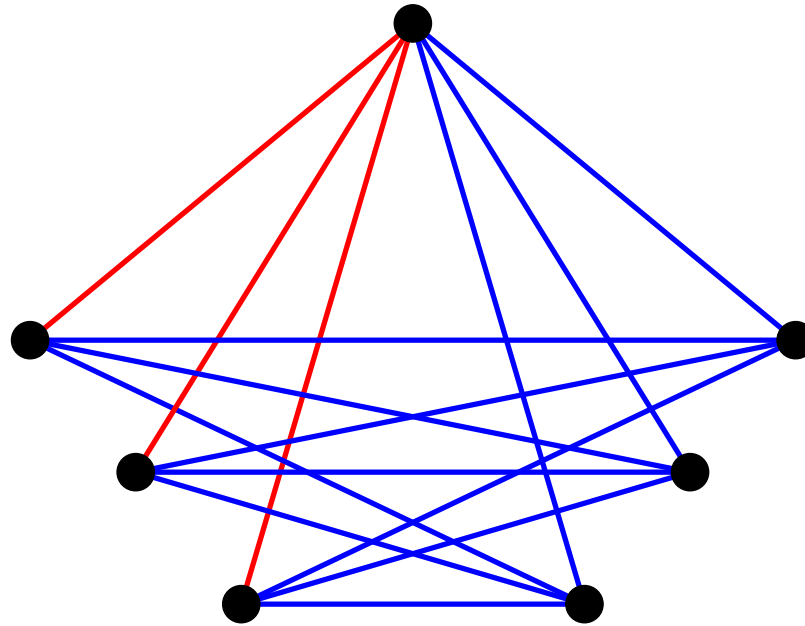
STARS

EXAMPLE: $n = 7$, $k = 3$, $r = 1$



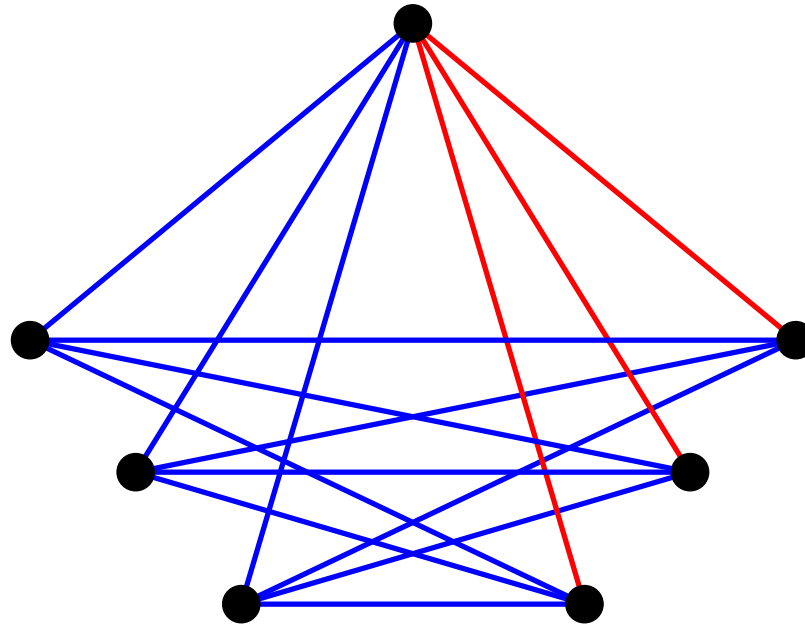
STARS

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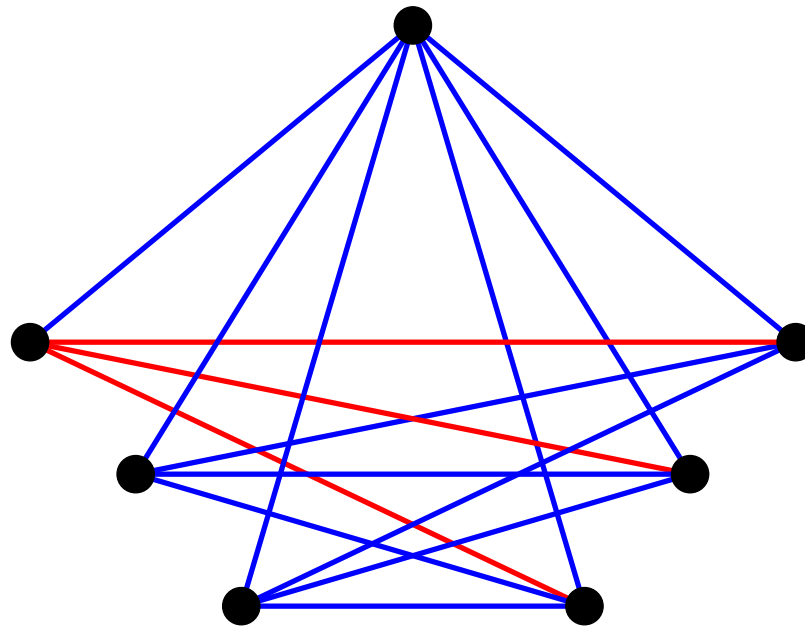
STARS

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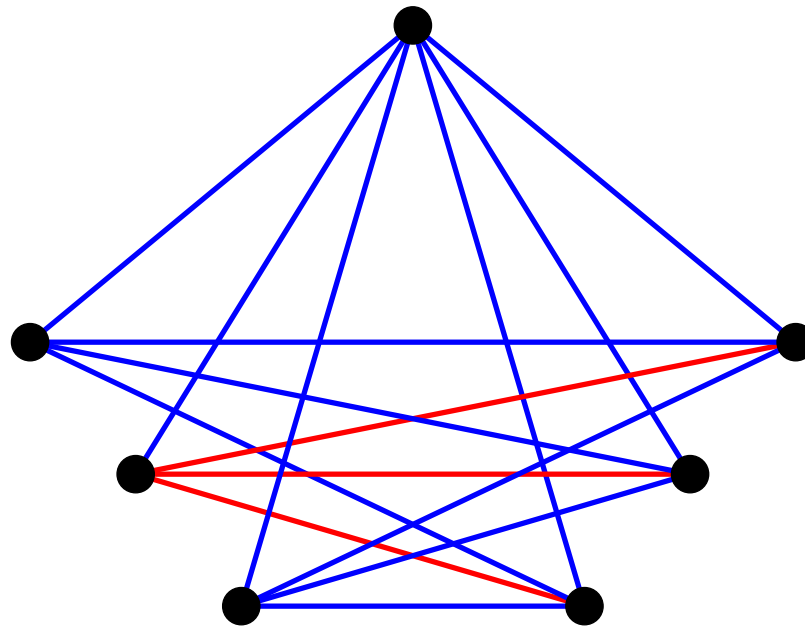
STARS

