

UPP-DAGs

J.-C. Bermond, M. Cosnard, S. Pérennes

MASCOTTE – I3S (CNRS & UNS) and INRIA Sophia Antipolis

Bordeaux Graph Workshop'2010 - Bordeaux - France

FOR ANDRE
ALL THE BEST

Introduction

- Routing, wavelength assignment and grooming in optical networks.
- Generic problem : satisfy a family of requests (or a traffic matrix) under capacity constraints
- request — dipath + wavelength (color)
- **Objectives :**
 - Minimize the **load of the routing** (number of paths sharing an edge)
 - Minimize the **number of wavelengths** (two dipaths sharing an arc have to be assigned different wavelengths)

Introduction

- minimum wavelength number \geq minimum routing load
- Minimizing the load or the number of wavelengths is a difficult problem (NP-hard).
- min wave number = min routing load if
 - General graph and multicast
 - Symmetric tree and all to all
- Even in the case of a family of dipaths, min wave number is NP-hard (= chromatic number of the conflict graph).

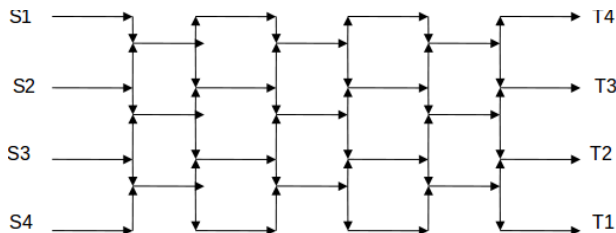
Introduction

- For directed trees and any sets of requests (family of digraphs), it can be shown that

$$\text{min wave numb} = \text{min routing load}$$

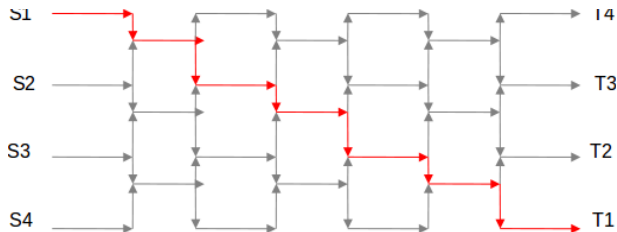
- Can this result be generalized to arbitrary Directed Acyclic Graphs ?

A pathological example



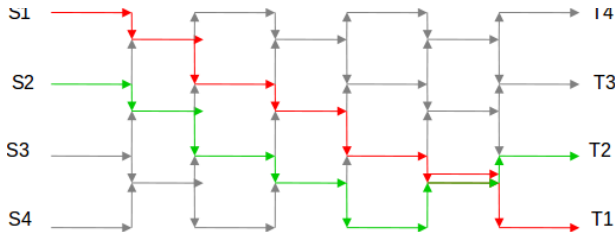
- Requests : $(S1, T1)$, $(S2, T2)$, $(S3, T3)$, $(S4, T4)$
- Minimum load ≥ 2 : each path will cross at least another one
- Minimum number of wavelengths ≥ 4 : each path will cross all the other paths.

A pathological example



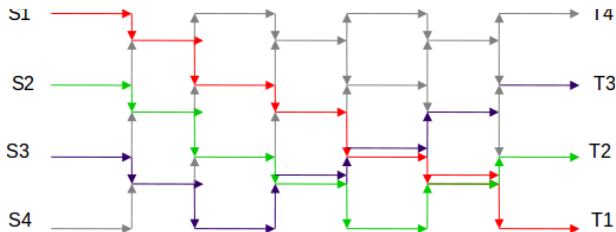
- Requests : $(S1, T1)$, $(S2, T2)$, $(S3, T3)$, $(S4, T4)$

A pathological example



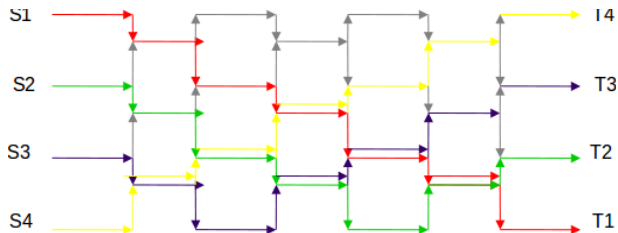
- Requests : $(S1, T1)$, $(S2, T2)$, $(S3, T3)$, $(S4, T4)$

