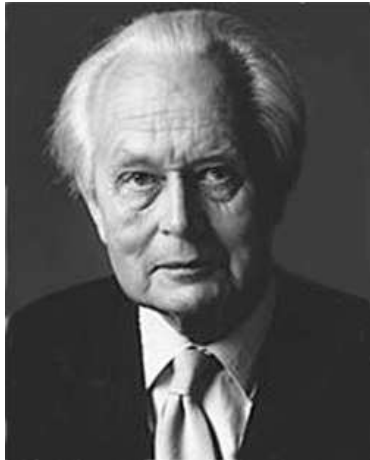
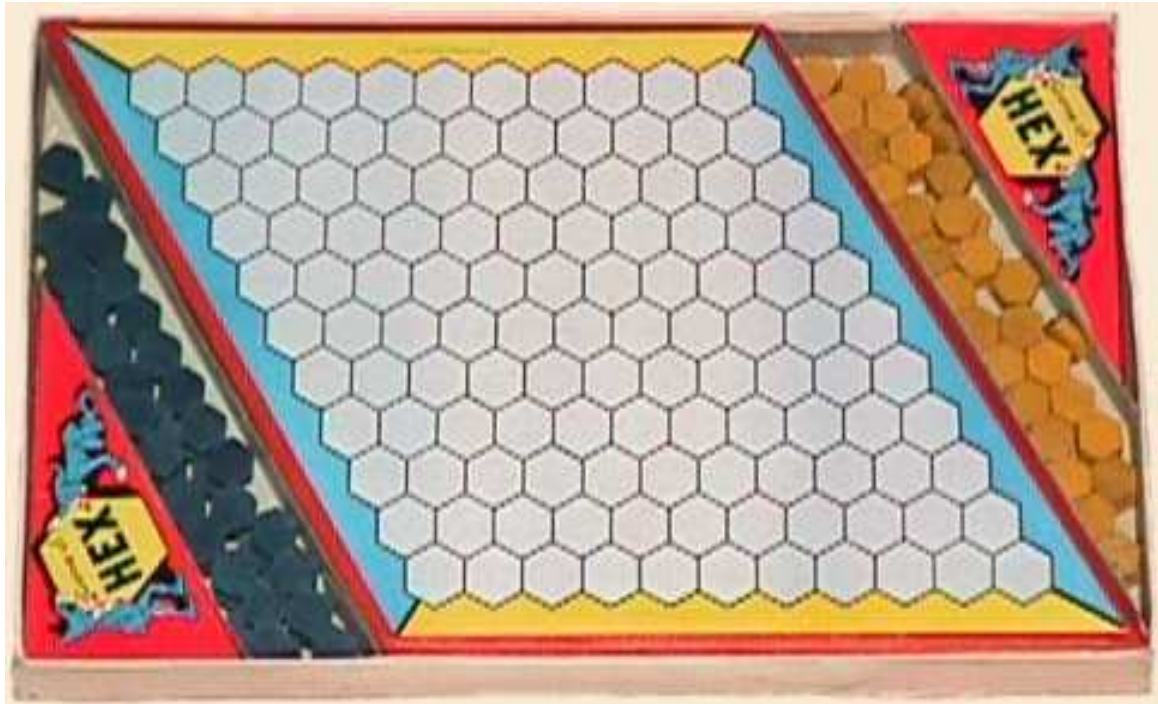


Misère (Reverse) Hex = Rex
Bordeaux Graph Workshop 2010
in honour of André Raspaud

Piet Hein
1942



John Nash
1948



Bjarne Toft
IMADA
The University of Southern Denmark
Odense, Denmark

Copenhagen Conference 1932



Heisenberg, Werner Karl; Hein, Piet; Bohr, N.; Brillouin, Leon Nicolas; Rosenfeld, Leon; Delbrück, Max; Heitler, Walter; Meitner, Lise; Ehrenfest, Paul; Bloch, Felix; Waller, Ivar; Solomon, Jacques; Fues, Erwin; Strømgren, Bengt; Kronig, Ralph de Laer; Gjelsvik, A; Steensholt, Gunnar; Kramers, Hendrik Anton; Weizsäcker, Carl Friedrich von; Ambrosen, J.P.; Beck, Guido; Nielsen, Harald Herborg; Buch-Andersen; Kalckar, Fritz; Nielsen, Jens Rud; Fowler, Ralph Howard; Hyllerås, Egil Andersen; Lam, Ingeborg; Rindal, Eva; Dirac, Paul Adrian Maurice; N.N.; Darwin, Charles Galton; Manneback, Charles; Lund, Gelius

Petersen Conference 1990



1. Just
2. Moving forward
3. Finite
4. Full information
5. Strategic
6. Decisive (no draw)

①

~~redfordigt~~

~~overblik~~

②

~~premadstridende~~

③

~~endeligt~~

④

~~overskueligt~~

⑤

~~strategisk~~

⑥

~~udslagsspilende~~

Pulseret

Remis.

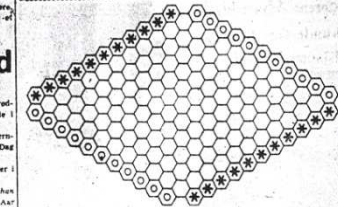
Politiken Dec. 26, 1942

26. December 1942

Vil De lære Polygon?

Piet Hein har konstrueret et Spil, der indeholder alle mulige Geometriske Former og som, der blot kan kaldes en Blyant.

„Politiken“ udskriver i Dag en Præmieopgave, der vil volde Hovedbrud for Bøgnere



Medledes er Polygon-Spilbrættet ud.

Det er dukket et helt nyt og smukt Talsystem frem. Fæstholdet er en Mæde for et På Lyg af den matematiske naturhistoriske Forening med det formelige Navn „Polygon“, hvortil man gerne havde en Medlemmer af „Blyantforeningen“, kaldt „Fæstholdet“. Men det er først og fremmest et Spil, som kan bruges til at træne sig i Geometri og som kan bruges til at træne sig i Matematik, og som endelig kan bruges til at træne sig i at tænke sig selv til at løse et Problem, som kan være meget svært at løse.

En Opgave som er et Spil. Her kommer saa den første Opgave (Se Tegning 1). Som man ser har Sort faktisk mange flere Felter som hvide end hvid. Sort har endda en hel Række som forlinder begge Sorts Felter — med Udtagelse af et enkelt Hvid, hvor Hvidens er brudt af et hvidt Felt. Hvis Sort har en hvid Felt, saa har Sort jo vundet.

Hvid begynder. Hvid er det første Hvid som Hvidt kan bevæge først (Det som hurtigt bringer Sort eller hvidt). Nu der! Og hvem vinder, naar begge spiller meget nøjagtigt?

Den bedste Maade at løse Opgaven er at spille i Gange.

Det menes at, at det nye Spil indeholder alle mulige Geometriske Former og som blot kan kaldes en Blyant.

I den følgende Artikel vil Piet Hein sige om det første Indblik i det nye Spil. Dette er en Blyant som alle skal have en Løsning og som slipper over alle ting, saa Løsningen er en Blyant.

En mulig Afsnit af Spillet — det er nu Hvids Tur. Hvid vil, og det betyder saa om at være på et af de to Felter som Hvid er i. Hvid vil, og det betyder saa om at være på et af de to Felter som Hvid er i.


Spillet eneste Regel. Den ene Spiller Hvid har Markert et Felt, den anden Sort har Markert et Felt.

En mulig Afsnit af Spillet — det er nu Hvids Tur. Hvid vil, og det betyder saa om at være på et af de to Felter som Hvid er i. Hvid vil, og det betyder saa om at være på et af de to Felter som Hvid er i.