EXAMPLE: $n = 7$, $k = 3$, $r = 1$



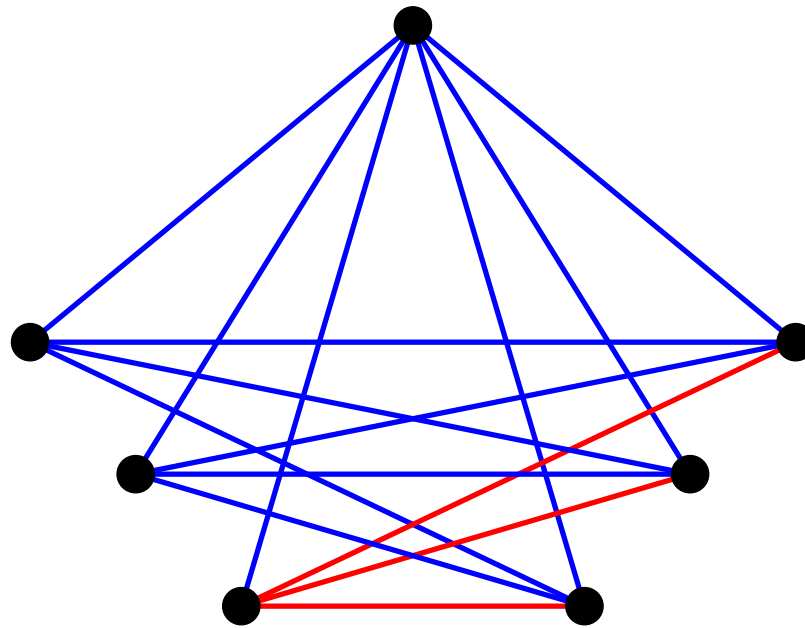
STARS

EXAMPLE: $n = 7$, $k = 3$, $r = 1$

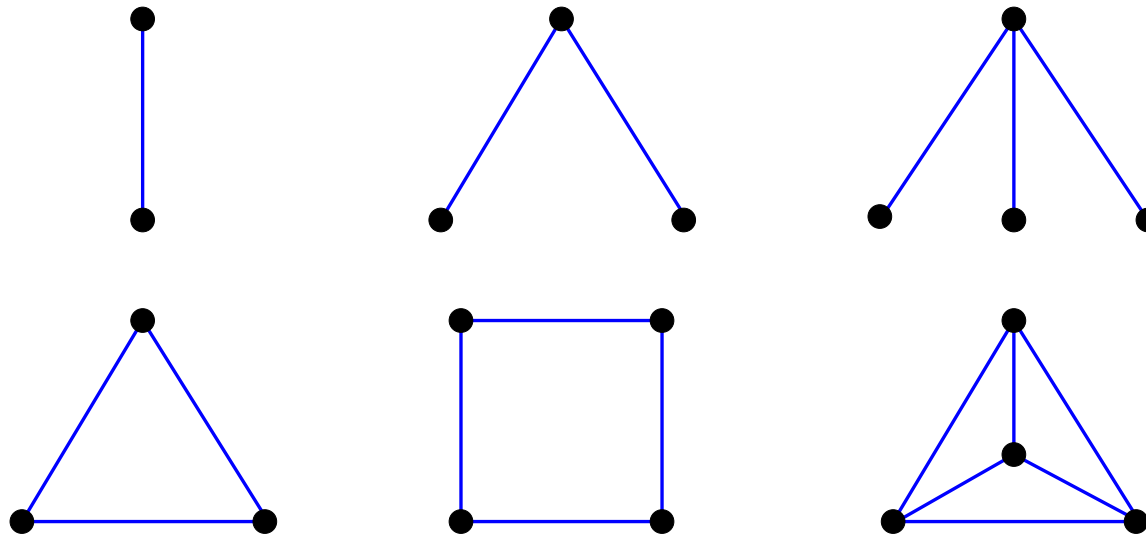


STARS

EXAMPLE: $n = 7$, $k = 3$, $r = 1$

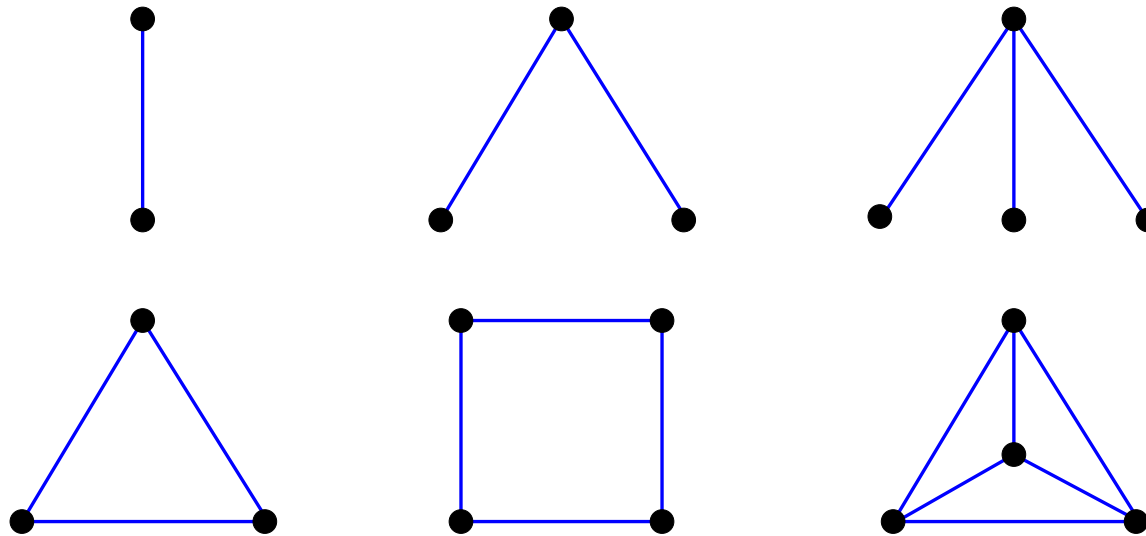


SMALL GRAPHS



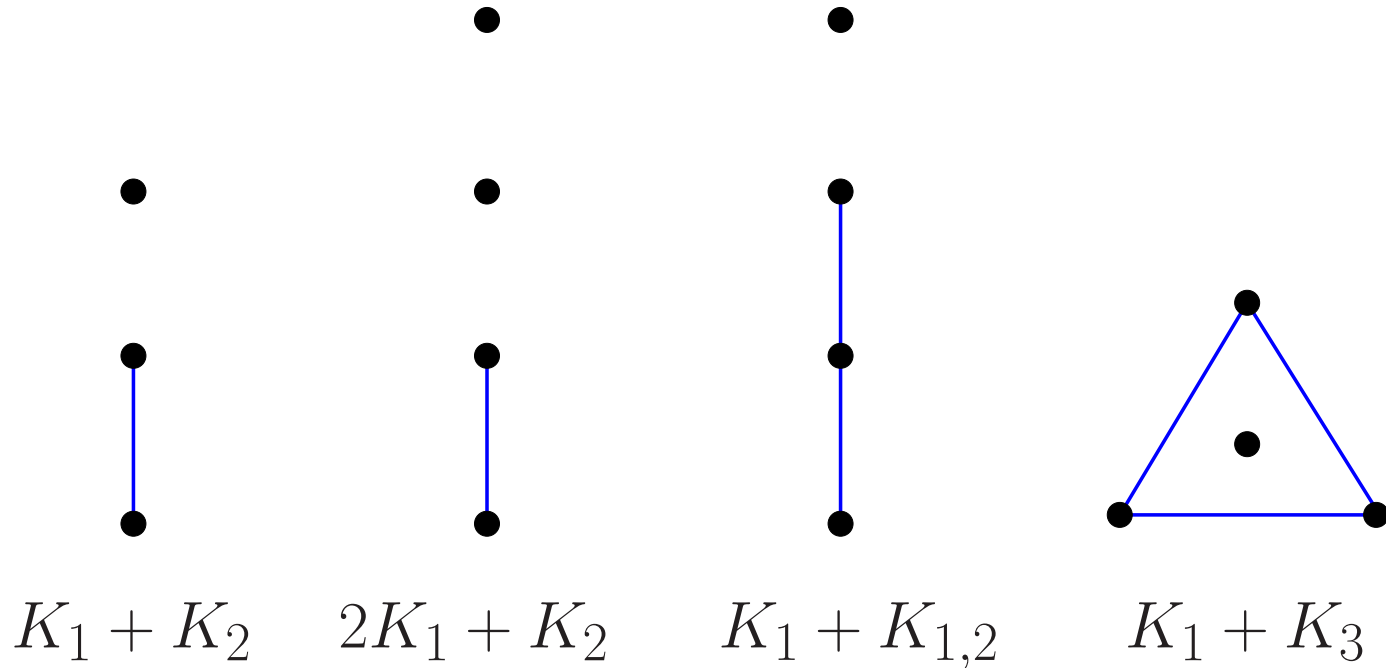
STARS, CYCLES AND COMPLETE GRAPHS

SMALL GRAPHS



STARS, CYCLES AND COMPLETE GRAPHS ✓

SMALL GRAPHS



GRAPHS WITH ISOLATED VERTICES

SMALL GRAPHS



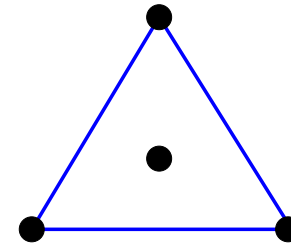
$K_1 + K_2$



$2K_1 + K_2$



$K_1 + K_{1,2}$



$K_1 + K_3$

GRAPHS WITH ISOLATED VERTICES ✓

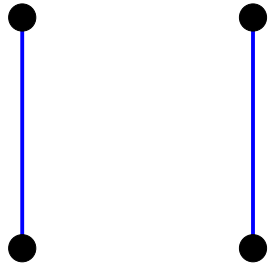
SMALL GRAPHS

Extremal graphs for small graphs with isolated vertices:

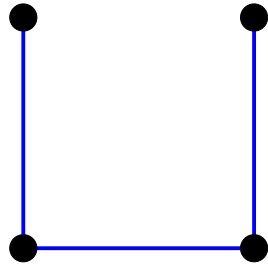
- $K_1 + K_2$: $K_1 + K_{n-1}$
- $K_1 + K_3$: $K_1 + K_{n-1}$, $n \equiv 2, 4 \pmod{6}$...
- $2K_1 + K_2$: $2K_1 + K_{n-2}$
- $K_1 + K_{1,2}$: $K_1 + K_r[2]$, $n = 2r + 1$, or P_5 ...

