

On some Graph Colo(u)ring Problems

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Quick Overview - Acyclic Coloring

Definition

Acyclic coloring of a graph G :

- Proper vertex coloring of G (no two neighbors are assigned the same color)
- Any two classes of colors induce an acyclic graph (=a forest)
- Equivalently, no bicolored cycle in G
- $a(G)$: smallest number of colors necessary to acyclically color G

Acyclic Coloring

- (Asymptotically tight) bounds on $a(G)$ for $G=d$ -dimensional grid, $d \geq 2$
- $a(G) \leq 9$ for any graph of max. degree 5 (recently improved to 8)

Quick Overview - Star Coloring

Definition

Star coloring of a graph G :

- Proper vertex coloring of G
- Any two classes of colors induce a forest **of stars**
- Equivalently, no bicolored path of length 3 in G
- $\chi_s(G)$: smallest number of colors necessary to star color G

Star Coloring

- Lovász Local Lemma \Rightarrow general upper bound on χ_s for graphs of max degree Δ
- $\chi_s(G) = O(\Delta^{\frac{3}{2}})$
- Lower bound: $\chi_s(G) = \Omega\left(\frac{\Delta^{\frac{3}{2}}}{\log \Delta^{\frac{1}{2}}}\right)$

Quick Overview - Oriented Coloring

Definition

Oriented coloring of a graph G :

- Let \vec{G} be an orientation of G
- Proper vertex coloring of \vec{G}
- If $(u, v), (x, y) \in A(\vec{G})$, we cannot have $c(u) = c(y)$ and $c(v) = c(x)$
- $\chi_o(G)$: smallest number of colors necessary to color any orientation \vec{G}

Oriented Coloring

- Bounds for the oriented coloring of 2-D grids:
$$7 \leq \chi_o(G(n_1, n_2)) \leq 11$$
- Conjecture: $\chi_o(G(n_1, n_2)) = 7$ by homomorphism to $T(7; 1, 2, 4)$

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Quick Overview - $L(p, q)$ -labeling

Definition

$L(p, q)$ -labeling of a graph G :

- Vertex labeling of G , in the (integer) range $[0; M]$
- Neighbors in G must be assigned values that differ by at least p
- Vertices at distance 2 in G must be assigned values that differ by at least q
- $\lambda_q^p(G)$: smallest value of M necessary to $L(p, q)$ -label G

Quick Overview - $L(p, q)$ -labeling

$L(p, q)$ -labeling

Resp. values of p and q	$\lambda_q^p(G_d) \geq$	$\lambda_q^p(G_d) \leq$	Gap	No-hole Property
$p \neq 0 ; q = 0$	p		0	No (except for $p = 1$)
$p = 0 ; q \neq 0$	$(2d - 1)q$		0	No (except for $q = 1$)
$p, q \geq 1 ; 2p < q$	$2p + (2d - 2)q$	$2dq$	$2q - 2p$	No
$q = 1 ; 1 \leq p \leq 2d$	$2p + 2d - 2$		0	Yes (Prop 4)
$p, q \geq 1 ; q \leq 2p \leq 4dq$	$2p + (2d - 2)q$	$2p + (2d - 1)q - 1$	$q - 1$	Yes (Prop 4)
$p, q \geq 1 ; p = \alpha q ; \alpha \leq 2d$	$2p + (2d - 2)q$		0	No
$p, q \geq 1 ; p = \alpha q + \beta$ $1 \leq \beta \leq q - 1 ; p \leq 2dq + \beta - q$	$2p + (2d - 2)q$	$2p + 2dq - 2\beta$	$2(q - \beta)$	No
$p, q \geq 1 ; p \geq 2dq + 1$	$p + (4d - 2)q$		0	No

Table 1: $L(p, q)$ labeling of G_d : Summary of the results

The S -labeling problem (joint work with S. Vialette)