Spillet eneste Regel. Den ene Spiller Hvid har Markert et Felt, den anden Sort har Markert et Felt.

Kr. Løsningerne skal være indsendt til Forretningen af Frugt-krestau, og den 30. ds., og Konvoluten skal være tydeligt mærket „Polygon“

Polygonspillet's Opfinder, Piet Hein, præsenterer Spillet



Kontakt, Vinkel

Idéens to Halvdele.

Men herefter giver vi Ordet til Piet Hein som Opfinder af Polygon-Spillet: Spillet bygger paa det enkle Faktum, at to Linjer inden for en Firkant, som hver forbinder et Par modstaaende Sider

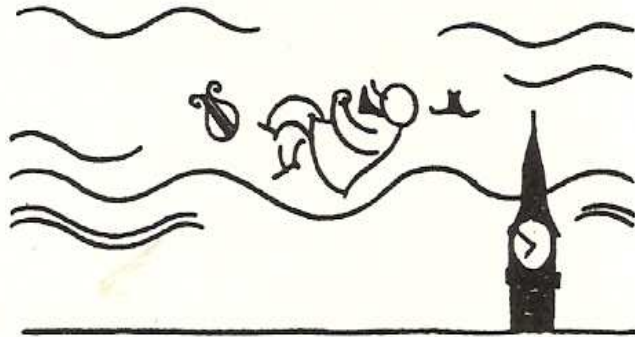
Den første Polygon-Opgave

Maade, at de kor sammenhængende, Forbindelse mellem to Fronter

Nu kan enhver staaende Spillebrædder

En at de Erfarer at man ikke altid ligger liggende klædt takt — se Tegning bindelsen mellem Felter ligger i Vinkel, og de to me ubesatte, saa er For det er tidnok temmeliggyde Felter det andet. Den stilling, er noget

Life as a game of Hex



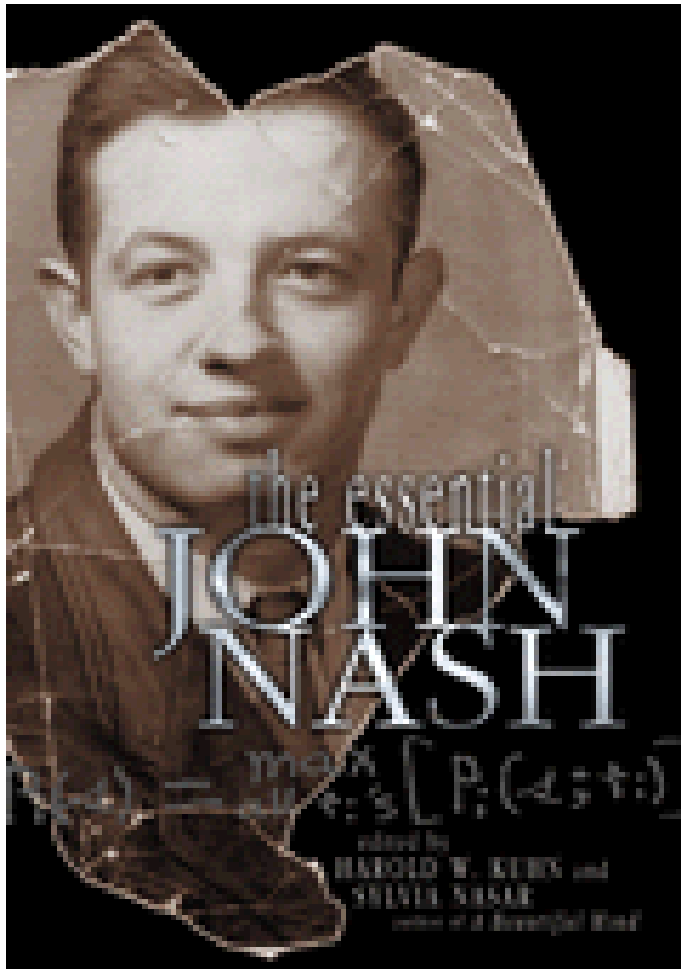
**Life is almost
like a play
Easy – hard
Decide your way
With the simplest
Rules you start
Most easy then
To make it hard.
(transl. BT)**

LIVET BETRAGTET SOM SPIL

Gruk ved et spil Polygon.

Livet er ganske
som sådan et spil:
let eller svært
som man gør det til,
og: når de enkleste
regler er lært,
så er det lettest
at gøre det svært.

John Nash, b. 1928, (a beautiful mind)
discovered Hex in 1948-1949



Non-cooperative Games

John F. Nash Jr. (21 år gammel)

A DISSERTATION

Presented to the Faculty of Princeton
University in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance by the
Department of Mathematics

May, 1950

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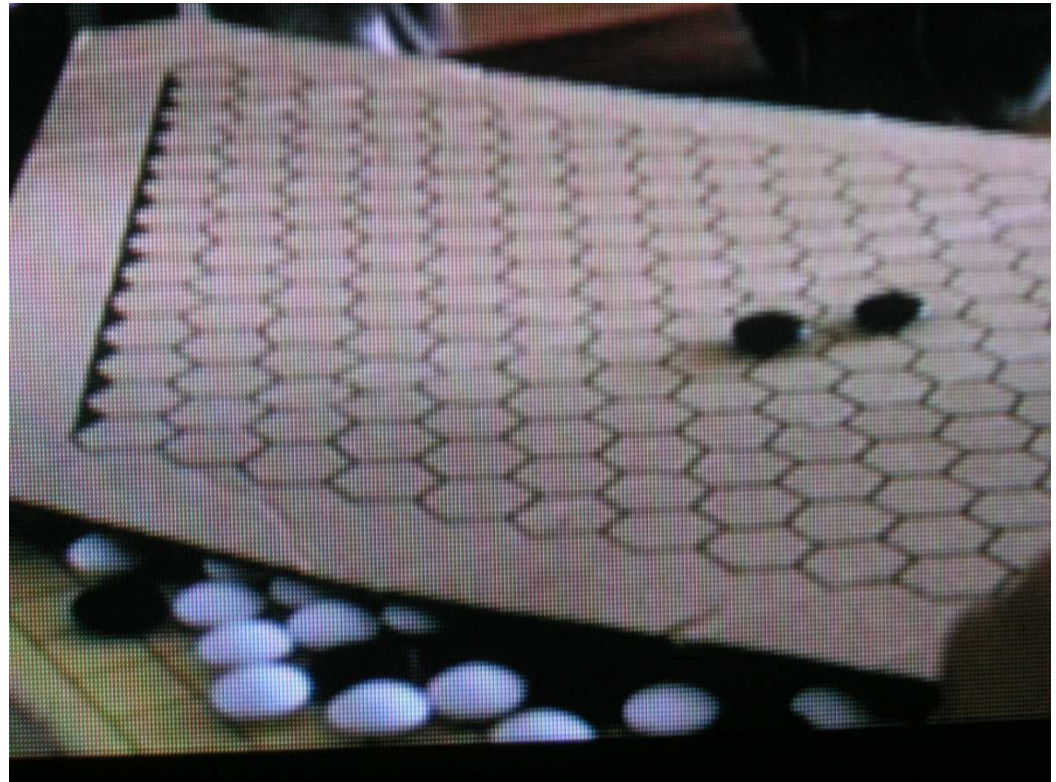
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Stockholm 1994



Removed scenes from the movie

A Beautiful Mind



John Nash's Hex theorem

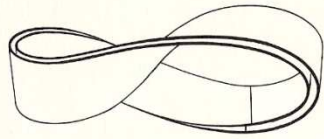
- **The first player has a winning strategy**

(but a winning first move for for the first player in $n \times n$ Hex is **not known with mathematical certainty** – not then and not now!)