SMALL GRAPHS

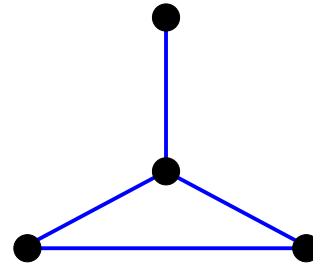
Remaining small graphs:



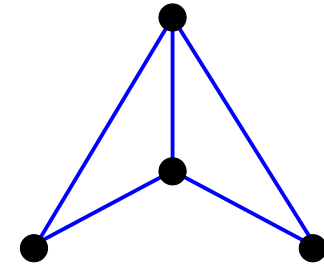
$2K_2$



P_4



$K_{1,3} + e$



$K_4 \setminus e$

This is where the fun starts!

SMALL GRAPHS: $2K_2$

THEOREM For $k \geq 3$,

$$ex[3k, 2K_2] \geq 3k(k - 1)$$

SMALL GRAPHS: $2K_2$

THEOREM For $k \geq 3$,

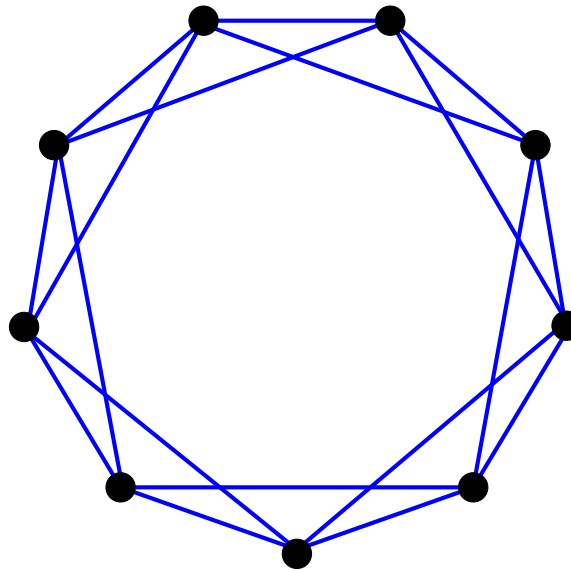
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EXAMPLE: C_{3k}^{k-1}

SMALL GRAPHS: $2K_2$

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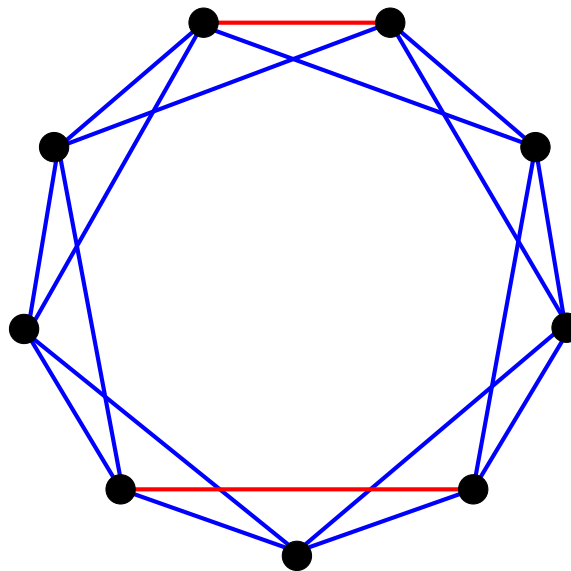
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SMALL GRAPHS: $2K_2$

THEOREM For $k \geq 3$,

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SMALL GRAPHS: $2K_2$

THEOREM For $k \geq 3$,

$$ex[3k, 2K_2] \geq 3k(k - 1)$$

EXAMPLE: C_{3k}^{k-1}

Similar constructions and bounds for $n = 3k + 1$ and $n = 3k + 2$.

SMALL GRAPHS: $2K_2$

THEOREM If G admits an induced $2K_2$ decomposition, then

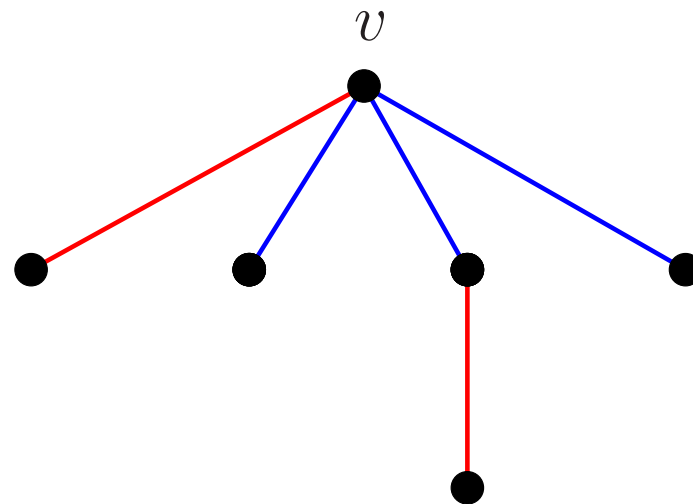
$$\Delta \leq \binom{n - \Delta - 1}{2}$$

Proof For any vertex v , and in particular a vertex of maximum degree, the edges incident to v must be paired with edges in the subgraph induced by the non-neighbours of v .

SMALL GRAPHS: $2K_2$

THEOREM If G admits an induced $2K_2$ decomposition, then

$$\Delta \leq \binom{n - \Delta - 1}{2}$$

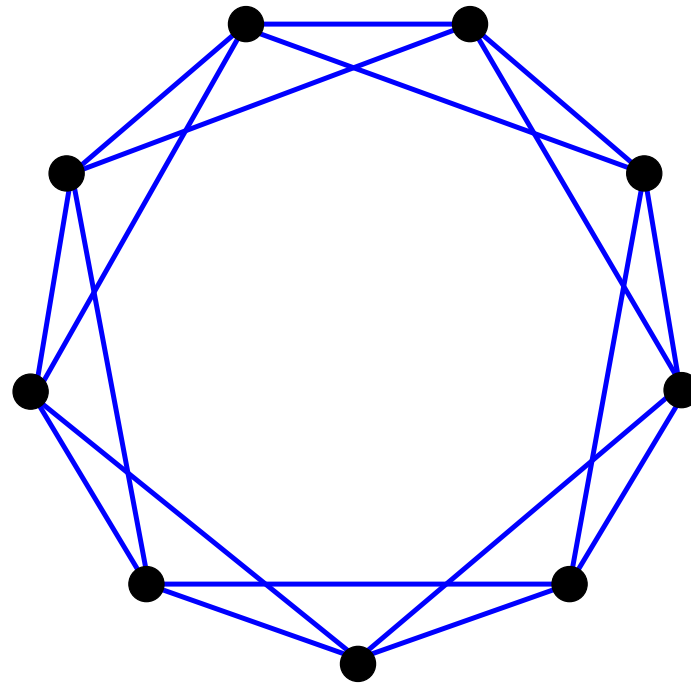


EXTREMAL GRAPHS FOR $2K_2$

- $n = 4$: $2K_2$
- $n = 5$: $K_1 + 2K_2$
- $n = 6$: $2K_3, C_6$
- $n = 7$: $2K_3$ plus a vertex joined to one vertex in each K_3
- $n = 8$: $2K_4, Q_3$, two copies of $K_4 \setminus e$ joined by two edges

