

# On the problem of finding disjoint cycles and dicycles in a digraph

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# Edge disjoint spanning trees in graphs

Theorem (Tutte 1961, Nash-Williams 1961)

*A graph  $G$  contains  $k$  edge-disjoint spanning trees if and only if*

$$|E_G(\mathcal{P})| \geq k \cdot (|\mathcal{P}| - 1)$$

*holds for all partitions  $\mathcal{P}$  of  $V(G)$ .*

$E_G(\mathcal{P})$ : set of edges in  $G$  between distinct parts of  $\mathcal{P}$ .

## Theorem (Edmonds 1973)

*For a vertex  $r$  of a digraph  $D$  there exists  $k$  arc-disjoint branchings with root  $r$  if and only if*

$$d^+(X) \geq k$$

*for every proper subset  $X$  of  $V(D)$  containing  $r$ .*

$d^+(X)$ : number of arcs in  $D$  from some  $x \in X$  to some  $y \in V(D) - X$ .

# An intermediate problem

## Problem (Thomassé, Egres Open Problems List 2008)

*Find a good characterization of the digraphs  $D$  such that there exist edge-disjoint  $S, T$ , where  $S$  is a spanning tree of  $UG(D)$  and  $T$  is an out-branching of  $D$ .*

$UG(D)$ : underlying undirected graph; technically:  
same vertices and edges, different incidence relation

- || Obv. necessary: two edge-disjoint spanning trees in  $UG(D)$ .
- || Obv. sufficient: two arc-disjoint out-branchings in  $D$ .

# A problem on mixed paths

Let  $D$  be a digraph and  $r \in V(D)$ .

If there are edge-disjoint  $S, T$ , where  
 $S$  is a spanning tree of  $UG(D)$  and  
 $T$  is an out-branching of  $D$  rooted at  $r$

then for each  $s \in V(D)$  there exist edge-disjoint  $P, Q$  where  
 $P$  is an  $(r, s)$ -path in  $UG(D)$  and  $Q$  is an  $(r, s)$ -path in  $D$ .

## Problem (MIXED-EDGE-DISJOINT-PATHS)

*Given a digraph  $D$  and  $r, s \in V(D)$ ,  
decide if there exist edge-disjoint  $P, Q$ , where  
 $P$  is an  $(r, s)$ -path in  $UG(D)$  and  $Q$  is an  $(r, s)$ -path in  $D$ .*

|| Obv. necessary: two edge-disjoint  $(r, s)$ -paths in  $UG(D)$ .

|| Obv. sufficient: two arc-disjoint  $(r, s)$ -paths in  $D$ .

# The unmixed versions

## Theorem (Menger 1927)

*Given two vertices  $r \neq s$  of a graph or digraph  $D$ , there exist  $k$  edge-disjoint  $(r, s)$ -paths if and only if there no  $(r, s)$ -cut  $X$  with  $|X| < k$  in  $D$ .*

$X$  an  $(r, s)$ -cut: every  $(r, s)$ -path in  $D$  contains an arc from  $X$ .

# The intermediate version is difficult

Theorem (Bang-Jensen & Kriesell 2009)

MIXED-EDGE-DISJOINT-PATHS is  $\mathcal{NP}$ -complete.

# Mixed homeomorphisms

Let  $H$  be a fixed mixed graph and  $D$  be any digraph.

A *mixed homeomorphism*  $f$  from  $H$  into  $D$  maps

- each vertex of  $H$  to a vertex of  $D$ ,
- each directed edge  $xy$  to a nontrivial  $(f(x), f(y))$ -path in  $D$ , and
- each undirected edge  $xy$  to a nontrivial  $(f(x), f(y))$ -path in  $UG(D)$

such that

- $f(x) \neq f(x')$  for  $x \neq x'$  in  $V(H)$  and
- $Int(f(e)) \cap f(e') = \emptyset$  for  $e \neq e'$  in  $E(H)$ .

In this definition, a cycle through  $f(x)$  is considered as a nontrivial  $(f(x), f(x))$ -path with end vertex  $f(x)$  in  $D$  or  $UG(D)$ .

$Int(f(e))$  is the set of all vertices of  $f(e)$  except its end(s).

Homeomorphisms from *graphs* into *graphs* are defined accordingly.

# A class of homeomorphism extension problems.

Fix a mixed graph  $H$ .