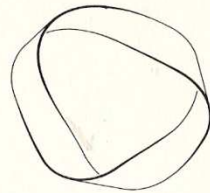
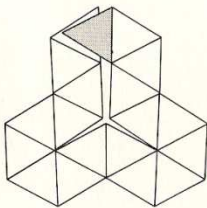
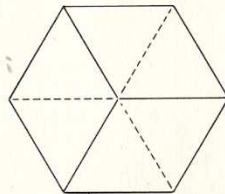
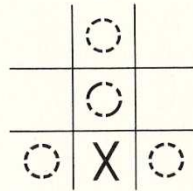
Proof: Strategy stealing.

Martin Gardner 1957

SCIENTIFIC AMERICAN



The **SCIENTIFIC AMERICAN** *Book of*



ILLUSTRATED WITH DRAWINGS AND DIAGRAMS

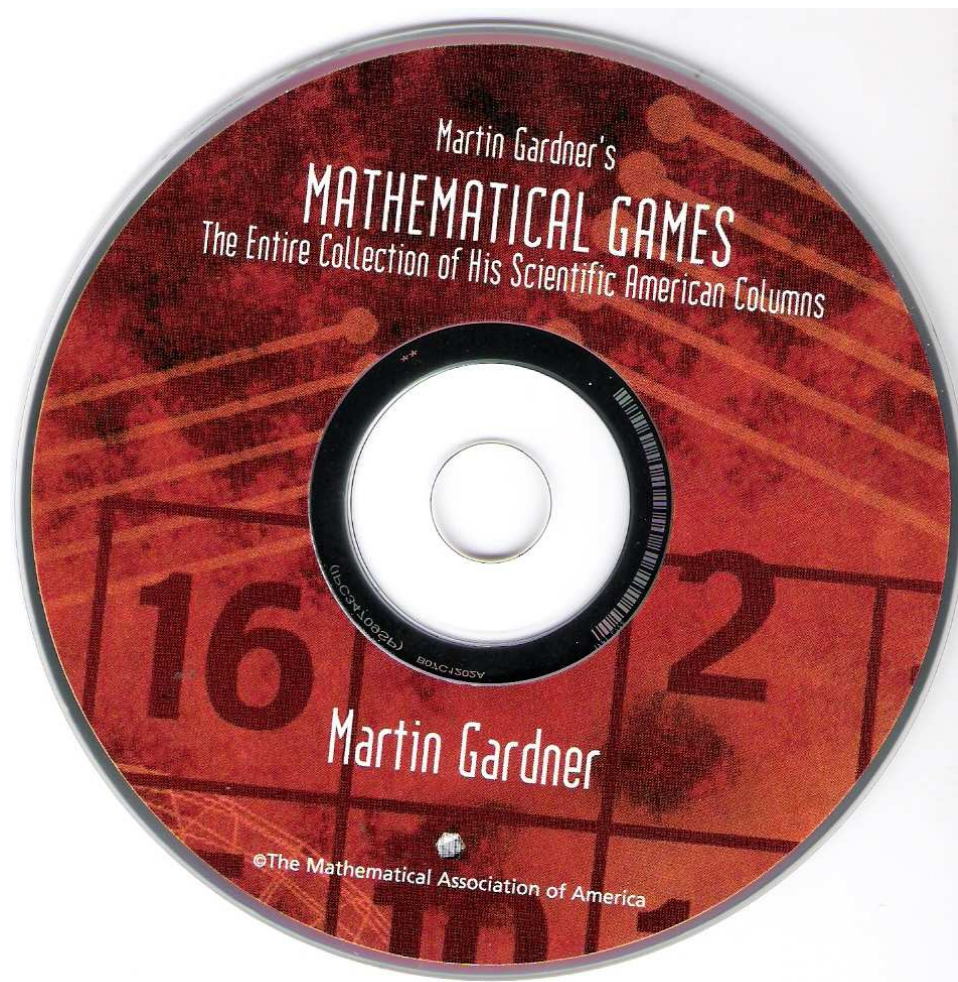
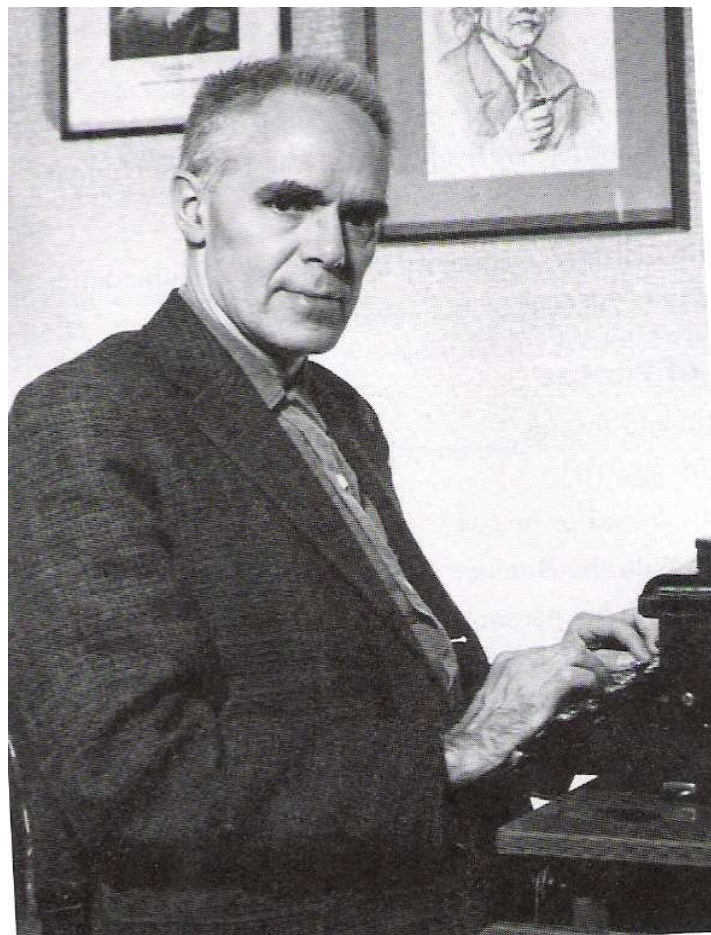
BY MARTIN GARDNER

Mathematical Puzzles & Diversions

Paradoxes and Paperfolding, Moebius Variations and Mnemonics, Fallacies, Brain-Teasers, Magic Squares, Topological Curiosities, Probability and Parlor Tricks, and a variety of ancient and new games and problems, from Polyominoes, Nim, Hee and the Tower of Hanoi to Four-Dimensional Ticktacktoe.
Together with
mathematical commentaries
by Mr. Gardner and addenda from readers of Scientific American. Plus bibliographies and, of course, solutions.