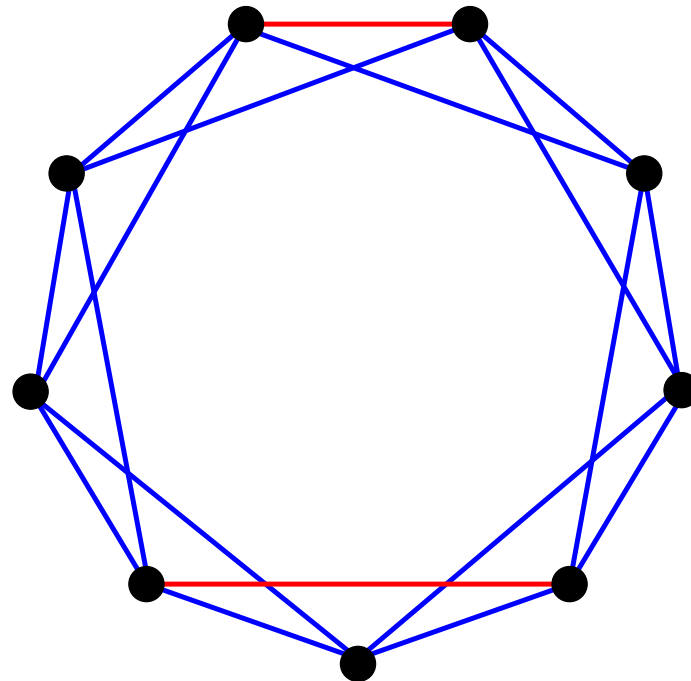
SMALL GRAPHS: $2K_2$

EXAMPLE 1: C_9^2



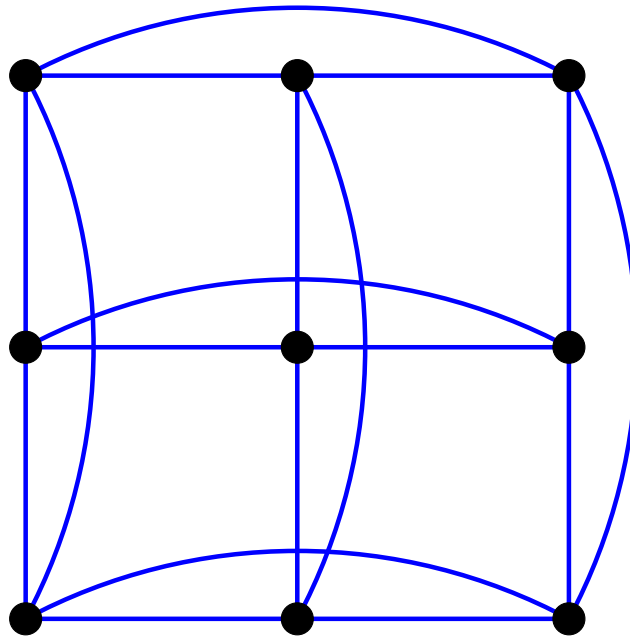
SMALL GRAPHS: $2K_2$

EXAMPLE 1: C_9^2



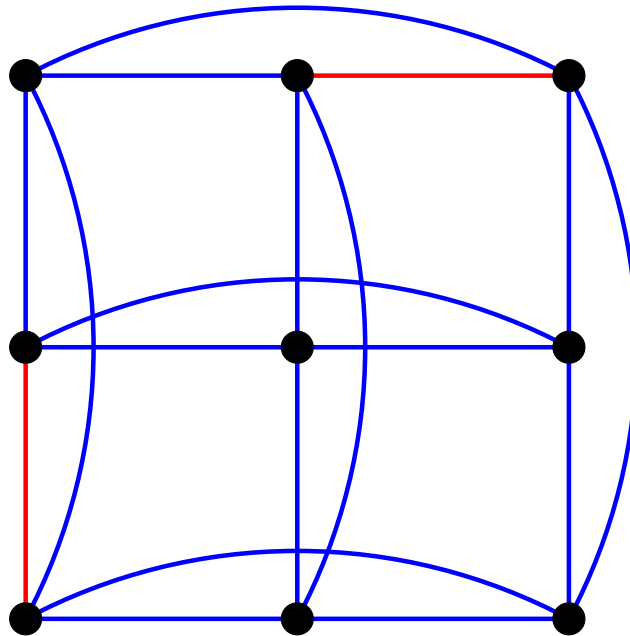
SMALL GRAPHS: $2K_2$

EXAMPLE 2: $K_3 \square K_3$



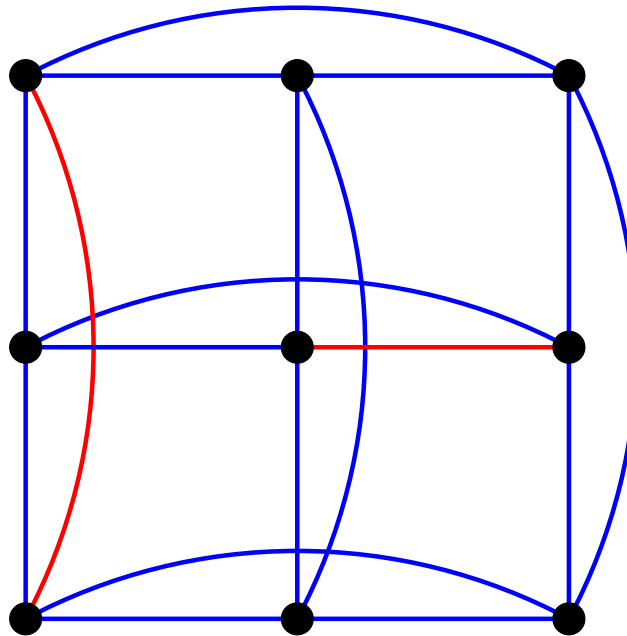
SMALL GRAPHS: $2K_2$

EXAMPLE 2: $K_3 \square K_3$



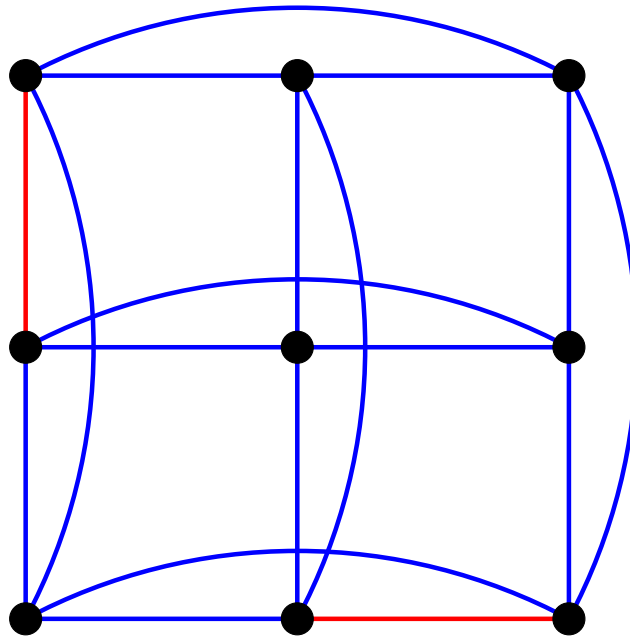
SMALL GRAPHS: $2K_2$

EXAMPLE 2: $K_3 \square K_3$



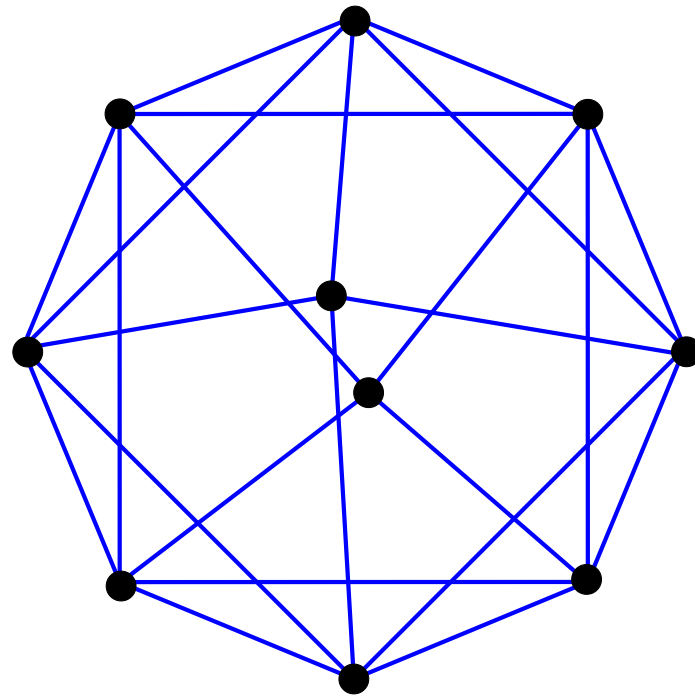
SMALL GRAPHS: $2K_2$

EXAMPLE 2: $K_3 \square K_3$



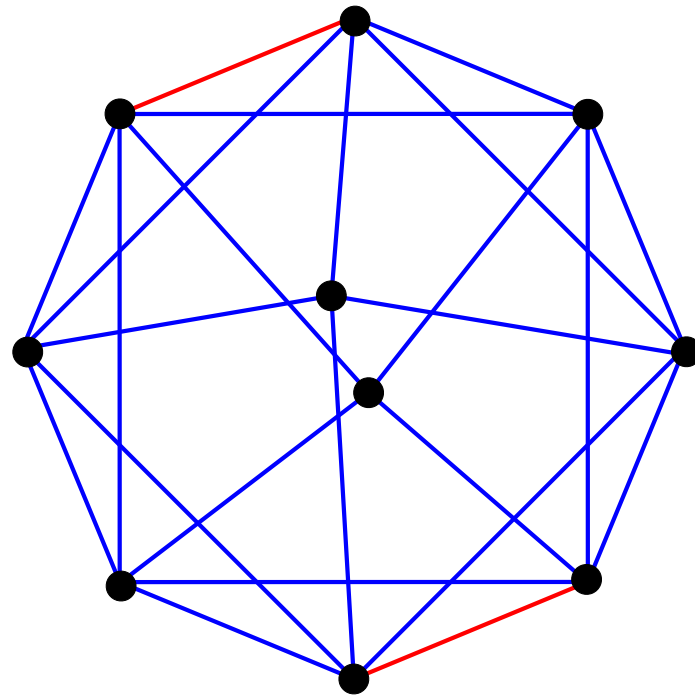
SMALL GRAPHS: $2K_2$

EXAMPLE 3: The Verre à Pied Graph



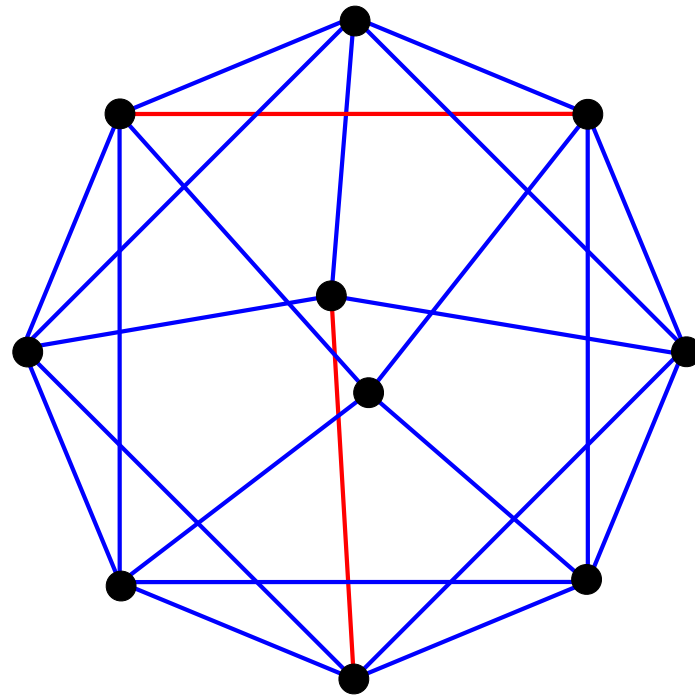
SMALL GRAPHS: $2K_2$

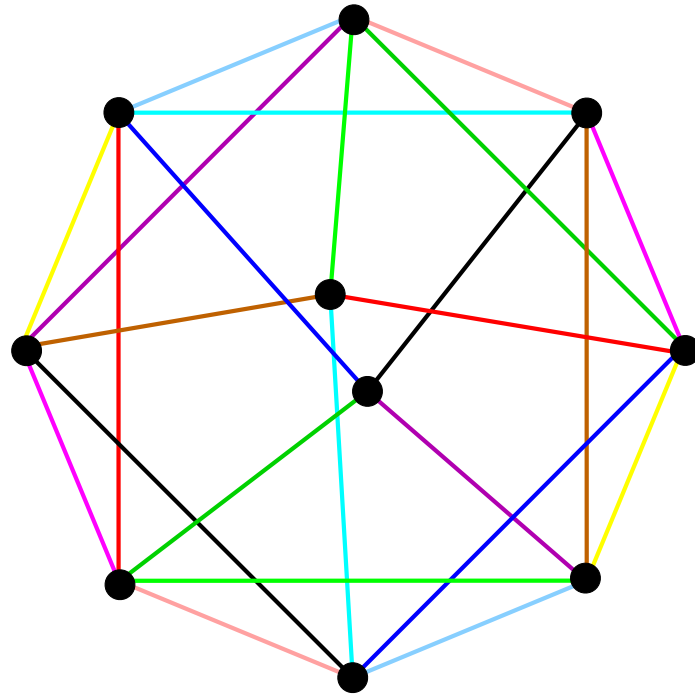
EXAMPLE 3: The Verre à Pied Graph



SMALL GRAPHS: $2K_2$

EXAMPLE 3: The Verre à Pied Graph





THE VERRE À PIED GRAPH

EXTREMAL GRAPHS FOR $2K_2$

- $n = 9$: C_9^2 , $K_3 \square K_3$