## Problem (MIXED-HOMEOMORPHISM-EXTENSION)

*Given a digraph  $D$  and an injection  $f : V(H) \rightarrow V(D)$ , decide if  $f$  extends to a mixed homeomorphism from  $H$  into  $D$ .*

Roughly, we look for a subdivision of  $H$  in  $D$ , where we fix the principal vertices and do not care about the direction of the edges of the subdivision paths or cycles corresponding to undirected edges of  $H$ .

# The undirected case

Fix a graph  $H$ .

## Problem (HOMEOMORPHISM-EXTENSION)

*Given a graph  $G$  and an injection  $f : V(H) \rightarrow V(G)$ , decide if  $f$  extends to a homeomorphism from  $H$  into  $G$ .*

To solve this, it is sufficient to solve polynomially many linkage problems. Each of these takes polynomial time by Graph Minors XIII (Robertson & Seymour 1995).

# The directed case

Fix a digraph  $H$ .

## Problem (DIGRAPH-HOMEOMORPHISM-EXTENSION)

*Given a digraph  $D$  and an injection  $f : V(H) \rightarrow V(D)$ , decide if  $f$  extends to a [mixed] homeomorphism from  $H$  to  $D$ .*

There is a classic dichotomy stating that DIGRAPH-HOMEOMORPHISM-EXTENSION is solvable in polynomial time if all edges of  $H$  have the same initial vertex or they all have the same terminal vertex, whereas, in all other cases, it is  $\mathcal{NP}$ -complete (Fortune & Hopcroft & Wyllie 1980).

## Theorem (Bang-Jensen & Kriesell 2009)

MIXED-HOMEOMORPHISM-EXTENSION for  $H$  is in  $\mathcal{P}$  if

- all edges of  $H$  are undirected, or
- all edges of  $H$  are directed and all have the same initial vertex or all have the same terminal vertex,

and it is  $\mathcal{NP}$ -complete in all other cases.

# The special case of two cycles

The case that  $H$  consists of a directed and an undirected loop at distinct vertices can be rephrased:

## Problem

*Given a digraph  $D$  and vertices  $x \neq y$ , decide if there is a cycle  $B$  in  $D$  and a cycle  $C$  in  $UG(D)$  such that  $x \in V(B)$ ,  $y \in V(C)$ , and  $V(B) \cap V(C) = \emptyset$ .*

The problem is  $\mathcal{NP}$ -complete — even if we do not prescribe  $y$ .

It is likely that this changes if we do neither prescribe  $y$  nor  $x$ .

## Problem (DISJOINT-CYCLES)

*Decide if a given (di)graph  $G$  has two disjoint cycles.*

In  $\mathcal{P}$  for graphs by classic results (Lovász 1965, Dirac 1963).

In  $\mathcal{P}$  for directed graphs (McCuaig 1993); difficult.

# The mixed disjoint cycles problem

## Problem (MIXED-DISJOINT-CYCLES)

*Decide if, for a given digraph, there exists cycles  $B$  in  $D$  and  $C$  in  $UG(D)$  such that  $V(B) \cap V(C) = \emptyset$ .*

**Theorem.** (Bang-Jensen & Kriesell 2009)

MIXED-DISJOINT-CYCLES restricted to strongly connected digraphs  $D$  is in  $\mathcal{P}$ , and  $B, C$  as there can be found in polynomial time if they exist.

## Conjecture (Bang-Jensen & Kriesell 2009)

MIXED-DISJOINT-CYCLES *is in*  $\mathcal{P}$ .

When do there exist disjoint cycles  $B$  in  $D$  and  $C$  in  $UG(D)$  ?

- || Obv. necessary: two disjoint cycles in  $UG(D)$ .
- || Obv. sufficient: two disjoint cycles in  $D$ .

Concentrate on digraphs without two disjoint cycles.  
=: intercyclic

# Cycle transversal number

Suppose  $D$  is strongly connected.

Distinguish cases according to the cycle transversal number

$$\tau(D) := \min\{|T| : T \subseteq V(D), D - T \text{ acyclic}\}.$$

$\tau(D)$  can be determined in polynomial time (McCuaig 1993).

$\tau(D)$	Method to answer
0	“No” (as $D$ is acyclic)
1	Decision Algorithm
2	Characterization (partly topological argument)
3	“Yes” (using a result of McCuaig)
$\geq 4$	“Yes” (as $D$ is not intercylic (McCuaig 1993))

# Cycle transversal number 1

Outline of our decision algorithm

for a strongly connected input digraph  $D_0$  with  $\tau(D_0) = 1$ .