SIMON AND SCHUSTER • NEW YORK • 1959

Martin Gardner 1914-2010



Claude Berge playing Hex 1974



*Claude Berge
Jean-Marie Pla
Neil Grabois
1974*

© Michel
Las Vergnas

Yōhei Yamasaki, Osaka University

まず白が最初に図5-1の●の一つに打てば、黒は最終着を◎に指定して勝つことができます。例えば図5-2のようなA-a, B-b……のようなミアイをつけておけばよいのです。

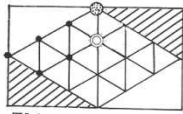


図5-1

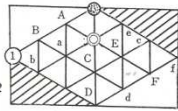


図5-2

白1が図6-1の●の一つであれば、今度は最終着を◎とし、例えば図6-2のように勝てます。

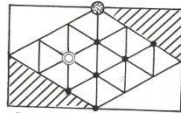


図6-1

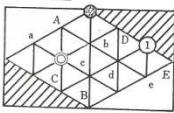


図6-2

従って、次の四通りを考えると十分です。

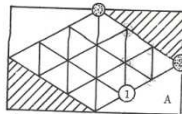


図7-1

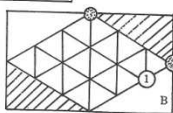


図7-2

106

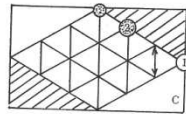


図7-3

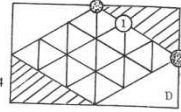


図7-4

Aについては代りにA'を考えます。このように最後に着手する者をかえないまま、黒石を追加した状態から始めても黒の必勝が結論できれば十分だからです。こういう設定はA>A'のように表し、以後も用いることにします。A'はA'に到着し

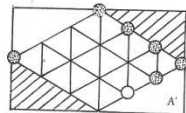


図8-1

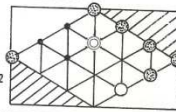


図8-2

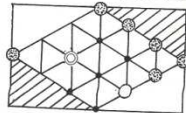


図8-3

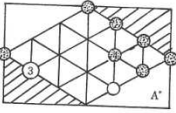


図8-4

次のように解決します。

Publ. RIMS, Kyoto Univ.
14 (1976), 337-358

Theory of Division Games

—Dedicated to the memory of Professor Taira Honda—

By

Yōhei YAMASAKI*

yamasaki@math.sci.osaka-u.ac.jp

Abstract

Lehman [9] generalized the Shannon's switching games to those on Boolean functions. We discuss in this paper, the theory of division games which contain Lehman's games and their reverse games. The theory of Lehman's games is analyzed and rearranged in this paper. Our results for division games contain naturally those for the classical ones as Hex, Bridg-it and their reverse games. Especially for the reverse games, we introduce several concepts dual to those introduced before, and obtain concise reformulations.

The most significant feature of this paper is that we adopt the notion "assignment" to deal with generalized games in which the principle of alternate move does not necessarily hold. The classical results will be accordingly purified and divided into four theorems, i. e., Theorems 2, 3, 4 and 5, which indicate the dualities about points and about assignments, with respect to signature and to players. It is also a virtue of this generalization that we can characterize in a game theoretical argument the triple systems and the block designs W_{11} and W_{12} whose automorphism groups are Mathieu groups M_{11} and M_{12} .

Introduction

Consider Hex. Two players, say White and Black play on a Hex board consisting of n rows and n columns of hexagons arranged in a rhombus. We colour each pair of opposite side of edges white and black respectively. They play alternately, placing a counter of their colour on an unoccupied hexagon. Each player wins if he forms a chain of adjacent counters that joins his side of the board to the opposite side. A game in case $n=4$ is illustrated in Figure 1, where Black has won.

Communicated by S. Hitotumatu, February 14, 1977.

* Department of Mathematics, Osaka University, Toyonaka, Osaka 560, Japan.

Claude Berge and Ryan Hayward in Marseilles 1992



ICGA Tournaments

Tournaments between computer programs: chess, draughts, checkers, Go, backgammon, and more

Home / Beijing 2008 / Hex

13th Computer Olympiad, Hex

Sept. 28 - Oct. 5
2008

See also

- Games / Hex
- Computer Olympiad

Result

Rank	Program	Origin	Score	Games	SOS	SoDOS	Title
1	Wolve 2008	CAN	9.0	12	60.0	36.00	Gold medal
2	MoHex 2008	CAN	8.0	12	64.0	44.00	Silver medal
3	Six	HUN	4.0	12	80.0	22.00	Bronze medal
4	Yopt	FRA	3.0	12	84.0	16.00	

RYAN HAYWARD
PHILIP HENDERSON
GÁBOR MELIS

- SOS: Sum of Opponent Scores
- SoDOS: Sum of Defeated Opponent Scores

PS: THE FIRST HEX PLAYING MACHINE (ANALOG) WAS CREATED BY CLAUDE SHANNON AND E.F.MORE IN 1953 AT BELL LABS

Paris juli 2004



MARTIN GARDNER 1957:

There are a number of variations on the basic theme of Hex, including a version in which each player tries to force his opponent to make a chain. According to a clever proof devised by Robert Winder, a graduate student of mathematics at Princeton, the first player can always win this game on a board which has an even number of cells on a side, and the second player can always win on a board with an odd number.

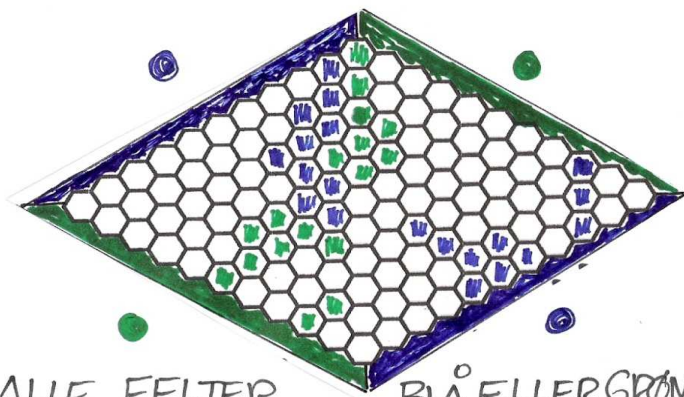
VARIATION: Rex

(Reverse Hex or Misère Hex)

- Objective: **Avoid creating a chain between your two sides!**
- The game cannot end in draw (hence either the first or the second player has a winning strategy)
- On an $n \times n$ board with n even the first player has a winning strategy (Evans 1974)
- On an $n \times n$ board with n odd the second player has a winning strategy (Lagarias and Sleator 1999)

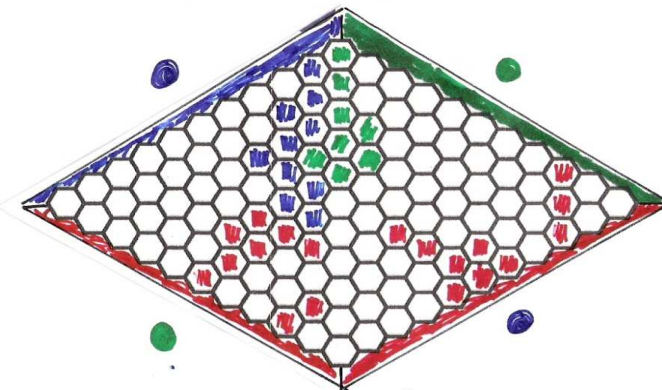
Not both can avoid making a chain, i.e. not both can win Misère Hex

IKKE BEGGE KAN TÅBE I HEX

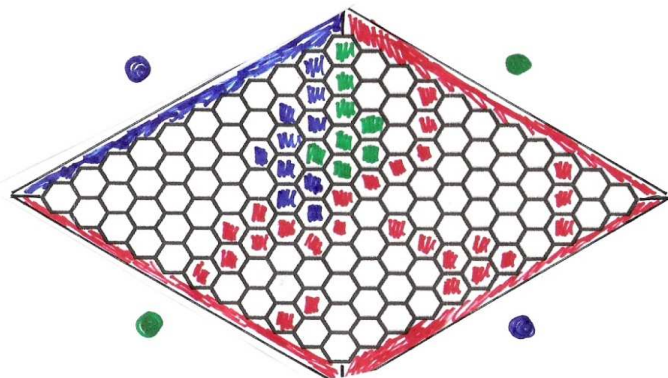


ALLE FELTER

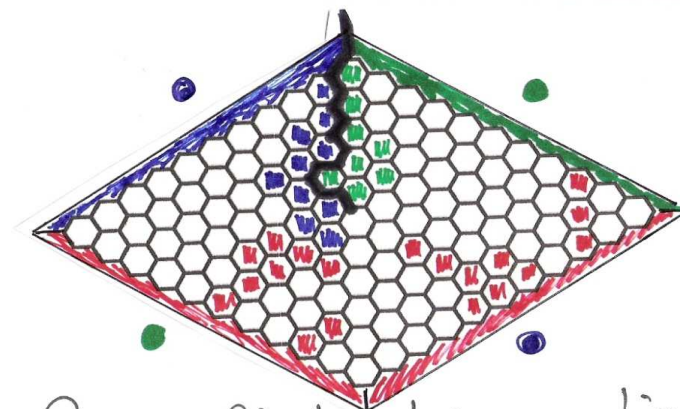
BLÅ ELLER GRØNNE



OMFARV!