- $n = 10$: Verre à Pied Graph C_8^2 plus two vertices joined to disjoint sets of four nonconsecutive vertices of C_8
- $n = 11$: C_9^2 plus two vertices joined to disjoint sets of four nonconsecutive vertices of C_9
- $n = 12$: C_{12}^3

SMALL GRAPHS: P_4

PROPOSITION

If F is a spanning subgraph of G , then

$$ex[n, F] \geq \frac{e(F)}{e(G)} ex[n, G]$$

COROLLARY

$$ex[n, P_4] \geq \frac{3}{4} ex[n, C_4]$$

Therefore

$$ex[2k, P_4] \geq 3 \binom{k}{2} \quad \text{and} \quad ex[2k+1, P_4] \geq 3 \binom{k}{2}$$

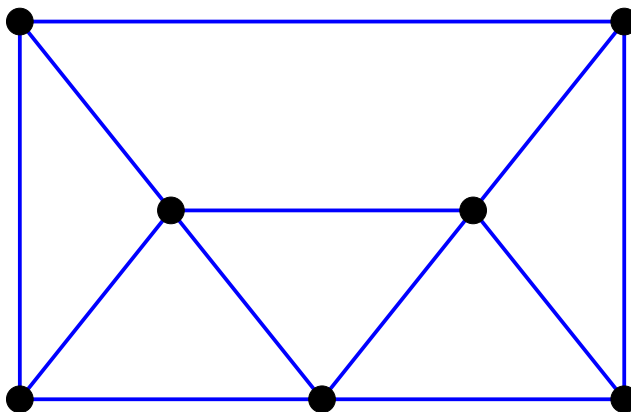
SMALL GRAPHS: P_4

Bound

$$ex[2k + 1, P_4] \geq 3 \binom{k}{2}$$

not sharp for $k = 3$:

$$ex[7, P_4] \geq 12$$



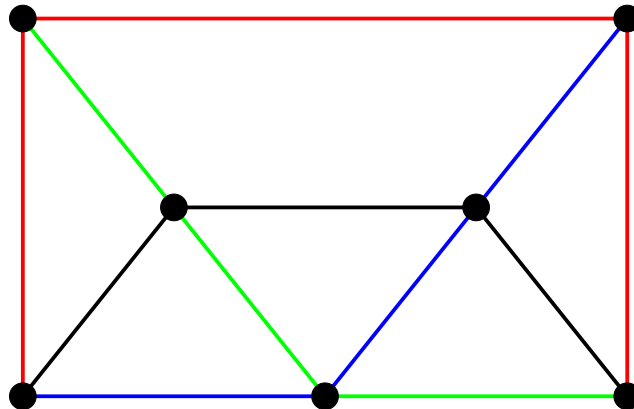
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SMALL GRAPHS: P_4

The best upper bound on $ex[n, P_4]$ that we are able to obtain, *even when the problem is restricted to regular graphs*, is

$$ex[n, P_4] \leq \binom{n}{2} - cn$$

where c is a constant, $c < 1$. The lower and upper bounds are thus very far apart.