A pathological example



- Requests : $(S1, T1)$, $(S2, T2)$, $(S3, T3)$, $(S4, T4)$

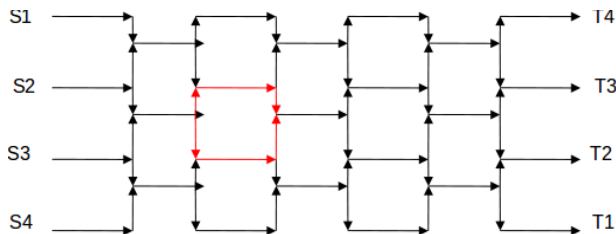
A pathological example



- Requests : $(S1, T1)$, $(S2, T2)$, $(S3, T3)$, $(S4, T4)$
 - A solution with maximum load = 2 and minimum number of wavelengths = 4.
 - Can be generalized to max load = 2 and minimum number of wavelengths = n.

Definitions

- A **DAG (Directed Acyclic Graph)** is a digraph with no directed cycle.
- An **(oriented) cycle** in a DAG consists therefore of an even sequence of dipaths P_1, P_2, \dots, P_{2k} alternating in direction
- An **internal cycle** of a DAG G is an oriented cycle such that no vertex is a source or a sink.



Definitions

- Given a digraph G and a family of dipaths \mathcal{P} , the load of an arc e is the number of dipaths of the family containing e

$$\text{load}(G, \mathcal{P}, e) = |\{P : P \in \mathcal{P}; e \in P\}|$$

- The load of G for \mathcal{P} , $\pi(G, \mathcal{P})$, is the maximum over all the arcs of G .
- Two dipaths are in conflict (or intersect) if they share an arc.
- $w(G, \mathcal{P})$ is the minimum number of colors needed to color the dipaths of \mathcal{P} in such a way that two dipaths in conflict have different colors.

$$\pi(G, \mathcal{P}) \leq w(G, \mathcal{P}).$$

Definitions and properties

- The conflict graph of (G, \mathcal{P}) is a graph whose vertices are the dipaths of \mathcal{P} , two vertices being joined if their associated dipaths are in conflict.
- w is the chromatic number of the conflict graph
- π is upper bounded by the clique number of the conflict graph.

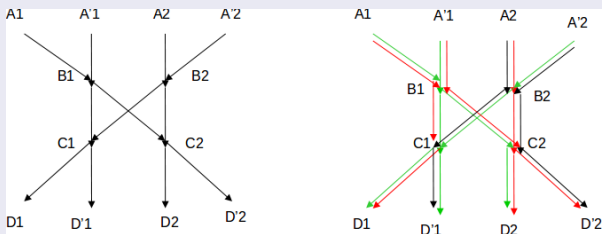
Problems

- Consider a simplified problem: unique routing. Can we solve the problem of finding the minimum number of wavelenths ?
- The answer is unknown (pathological example).
- Given a DAG G and a family of dipaths \mathcal{P} , what is the relation between the load of G for \mathcal{P} and the minimum number of wavelenths?
- Is it possible to characterize the DAGs for which load is equal to the min wave number?

Unique Path Property DAG

Proof.

$$w(G, \mathcal{P}) = \lceil (4/3\pi(G, \mathcal{P})) \rceil$$



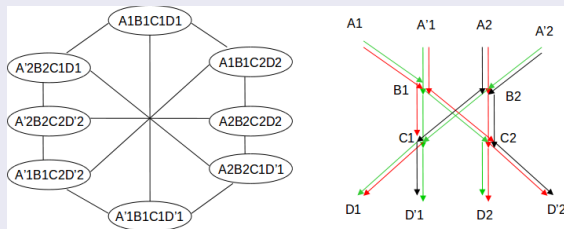
Dipaths : (A1B1C1D1), (A1B1C2D2), (A2B2C2D2),
(A2B2C1D1), (A'1B1C1D1), (A'1B1C2D2), (A'2B2C2D2),
(A'2B2C1D1)