1. mulighed : Grøn vinder
Tilsvarende : Blå vinder



2. mulighed : umulig!!

Hex and Brouwer's Fix-point Theorem

THE GAME OF HEX AND THE BROUWER FIXED-POINT THEOREM

DAVID GALE

1. **Introduction.** The application of mathematics to games of strategy is now represented by a voluminous literature. Recently there has also been some work which goes in the other direction, using known facts about games to obtain mathematical results in other areas. The present paper is in this latter spirit. Our main purpose is to show that a classical result of topology, the celebrated Brouwer Fixed-Point Theorem, is an easy consequence of the fact that Hex, a game which is probably familiar to many mathematicians, cannot end in a draw. This fact is of some practical as well as theoretical interest, for it turns out that the two-player, two-dimensional game of Hex has a natural generalization to a game of n players and n dimensions, and the proof that this game must always have a winner leads to a simple algorithm for finding approximate fixed points of continuous mappings. This latter subject is one of considerable current interest, especially in the area of mathematical economics. This paper has therefore the dual purpose of, first, showing the equivalence of the Hex and Brouwer Theorems and, second, introducing the reader to the subject of fixed-point computations.

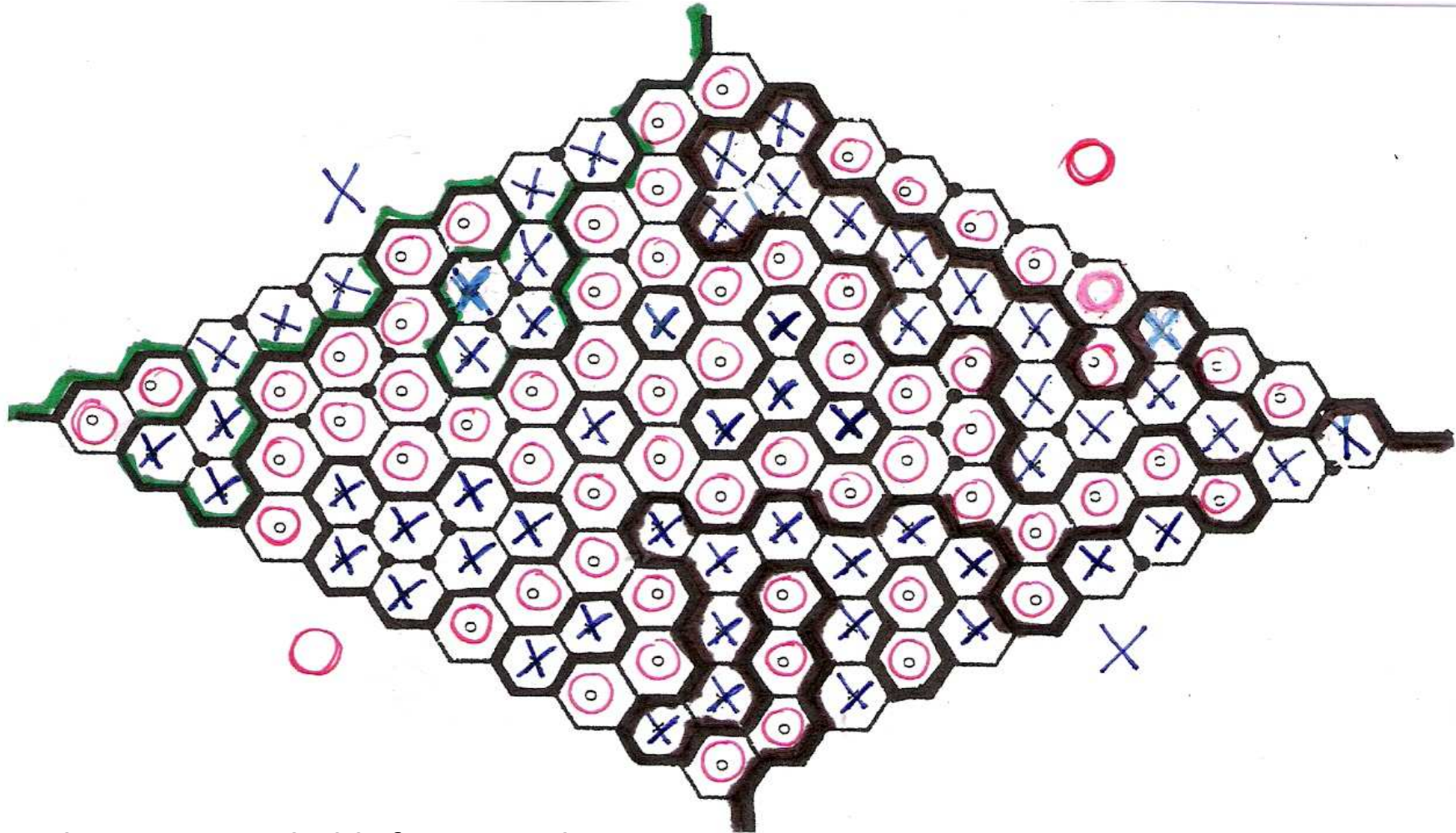
I should say that over the years I have heard it asserted in "cocktail conversation" that the Hex and Brouwer Theorems were equivalent, and my colleague John Stallings has shown me an argument which derives the Hex Theorem from familiar topological facts which are equivalent to the Brouwer Theorem. The proof going in the other direction only occurred to me recently, but in view of its simplicity it may well be that others have been aware of it. The generalization to n dimensions may, however, be new.

In the next section we will present the relevant facts about Hex. In Section 3 we prove the equivalence of the Hex and Brouwer Theorems. The general Hex Theorem and fixed-point algorithm are presented in the final section.

- Brouwer's Fix-point Theorem was an important tool for John Nash when the theory of the Nash equilibrium was developed.
- David Gale proved in 1979 that **NOT BOTH CAN AVOID A CHAIN** is equivalent to **Brouwer's Fixpoint Theorem**.

David Gale's proof (1979)

NOT BOTH CAN AVOID A CHAIN



This argument holds for general Hex, i.e. any
PLANAR 3-REGULAR 2-CONNECTED GRAPH

Evans 1974: For Rex and n even, the acute corner is a winning first move for the first player

A Winning Opening in Reverse Hex

Ronald Evans

J. Recreational
Math. 7 (1974)
189-192.

We begin with a brief discussion of the game of Hex, discovered about 30 years ago. Two players, White and Black, play on a board consisting of n rows and columns of hexagons arranged in a rhombus. The 4×4 hexboard is illustrated in Figures 1, 2, 3, 4, and 5. White plays first and places a white marker in some hex (hexagon). Then Black places a black marker in another hex. White and Black continue to take turns, each placing a marker of his color in an unoccupied hex. White wins if he joins the top edge of the hexboard to the bottom edge with a chain of adjacent hexes containing white markers. (Two hexes are adjacent if they have a common side.) Similarly, Black wins if he forms a black chain joining the left and right edges.