- Find a cycle transversal  $\{a\}$  in  $D_0$ .
- Split  $a$ , i.e. replace  $a$  by  $a^+$ ,  $a^-$ ; for each edge from  $x$  to  $y$ , replace  $x$  by  $a^+$  if  $x = a$  and  $y$  by  $a^-$  if  $y = a$ .
- The result  $D$  is acyclic with source  $a^+$  and sink  $a^-$ .
- Find a largest system  $\mathcal{P}$  of openly disjoint  $(a^+, a^-)$ -paths.
- Study the  $\bigcup \mathcal{P}$ -bridges. Easy if  $|\mathcal{P}| \neq 2$ ; reduce or decide.
- For  $\mathcal{P} = \{P, Q\}$  reduce to the case that  $V(D) = V(P) \cup V(Q)$ ; reduce further or decide.

# Cycle transversal number 2 I

$D_0$  strongly connected, cycle transversal number is 2.

First:  $D_0$  simple, minimal in- and out-degree at least 2.

- If  $D_0$  is 2-regular then the answer is “no” if and only if  $D_0$  is not the square of an odd cycle (1-page argument).
- Find cycle transversal  $\{x, y\}$ , split into  $x^+, x^-$  &  $y^+, y^-$ .
- The result  $D$  is acyclic; sources:  $x^+, y^+$ , sinks:  $x^-, y^-$ .
- If  $D$  is not intercyclic then the answer is “yes”.
- Otherwise, find a plane embedding of  $D$  in the unit disc  $S$  where  $x^+, y^+, x^-, y^-$  map to  $(-1, 0), (0, -1), (1, 0), (0, 1)$  and these are the only points on  $\partial S$ .  
(McCuaig 1993, Metzlar 1989, Thomassen 1983/1985)
- Find cycles  $B, C$  as required (“yes”), unless  $D_0$  is a diwheel (“no”).

# Cycle transversal number 2 II

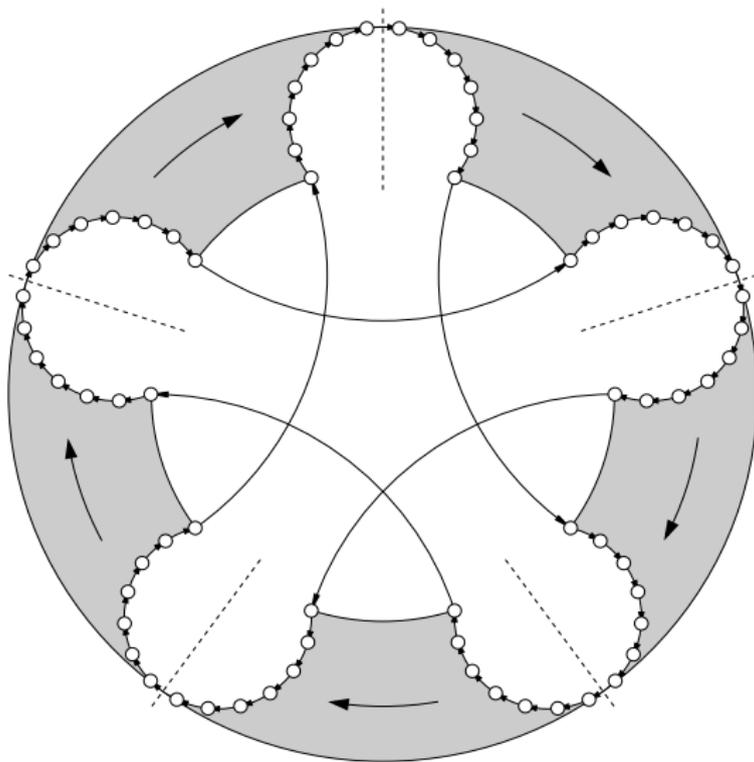
$D_1$  strongly connected, cycle transversal number is 2.

- If vertex  $x$  has only one outneighbor  $y$ : Contract  $xy$ .
- If vertex  $x$  has only one in-neighbor  $y$ : Contract  $yx$ .
- Repeat the first two steps as long as possible.  
The underlying simple digraph  $D_0$  of the result has minimum in- and out-degree  $\geq 2$ .
- Decide for  $D_0$ .  
If the answer for  $D_0$  is “yes” then the answer for  $D_1$  is “yes”.
- Otherwise analyze
  - the subdigraphs formed by the contracted edges, and
  - the edges connecting them.