A similar situation applies to the graph $K_{1,3} + e$. For $n \equiv 0 \pmod{5}$, we have:

$$\frac{2n^2}{5} - 2n < ex[n, K_{1,3} + e] < \binom{n}{2} - \frac{n}{4}$$

SMALL GRAPHS: $K_4 \setminus e$

UPPER BOUND:

$$ex[n, K_4 \setminus e] \leq \binom{n}{2} - \frac{n}{5}$$

SMALL GRAPHS: $K_4 \setminus e$

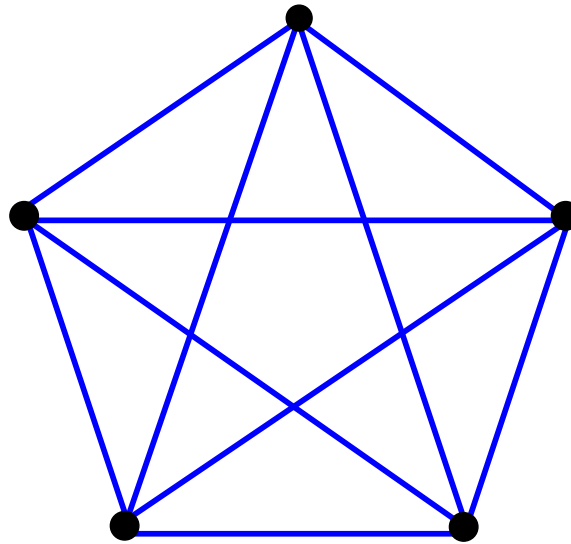
LOWER BOUND

Ingredients:

- P_3 decomposition of K_5
- Steiner triple system: K_3 decomposition of K_r , $r \equiv 1, 3 \pmod{6}$

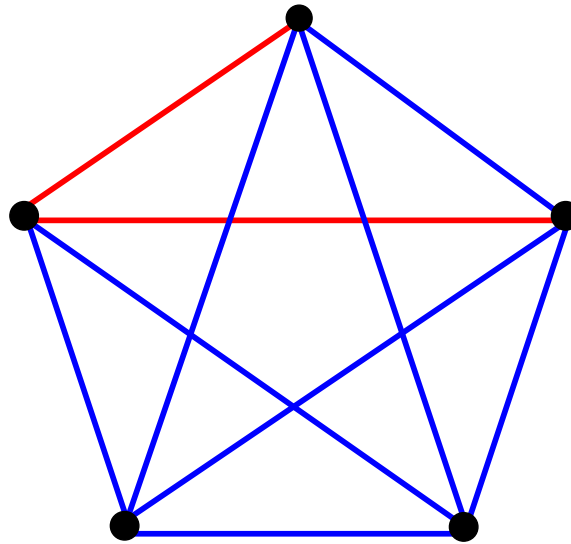
SMALL GRAPHS: $K_4 \setminus e$

P_3 decomposition of K_5 :



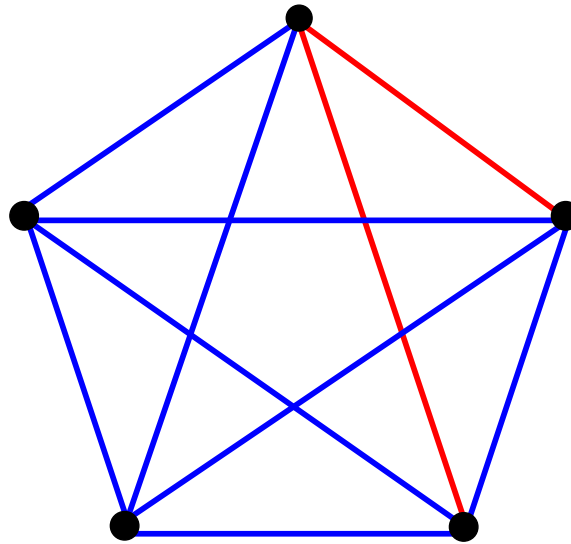
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P_3 decomposition of K_5 :



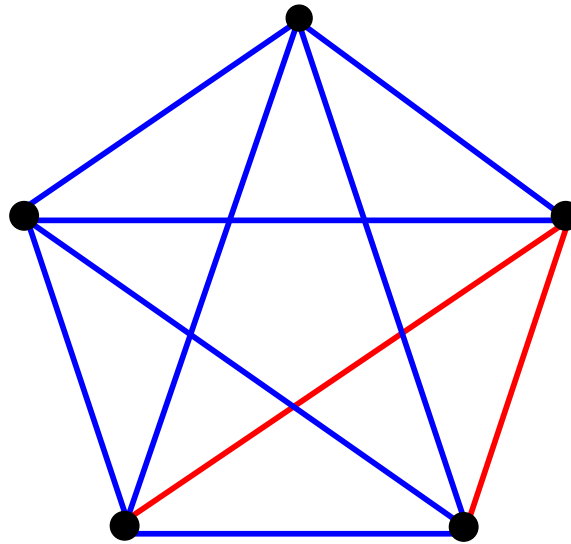
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P_3 decomposition of K_5 :



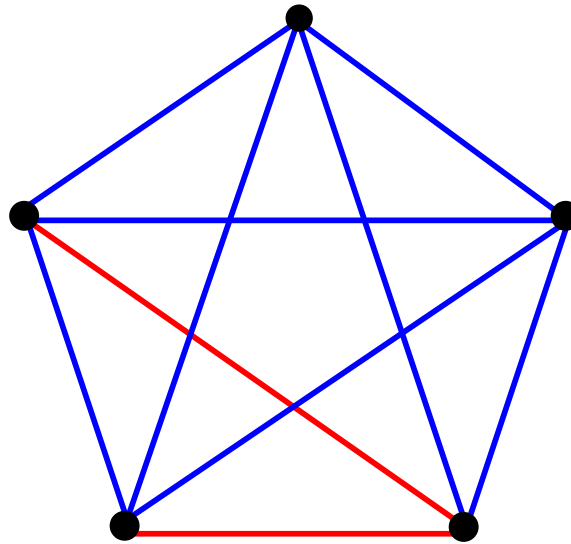
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P_3 decomposition of K_5 :



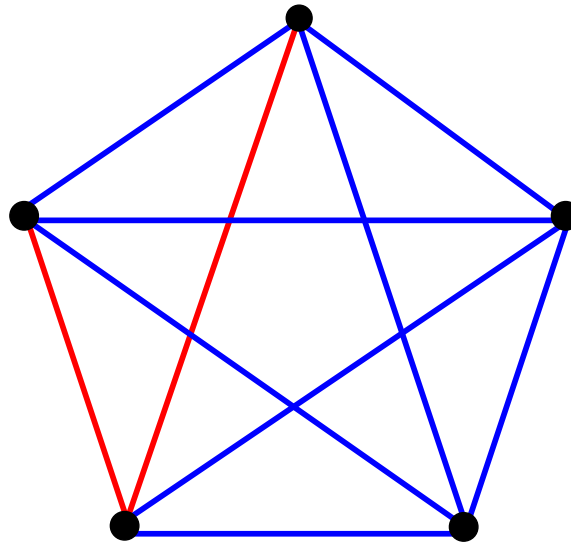
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P_3 decomposition of K_5 :



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SMALL GRAPHS: $K_4 \setminus e$

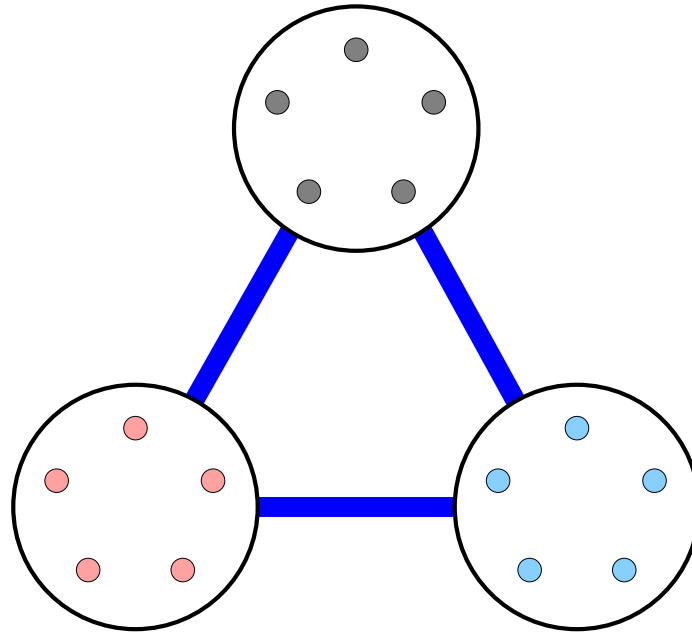
- P_3 decomposition of K_5 ✓

SMALL GRAPHS: $K_4 \setminus e$

- P_3 decomposition of K_5 ✓

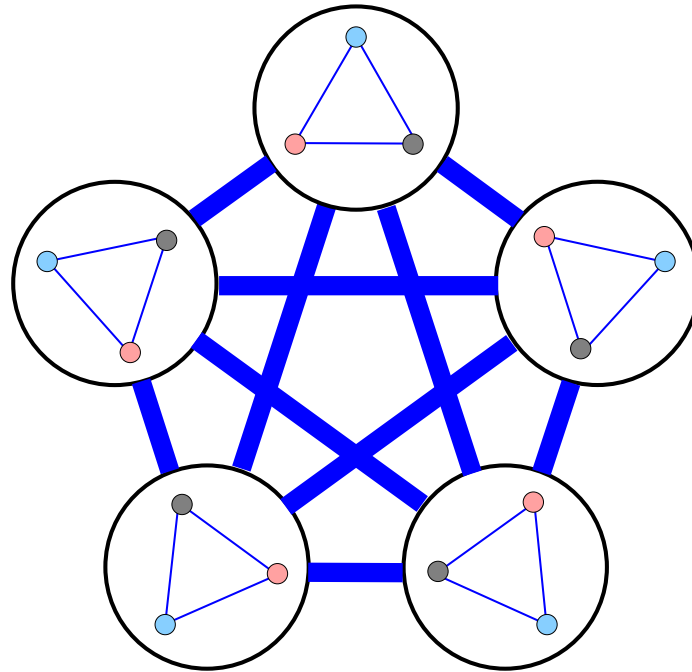
This decomposition gives rise to an induced $K_4 \setminus e$ decomposition of the complete tripartite graph $K_3[5]$.