It has been proven that White has a winning Hex strategy on any $n \times n$ hexboard; see [1, pp. 327-339]. However, the winning strategy is not known except for a few small values of n .

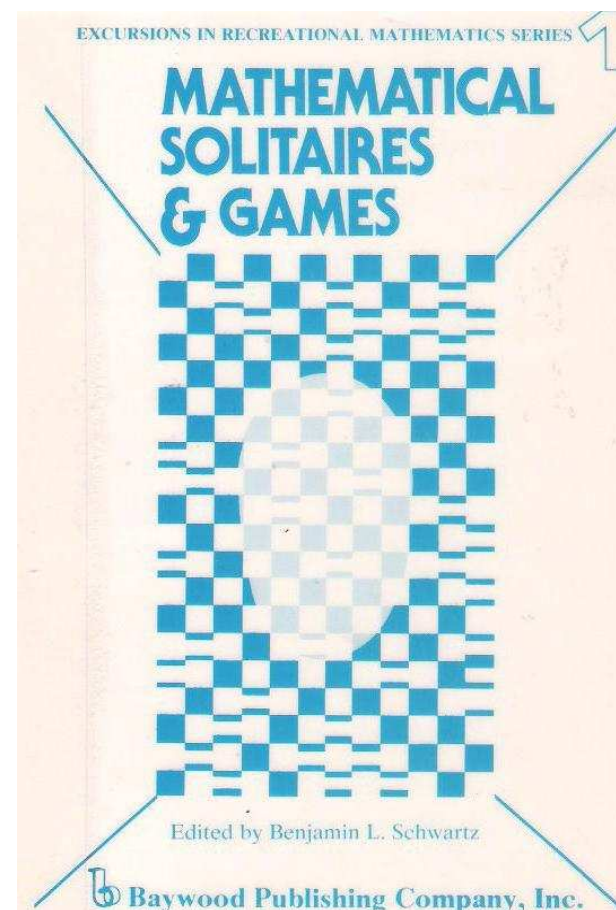
Now consider the game of Hex modified so that the object is to lose, that is, White (W) attempts to force Black (B) to form a black chain joining B's edges of the hexboard, and vice versa. This game is called Reverse Hex. We establish the convention that W always moves first. As noted in Reference 2, p. 78, when Reverse Hex is played on an $n \times n$ hexboard, W has a winning strategy when n is even, and B has a winning strategy when n is odd. These winning strategies are not known in general. Even in the case $n = 4$, W's winning strategy is apparently unknown (although it is known for Hex).

In Reverse Hex with n even, what are the opening moves for W which will leave him with a winning strategy? In this paper, we give a partial answer by showing that W has a winning strategy after opening with a marker in an acute corner of the hexboard.

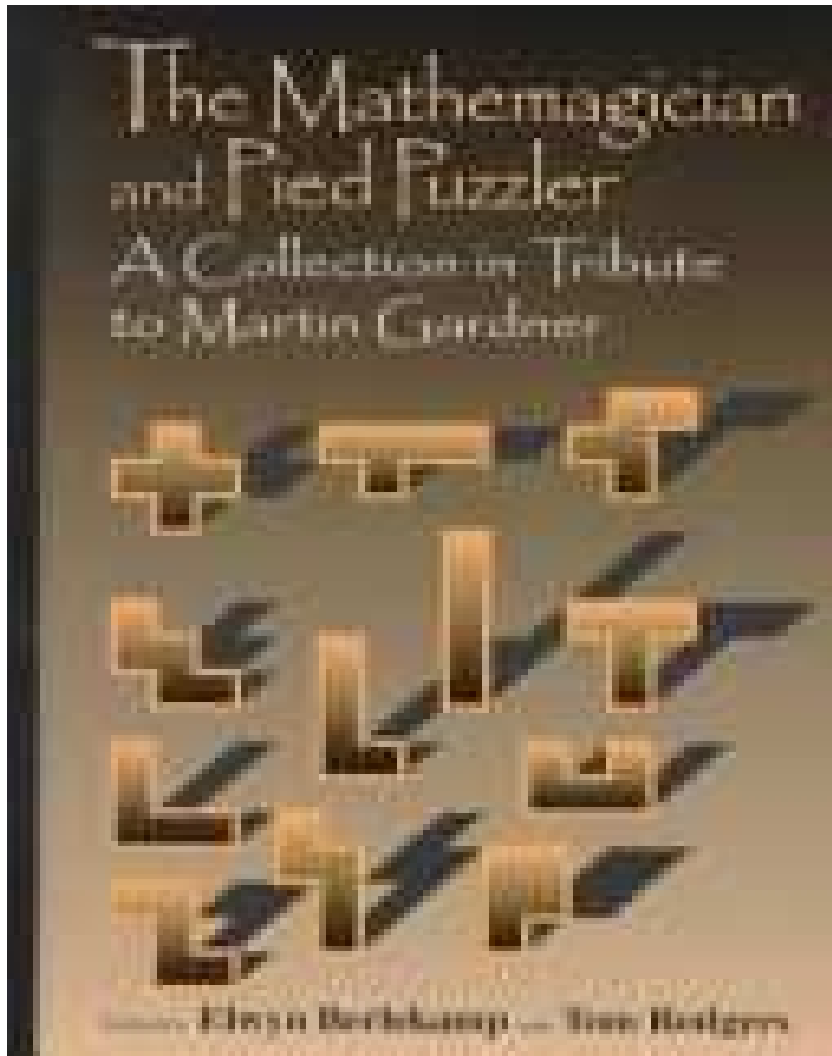
Remark 1: Beck [1, p. 334] has shown that if W opens in an acute corner in Hex, then B has a winning strategy.

Remark 2: In Reverse Hex with $n = 2$, if W opens in an obtuse corner, then B has a winning strategy. One might conjecture that this is true for all even n .

Number the rows and columns of the $n \times n$ hexboard from 1 to n as



Lagarias and Sleator 1999

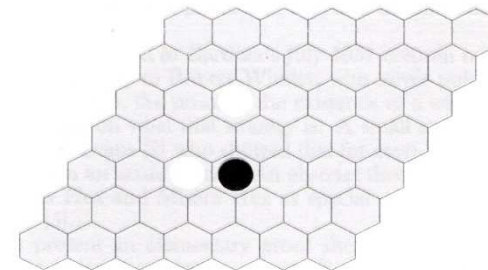


Who Wins *Misère* Hex?

Jeffrey Lagarias and Daniel Sleator

Hex is an elegant and fun game that was first popularized by Martin Gardner [4]. The game was invented by Piet Hein in 1942 and was rediscovered by John Nash at Princeton in 1948.

Two players alternate placing white and black stones onto the hexagons of an $N \times N$ rhombus-shaped board. A hexagon may contain at most one stone.



A game of 7×7 Hex after three moves.

White's goal is to put white stones in a set of hexagons that connect the top and bottom of the rhombus, and Black's goal is to put black stones in a set of hexagons that connect the left and right sides of the rhombus. Gardner credits Nash with the observation that there exists a winning strategy for the first player in a game of hex.

The proof goes as follows. First we observe that the game cannot end in a draw, for in any Hex board filled with white and black stones there must be either a winning path for white, or a winning path for black [1, 3]. (This fact is equivalent to a version of the Brouwer fixed point theorem, as shown by Gale [3].) Since the game is finite, there must be a winning strategy for either the first or the second player. Assume, for the sake of

Hayward, Toft and Henderson 2011?