(Reorganized representation of the input  $D_1$ .)

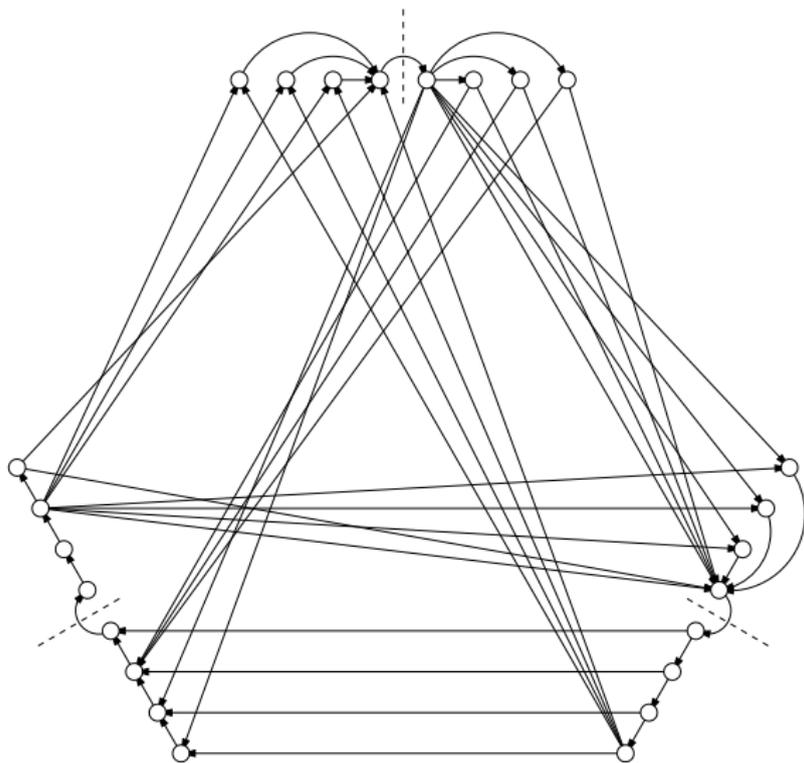
Difficult, many different “yes”- and “no”-instances.

# A vault



Typical example;  $D_0$  is the square of a 5-cycle.

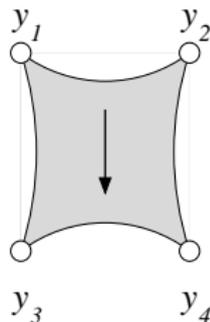
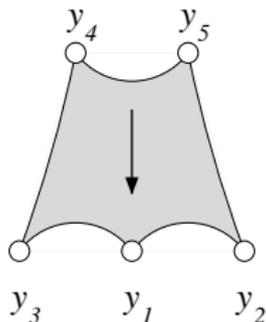
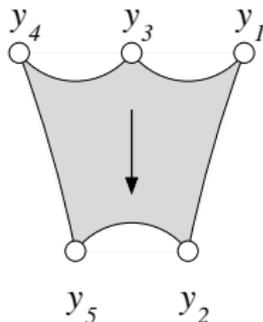
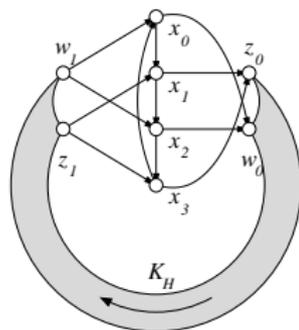
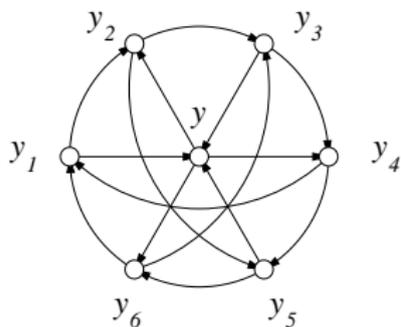
# A trivault



Typical example;  $D_0$  is the square of a triangle.

# Cycle transversal number $\geq 3$

Structure theorem for intercyclic digraphs  $D$  (McCuaig 1993) implies “yes” for  $\tau(D) \geq 3$ .



Conjecture (Bang-Jensen & Kriesell 2009)

MIXED-DISJOINT-CYCLES *is in*  $\mathcal{P}$

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## Metaproblem.