Notations

- Let G be a graph having n vertices and m edges
- An S -labeling $\theta : V(G) \rightarrow \{1, 2 \dots n\}$ is a bijective mapping of the vertices of G
- Let $\Theta(G)$ be the set of all S -labelings of G
- For any graph G and any $\theta \in \Theta(G)$,

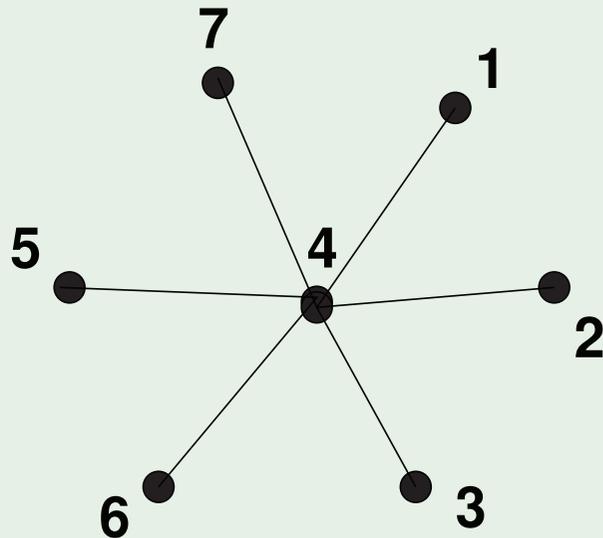
$$SL(G, \theta) = \sum_{e \in E(G)} \min\{\theta(u) : u \in e\}$$

The S -Labeling Problem

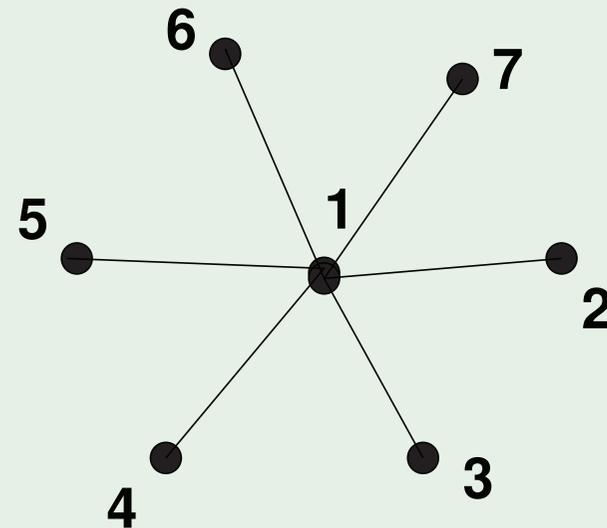
Given a graph G , find an S -labeling $\theta \in \Theta(G)$, such that $SL(G, \theta)$ is minimized.

S-labeling of $K_{1,6}$

Example



$$1+2+3+4+4+4= 18$$



$$1+1+1+1+1+1= 6$$

Figure: 2 different S-labelings - One of them is optimal



Overview of the Results

Theorem

The S-labeling problem is:

- **NP-complete** for general graphs [Vialette 06]
- **polynomial-time solvable** for split graphs

Overview of the Results

Definition

- **Split graph** = chordal graph whose complement is also chordal
- Vertices of a split graph can be partitioned into:
 - A clique K , and
 - A stable set S

Example

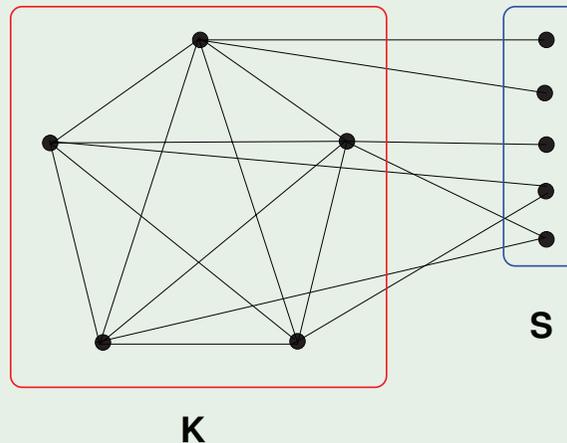


Figure: A split graph G and a partition $V(G) = K \cup S$

Overview of the Results

Theorem

For any graph G of maximum degree Δ ,

$$\frac{m(m + \Delta)}{2\Delta} \leq SL(G) \leq \frac{m(n + 1)}{3}$$

Theorem

Approximation ratios for the S -labeling problem:

- $r = \frac{4}{3}$ for general graphs
- $r = \frac{2\Delta}{3}$ for trees of max degree Δ
- $r = \frac{4\Delta}{3}$ for graphs of max degree Δ

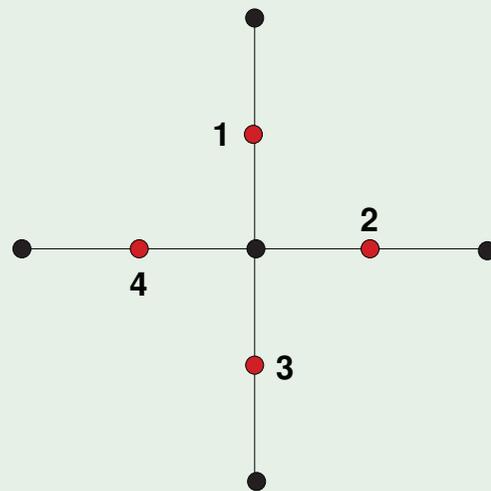


Overview of the Results

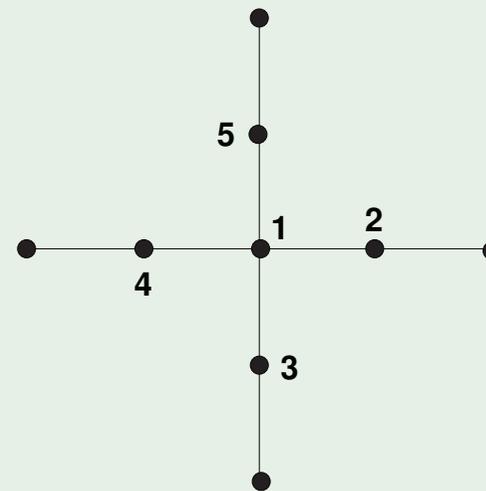
Definition (Contribution Vector)

For any S -labeling θ of a graph G , and for any $1 \leq i \leq |V(G)|$, $CV_\theta[i] =$ number of times label i contributes to the sum in $SL(G, \theta)$.

Example



$$(1+1)+(2+2)+(3+3)+(4+4)= 20$$



$$(1+1+1+1)+2+3+4+5= 18$$

Figure: $CV_\theta = [2, 2, 2, 2]$ (left) ; $CV_\theta = [4, 1, 1, 1, 1]$ (right)

Overview of the Results

Theorem

For any graph G and any optimal S -labeling θ of G , its contribution vector CV_θ is *monotone non-increasing*

Theorem

For any graph G of maximum degree Δ , any optimal S -labeling θ of G satisfies $CV_\theta[1] = \Delta$

Conclusion on the S -labeling problem

Some open questions

- Let k be s.t.
 - $CV_\theta[1] = CV_\theta[2] = \dots = CV_\theta[k] = \Delta$, and
 - $CV_\theta[k + 1] \neq \Delta$
 - Let G_Δ be the subgraph of G induced by the vertices of degree Δ
- Conjecture: there exists an optimal S -labeling θ s.t. the vertices associated to $CV_\theta[1 \dots k]$ form a **maximum independent set** in G_Δ
- Question 1: complexity of the S -labeling problem for trees ?
- Question 2: does the problem admit a PTAS ? a constant approximation ratio ?

Conclusion

Summary

- Glad and honored to pay tribute to André
- He's the one who taught me research
- Helped me at the beginning of my career
- Relations evolved from PhD supervisor, to colleague and friend

Acknowledgments

Thank you, André