How to Play Reverse Hex

Ryan B. Hayward¹, Bjarne Toft², and Philip Henderson³

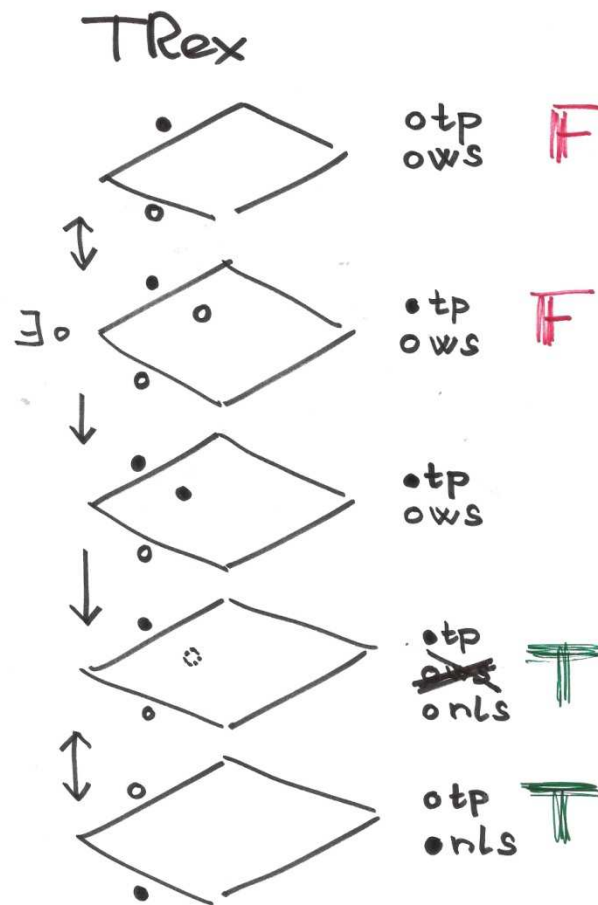
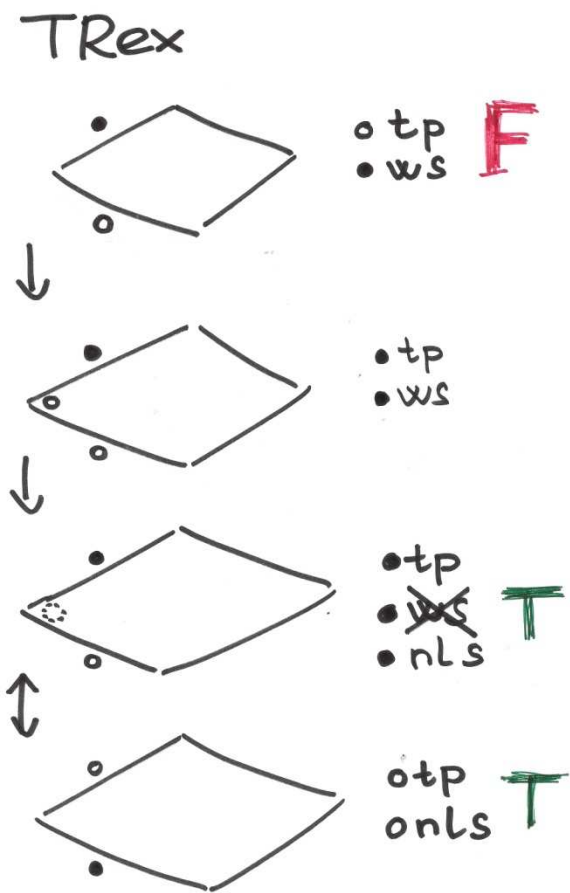
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Abstract. We present new results on how to play Reverse Hex on $n \times n$ boards. We give new proofs — and strengthened versions — of Lagarias and Sleator’s theorem (for $n \times n$ boards, each player can prolong the game until the board is full, so the first/second player can always win if n is even/odd) and Evans’s theorem (for even n , opening in the acute corner wins). Also, for even $n \geq 4$, we find another first-player winning opening (adjacent to the acute corner, on the first player’s side), and for odd $n \geq 3$ and for each first-player opening, we find a second-player winning reply. Finally, in response to comments by Martin Gardner, we give simple winning strategies for all board sizes up to, and including, 5×5 .

Terminated Rex (**TRex**) is Rex with the addition:
the game stops when there is just one empty
field left (i.e. there should always be a choice!).



Rex on an $n \times n$ -board with n even:

- Let the first player (White) play the non-losing strategy from **TRex**. **THIS IS A WINNING STRATEGY FOR THE FIRST PLAYER IN REX:**
- *Either* the second player (Black) creates a black chain *or* TRex ends with one empty field left. In the Rex game that field has to be chosen by Black and a black chain is formed!
- If also Black plays the non-losing strategy from **TRex**, then the Rex game will be decided only when the board is full.

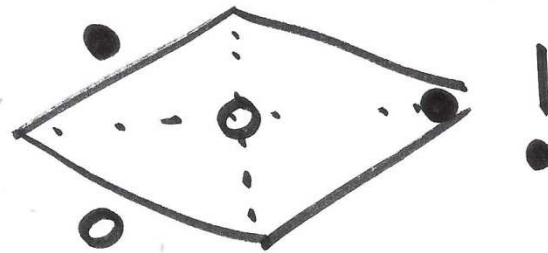
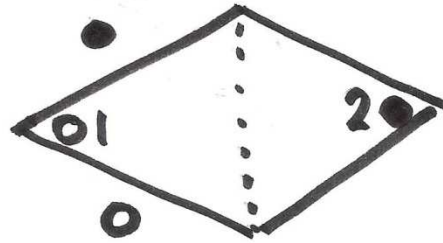
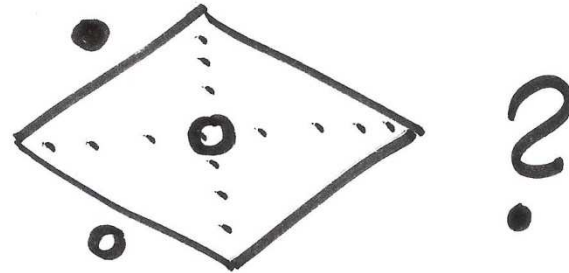
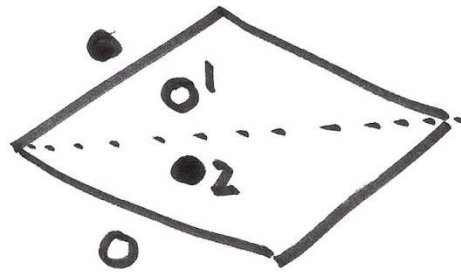
Rex on an $n \times n$ -board with n odd:

- Let the second player (Black) play the non-losing strategy from **TRex**. **THIS IS A WINNING STRATEGY FOR THE SECOND PLAYER IN REX:**
- *Either* the first player (White) creates a white chain *or* TRex ends with one empty field left. In the Rex game that field has to be chosen by White and a White chain is formed!
- If also White plays the non-losing strategy from **TRex**, then the Rex game will be decided only when the board is full.