Find more mixed problems in between their well-studied unmixed ancestors.

## Theorem (Bang-Jensen and Yeo 2010)

*The following problem is NP-complete: Given a directed graph  $D = (V, A)$  and a vertex  $s \in V$ ; does  $D$  have an out-branching  $B_s^+$  such that  $UG(D - A(B_s^+))$  is connected?*

# Sketch of proof:

First step: reduce 3-SAT to  $(s, t)$ -path in a digraph which avoids certain vertices.

Let  $W[u, v, p, q]$  be the digraph (the variable gadget) with vertices  $\{u, v, y_1, y_2, \dots, y_p, z_1, z_2, \dots, z_q\}$  and the arcs of the two  $(u, v)$ -paths  $uy_1y_2 \dots y_pv, uz_1z_2 \dots z_qv$ .

Let  $\mathcal{F}$  be an instance of 3-SAT with variables  $x_1, x_2, \dots, x_n$  and clauses  $C_1, C_2, \dots, C_m$ . The ordering of the clauses  $C_1, C_2, \dots, C_m$  induces an ordering of the occurrences of a variable  $x$  and its negation  $\bar{x}$  in these.

With each variable  $x_i$  we associate a copy of  $W[u_i, v_i, p_i, q_i]$  where  $x_i$  occurs  $p_i$  times and  $\bar{x}_i$  occurs  $q_i$  times in the clauses of  $\mathcal{F}$ . Identify end vertices of these digraphs by setting  $v_i = u_{i+1}$  for  $i = 1, 2, \dots, n - 1$ . Let  $s = u_1$  and  $t = v_n$ . Denote the resulting digraph by  $D'$ .

For each clause  $C_j = \{a_{j,1} \vee a_{j,2} \vee a_{j,3}\}$  we identify  $a_{j,i}$ ,  $i = 1, 2, 3$  with the vertex corresponding to that literal in  $D'$ .

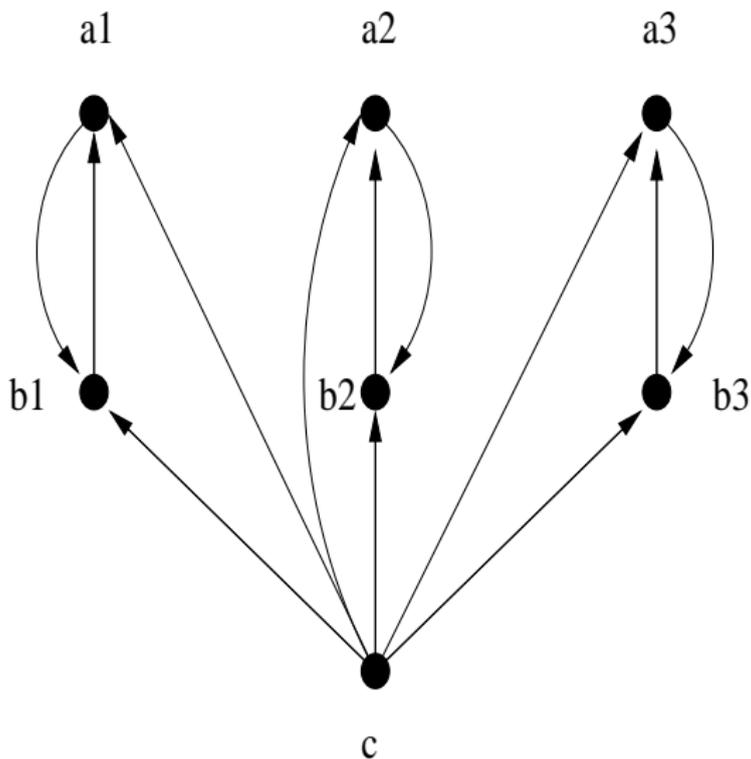
**Claim:**  $D'$  contains an  $(s, t)$ -path  $P$  which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$  if and only if  $\mathcal{F}$  is satisfiable.

Suppose  $P$  is an  $(s, t)$ -path which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$ . By construction, for each variable  $x_j$ ,  $P$  traverses either the subpath  $u_i y_{i,1} y_{i,2} \dots y_{i,p_i} v_i$  or the subpath  $u_i z_{i,1} z_{i,2} \dots z_{i,q_i} v_i$ .