SMALL GRAPHS: $K_4 \setminus e$



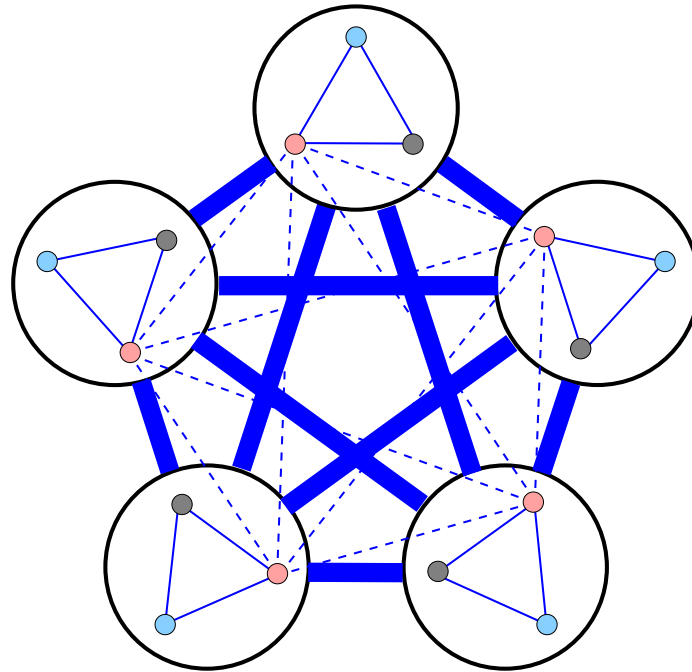
$K_3[5]$

SMALL GRAPHS: $K_4 \setminus e$



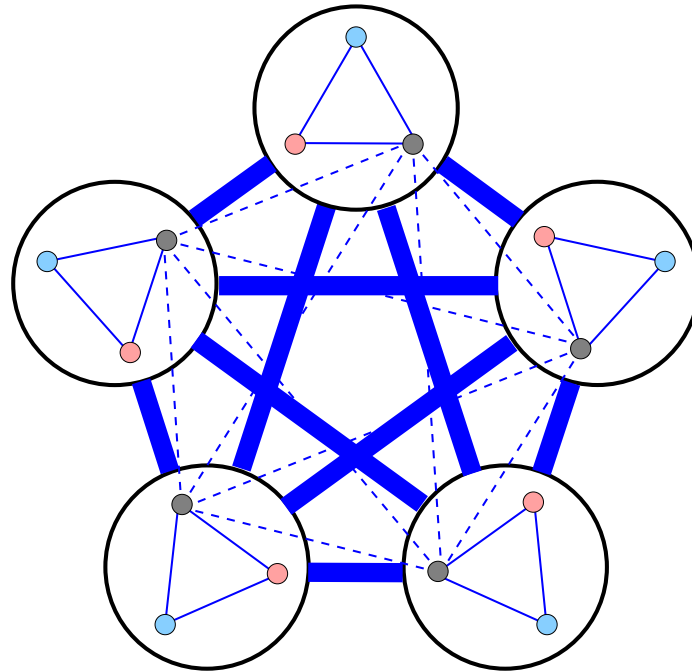
$$K_5[K_3]$$

SMALL GRAPHS: $K_4 \setminus e$



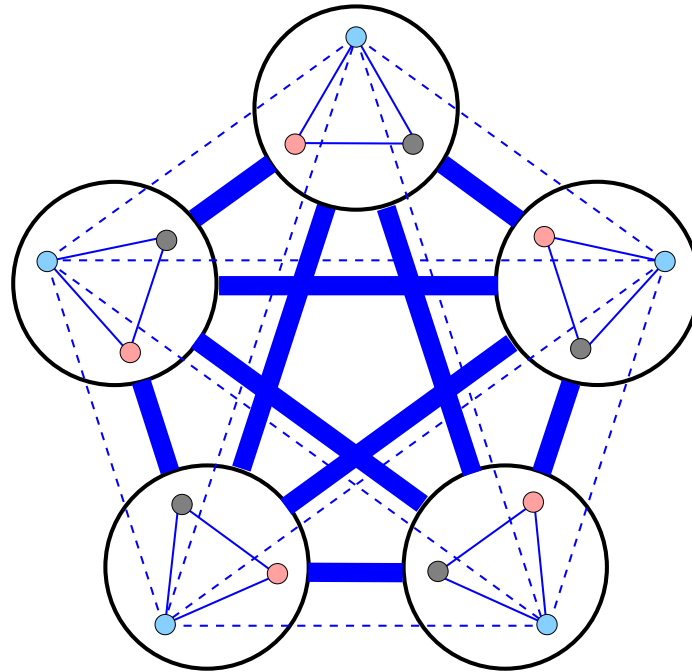
$K_3[5]$ redrawn

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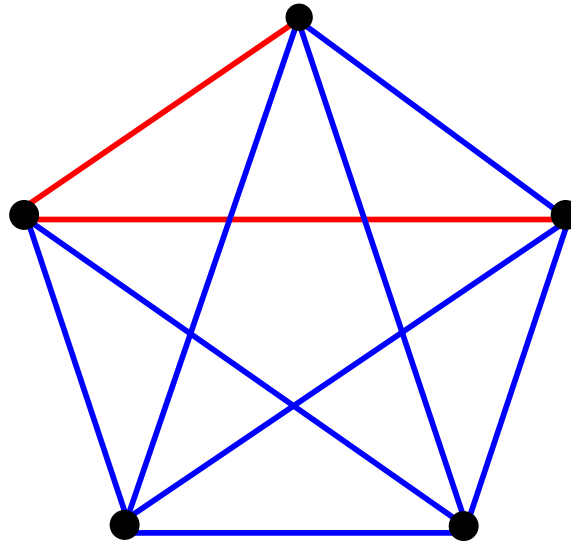
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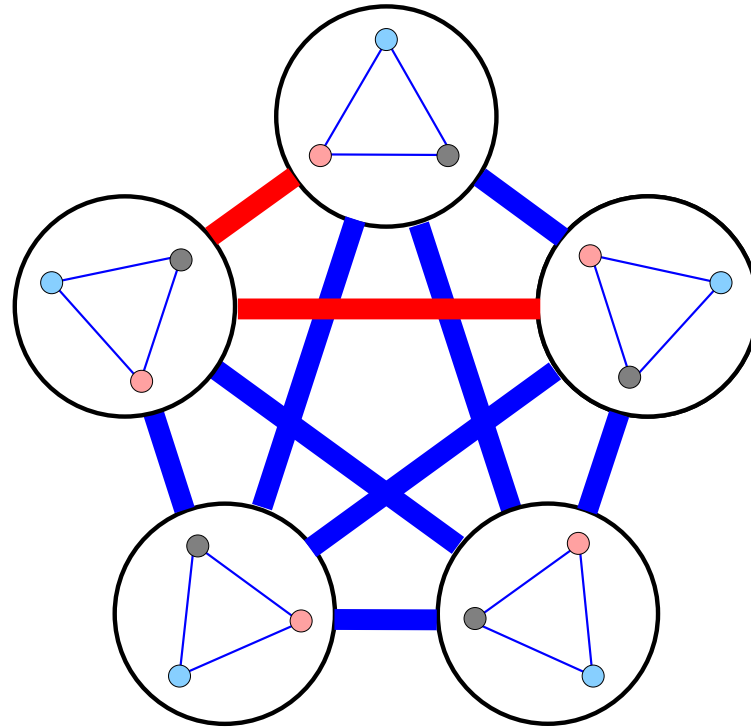