Rex on an $n \times n$ board with odd n : (second player has winning strategy)

- What is a winning move for the second player (Black) after the first player's (White's) first move?
- **ANSWER: Black plays symmetrically (around a diagonal) with the first move by White.**
- Proof: After these two moves the board is symmetric – we can use strategy stealing - and the TRex argument applies!
- **What if the first move by White is in the center?**

**Rex on an $n \times n$ board with odd n :
(second player has winning strategy)**



Rex on an $n \times n$ board with even n : (first player has winning strategy)

- What is a winning first move for the first player (White) ?
- **ANSWER (EVANS' THEOREM 1974): White plays the acute corner.**
- **IN ADDITION (Hayward, Toft and Henderson 2011): Also the cell adjacent to the acute corner on the white side is a winning first move for White.**
- **We have a new proof (shorter and simpler than Evans').**

Lemma (inspired by Evans)



Partially filled
 No chains
 Odd no. open cells
 • tp
 • ws S



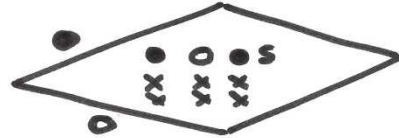
• tp
 • ws

Proof :

No. of open cells = 1 NO!
 " " " " = 3 OK!
 " " " " ≥ 5 :



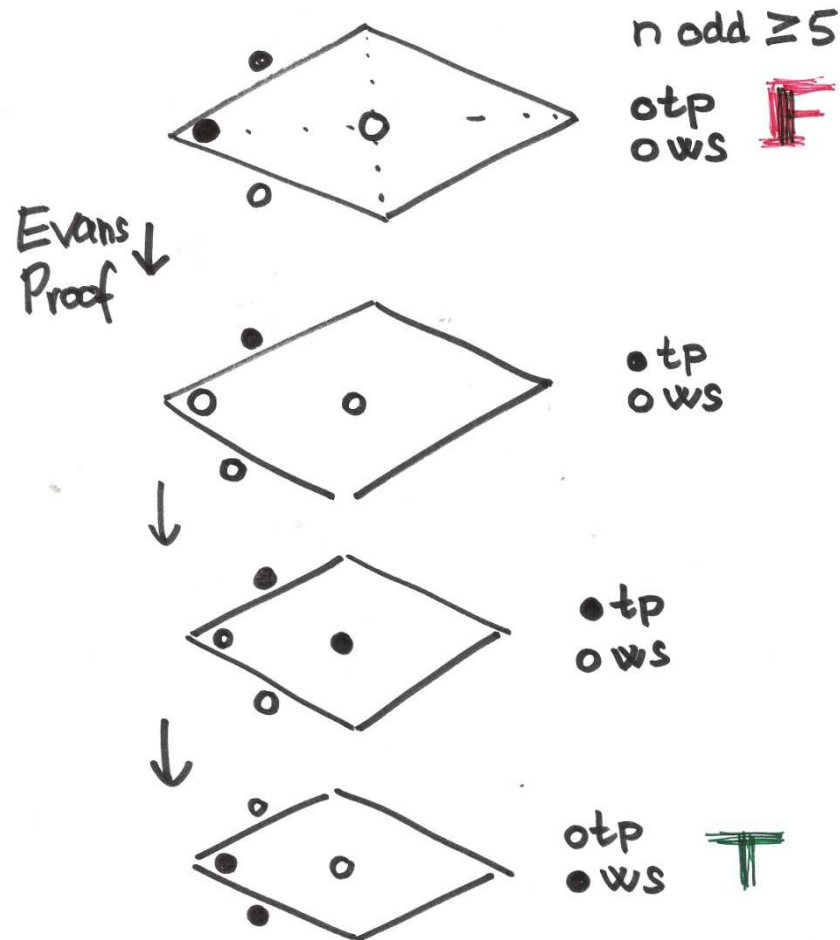
○ tp
 • ws ?

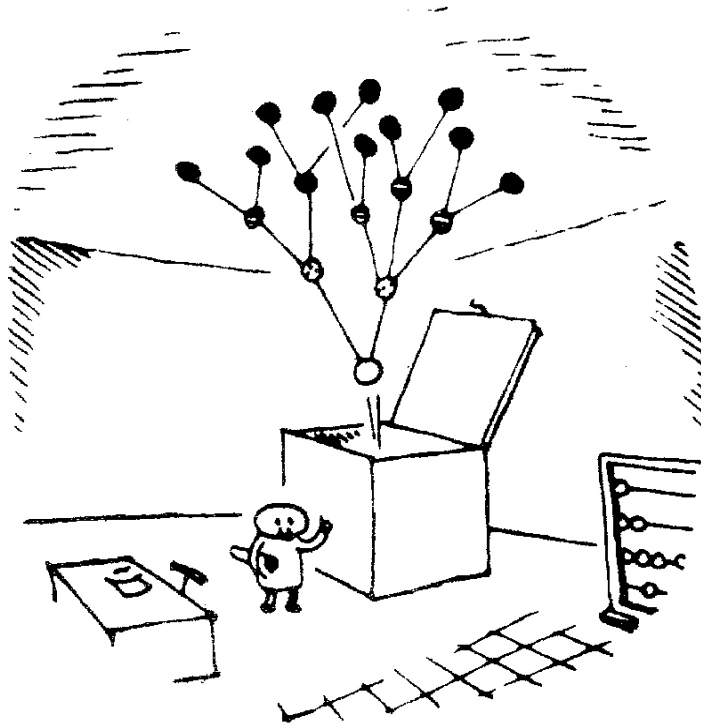


• tp
 • ws by induct.



If n is odd and the first move for the first player (White) is in the center, then the acute corner is a winning first move for the second player (Black).

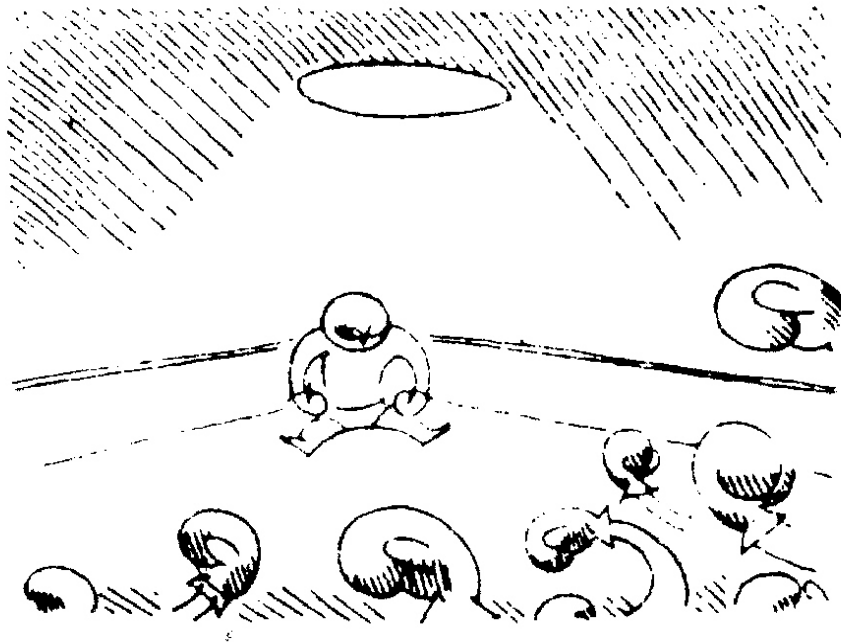




We shall have to evolve
problem solvers galore -
since each problem we solve
creates ten problems more.

Possible problems

- In our proofs, where we deduce from **F** to **T**, exactly where is the change from **F** to **T** ?
- For n odd, is the acute corner always a winning first move for the second player, no matter what the first player's first move is?
- Is the mathematical difficulty in Rex to determine losing moves (for the player with a winning strategy) rather than winning moves (and thus also in this respect reverse to Hex) ?



Probleme, die es wert sind,
in Angriff genommen zu werden,
beweisen ihren Wert,
indem sie sich sperren.

Ein Problem zeigt sich
des Angriffs wert
wenn es sich
dagegen wehrt

Problems worthy
of attack
prove their worth
by hitting back.

Tøv uforknytt løs
på problemerne, men
vår forberedt på
at de tøver igen.

Thank you very much for your attention!