Now define a truth assignment by setting  $x_j$  false precisely when the first traversal occurs for  $i^{31}$ . Conversely, given a truth assignment for  $\mathcal{F}$  we can form  $P$  by routing it through all the false literals in the chain of variable gadgets.

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<sup>31</sup>This is a satisfying truth assignment for  $\mathcal{F}$  since for any clause  $C_j$  at least one literal is avoided by  $P$  and hence becomes true by the assignment (the literals traversed become false and those not traversed become true). 



The clause gadget.



**Claim:**  $D_{\mathcal{F}}$  has an out-branching  $B_s^+$  such that  $D_{\mathcal{F}} - A(B_s^+)$  is connected if and only if  $D'$  contains an  $(s, t)$ -path  $P$  which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$ .

Suppose first that there exists  $B_s^+$  such that  $D - A(B_s^+)$  is connected. It follows from the structure of  $D_{\mathcal{F}}$  that the  $(s, t)$ -path  $P$  in  $B_s^+$  lies entirely inside  $D'$  and since  $tc_i$  is the only arc entering  $c_i$ , all arcs of the form  $tc_i$ ,  $i \in [m]$  are in  $B_s^+$ .

$P$  cannot contain all of  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for some clause  $C_j$  because that would disconnect the vertices of  $H_j - \{a_{j,1}, a_{j,2}, a_{j,3}\}$  from the remaining vertices.

Conversely, suppose that  $D'$  contains an  $(s, t)$ -path  $P$  which avoids at least one vertex from  $\{a_{j,1}, a_{j,2}, a_{j,3}\}$  for each  $j \in [m]$ . Then we form an out-branching  $B_s^-$  by adding the following arcs:

all arcs of the form  $tc_i$ ,  $i \in [m]$  and for each clause  $C_j$ ,  $j \in [m]$  and  $i \in [3]$  if  $P$  contains the vertex  $a_{j,i}$  we add the arc  $a_{j,i}b_{j,i}$  and otherwise we add the arcs  $c_jb_{j,i}$ ,  $b_{j,i}a_{j,i}$ . This clearly gives an out-branching  $B_s^+$  of  $D_{\mathcal{F}}$ .

It remains to show that  $D^* = D_{\mathcal{F}} - A(B_S^+)$  is connected. First observe that  $D^* \langle V(D') \rangle$  contains either all arcs of the path  $u_i y_{i,1} y_{i,2} \dots y_{i,p_i} v_i$  or all arcs of the subpath  $u_i z_{i,1} z_{i,2} \dots z_{i,q_i} v_i$  for each  $i \in [n]$  and hence it contains an  $(s, t)$ -path which passes through all the vertices  $u_1, u_2, \dots, u_n, t$ .

By the description of  $P$  above, for each clause  $C_j$ ,  $j \in [m]$  and  $i \in [3]$ , if  $P$  contains the vertex  $a_{j,i}$  then  $D^*$  contains the arcs  $c_j b_{j,i}, c_j a_{j,i}$  and if  $P$  does not contain the vertex  $a_{j,i}$  then  $D^*$  contains the arcs  $c_j a_{j,i}, a_{j,i} b_{j,i}$ . Now it is easy to see that  $D^*$  is connected and spanning.

## Theorem

*It is  $\mathcal{NP}$ -complete to decide whether a given digraph has an  $(s, t)$ -path  $P$  such that  $D - A(P)$  is connected for specified vertices  $s, t$ .*



## Theorem

*It is  $\mathcal{NP}$ -complete to decide for a given digraph  $D$  and distinct vertices  $s, t \in V(D)$  whether the underlying graph of  $D$  has an  $(s, t)$ -path  $Q$  such that  $D - A(Q)$  has an out-branching  $B_s^+$  rooted at  $s$ .*

## Theorem

*It is  $\mathcal{NP}$ -complete to decide for a given strongly connected digraph  $D$  whether  $D$  contains a directed cycle  $C$  such that  $UG(D - A(C))$  is connected.*

## Theorem

*It is  $\mathcal{NP}$ -complete to decide for a given strongly connected digraph  $D$  whether  $UG(D)$  contains a cycle  $C$  such that  $D - A(C)$  is strongly connected.*

## Theorem

*It is  $\mathcal{NP}$ -complete to decide whether a 2-regular digraph  $D$  contains a spanning strong subdigraph  $D'$  such that  $UG(D - A(D'))$  is connected.*

# Thank you very much for your attention!

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