Unique Path Property DAG

Proof.

conflict graph



If one copy of each dipath $\pi(G, \mathcal{P}) = 2$; $w(G, \mathcal{P}) = 3$ If k copies :
 $\pi(G, \mathcal{P}) = 2k$; $w(G, \mathcal{P}) = \lceil (8k/3) \rceil$ □

Main Result

Theorem

Let G be a DAG. Then, for any family of dipaths \mathcal{P} , $w(G, \mathcal{P}) = \pi(G, \mathcal{P})$ if and only if G does not contain an internal cycle.

Theorem

If a DAG G contains an internal cycle there exists a set \mathcal{P} of dipaths such that $\pi(G, \mathcal{P}) = 2$ and $w(G, \mathcal{P}) = 3$.

Unique Path Property DAG

Definition

A DAG has the UP Property if between two vertices there is at most one dipath. A digraph satisfying this property will be called an UPP-DAG.

Property

If G is an UPP-DAG and if a set of dipaths are pairwise in conflict, then their intersection is a dipath (Helly property). Hence the load is the clique number of the conflict graph. (Proof) (π = the clique number of the conflict graph.)

Unique Path Property DAG

Theorem

Let G be an UPP-DAG with only one internal cycle. Then, for any family of dipaths \mathcal{P} , $w(G, \mathcal{P}) \leq \lceil (4/3\pi(G, \mathcal{P})) \rceil$. If C is the number of internal cycles of the UPP-graph, then $w(G, \mathcal{P}) \leq \lceil ((4/3)C\pi(G, \mathcal{P})) \rceil$.

Unique Path Property DAG

Theorem

There exists an UPP-DAG with only one internal cycle and an infinite family of dipaths \mathcal{P} such that:

$$w(G, \mathcal{P}) = \lceil (4/3\pi(G, \mathcal{P})) \rceil$$

Good edge-labelling

edge-labelling: function $\phi : E(G) \rightarrow \mathbb{R}$.

A path is **increasing** if the sequence of its edges labels is non-decreasing.

An edge-labelling of G is **good** if, for any two distinct vertices u, v , there is at most one increasing (u, v) -path.

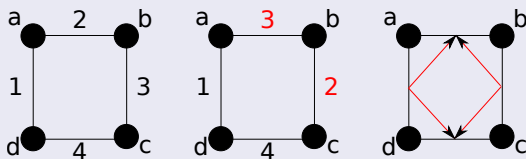
Good edge-labelling

edge-labelling: function $\phi : E(G) \rightarrow \mathbb{R}$.

A path is **increasing** if the sequence of its edges labels is non-decreasing.

An edge-labelling of G is **good** if, for any two distinct vertices u, v , there is at most one increasing (u, v) -path.

Example



UPP DAGs with load 2

Theorem

G UPP DAG with load 2. For any family of dipaths \mathcal{P} the conflict graph $C(G, \mathcal{P})$ has a good labeling

Theorem

Given a graph H with a good labeling there exists an UPP DAG G with load 2 and a family of dipaths \mathcal{P} such that $H = C(G, \mathcal{P})$

The end

Theorem

There exist a family of graphs with a good labeling and a chromatic number as large as we want

Corollary

There exist UPP digraphs with load 2 and w as large as we want

Proof.

- H of degree $\leq k$, $\text{girth} > 2K + 2$ and large X_i
- Partition the edges in $K + 1$ matchings Vizing's thm
- give edge of each matching a different label
- increasing path at most $K+1$ edges
- girth imply no 2 increasing paths



Open Problems

- G undirected is it possible to orient G to obtain an UPP DAG
- Bounds in terms of number of internal cycles
- Characterisation of graphs with good labeling
- (decision problem **NP hard** (Araujo, Cohen, Giroire, Havet))
- Quid UPP DAGS load 3 or more ??
- When $w = \pi$ (case for ALL TO ALL for each xy a dipath open question
- true for DAGS ??

MERCI