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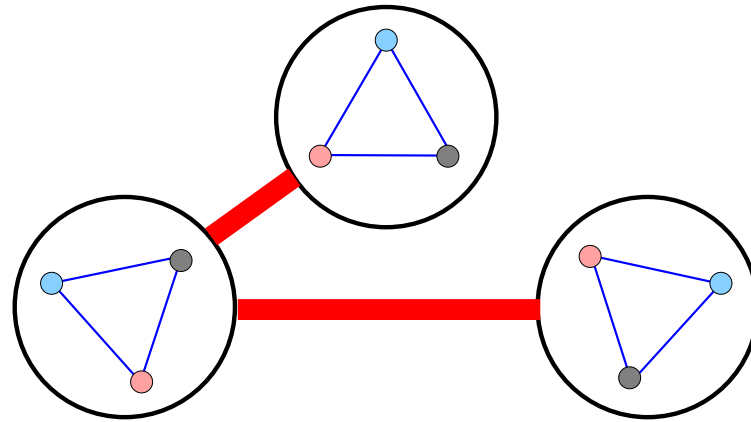
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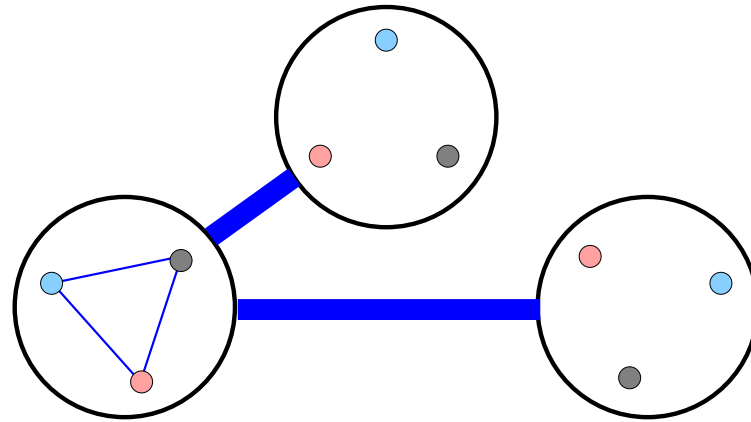
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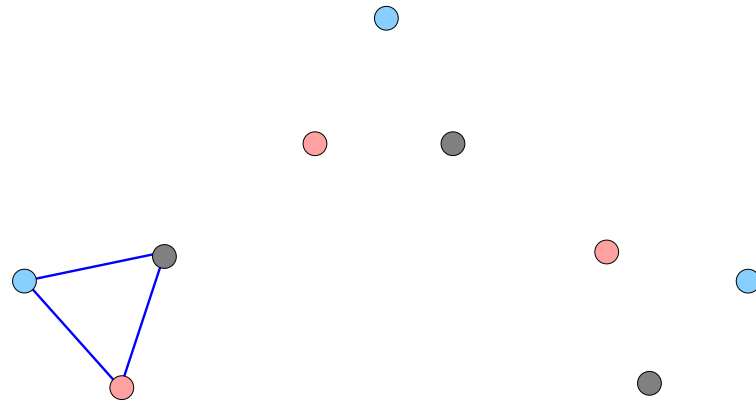
SMALL GRAPHS: $K_4 \setminus e$



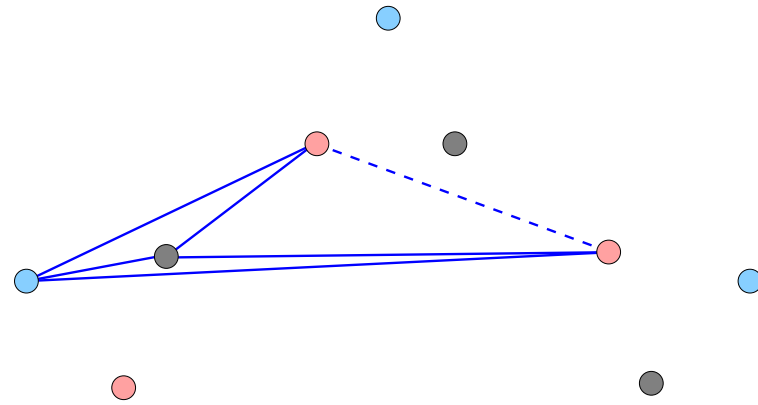
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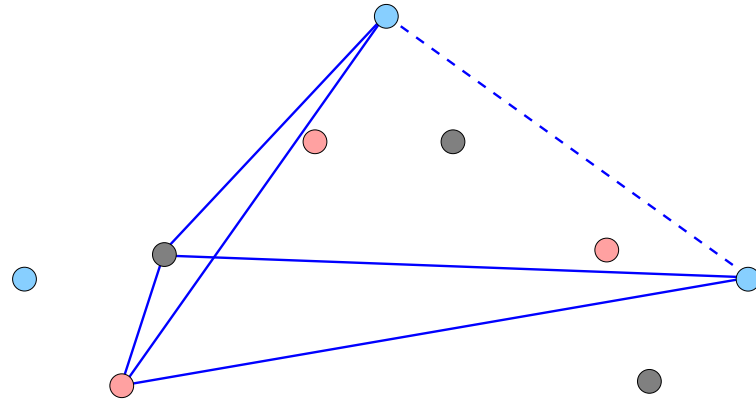
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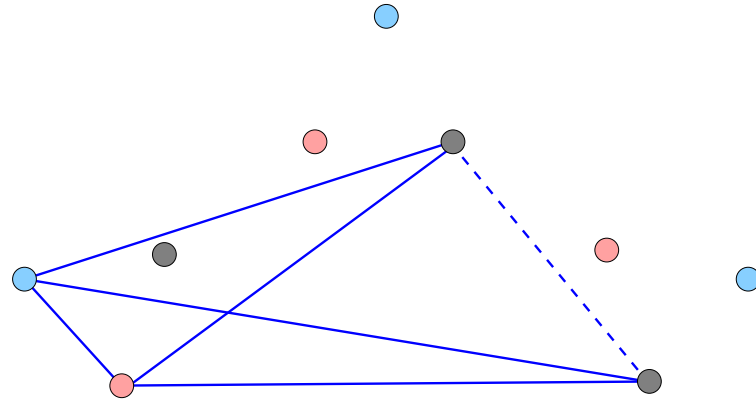
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An induced $K_4 \setminus e$ decomposition of the complete r -partite graph $K_r[5]$, for all $r \equiv 1, 3 \pmod{6}$, can be obtained by applying this construction to all the triangles in a K_3 decomposition of K_r .

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THEOREM

For $n = 5r$, where $r \equiv 1, 3 \pmod{6}$,

$$ex[n, K_4 \setminus e] \geq \binom{n}{2} - 2n$$

OPEN PROBLEMS

- *Reduce the gaps between the lower and upper bounds on $ex[n, F]$ when*
 - $F = 2K_2$
 - $F = P_4$
 - $F = K_{1,3} + e$

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Extremal graphs are often regular, so perhaps this will be easier.*

- *Given a fixed graph F , how **hard** is it to decide whether an input graph G admits an induced F -decomposition?*

The corresponding decision problem for standard decompositions was settled by **K. BRYŚ AND Z. LONC 2009**: *the problem is solvable in polynomial time if and only if every component of F has at most two edges.*

V. CHVÁTAL observed that the induced problem is also solvable in polynomial time in these cases.

REFERENCES

- [J.A. Bondy and J. Szwarcfiter](#), Induced decompositions of graphs. Submitted for publication.
- [A.E. Brouwer](#), Optimal packings of K_4 's into a K_n . *J. Combin. Theory Ser. A* **26** (1979), 278–297.
- [K. Bryś and Z. Lonc](#), Polynomial cases of graph decomposition: A complete solution of Holyer's problem. *Discrete Math.* **309** (2009), 1294–1326.
- [J. Spencer](#), Maximal consistent families of triples. *J. Combin. Theory* **5** 1968, 1–8.
- [R.M. Wilson](#), Decompositions of complete graphs into subgraphs isomorphic to a given graph. Proceedings of the Fifth British Combinatorial Conference, *Congressus Numerantium XV*, Utilitas Math., Winnipeg, Man., 1976, pp. 647–659.

THANK YOU

WELCOME TO THE CLUB, ANDRÉ!