

PERFECT MATCHINGS IN CUBIC GRAPHS

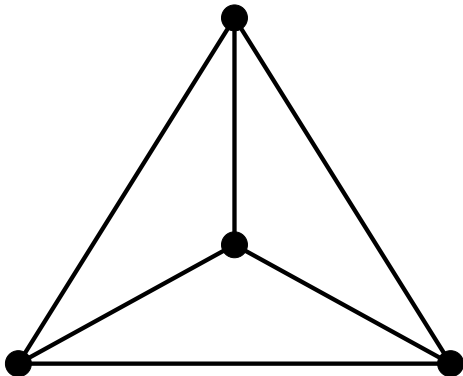
L. Esperet ¹ - F. Kardoš ² - A. King ³
D. Král' ⁴ - S. Norin ⁵

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- ² Pavol Jozef Šafárik University, Košice, Slovakia
- ³ Columbia University, New-York, NY, USA
- ⁴ Charles University, Prague, Czech Republic
- ⁵ Princeton University, Princeton, NJ, USA

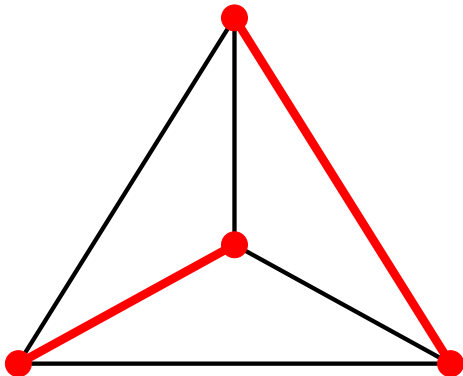
Graph Workshop in Bordeaux

November 20, 2010

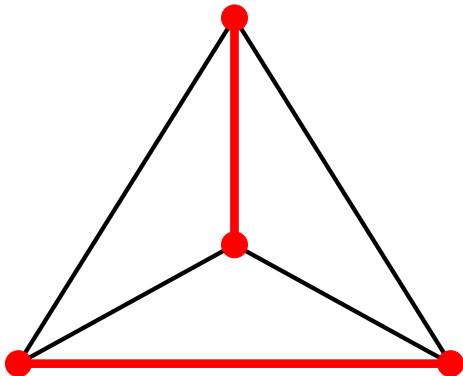
DEFINITIONS



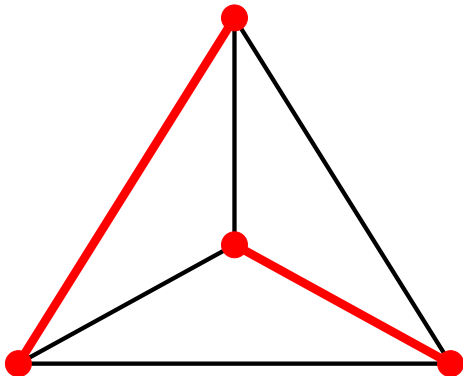
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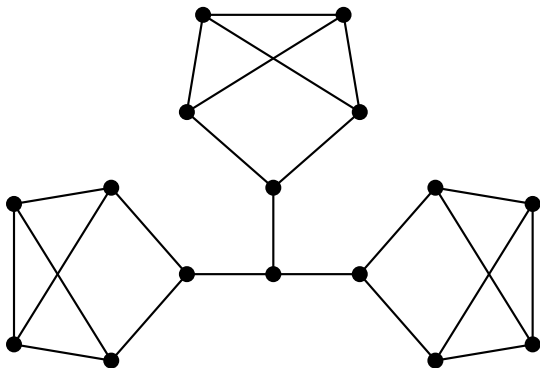
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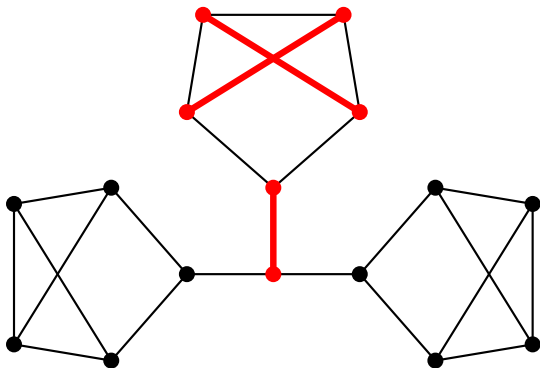
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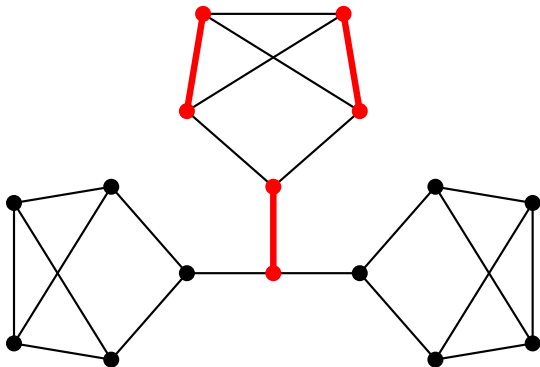
THE CONJECTURE



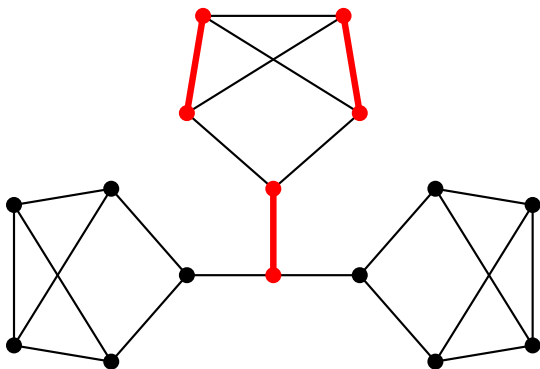
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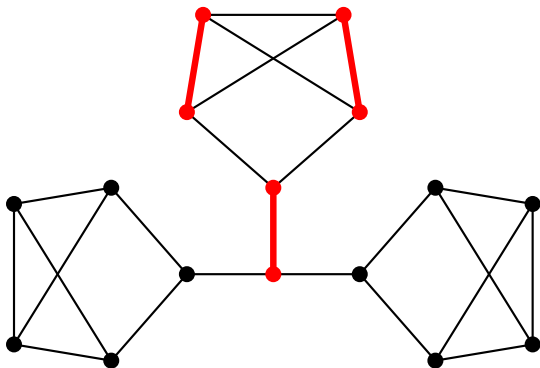
THE CONJECTURE



Theorem (Petersen 1891)

Every cubic bridgeless graph contains a perfect matching.

THE CONJECTURE



Conjecture (Lovász & Plummer 70's)

There exists a constant $c > 0$, such that any n -vertex cubic bridgeless graph contains at least 2^{cn} perfect matchings.

BIPARTITE GRAPHS

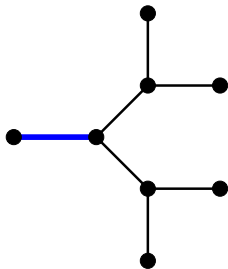
Theorem (Voorhoeve 1979)

Every bipartite cubic bridgeless graph with n vertices contains at least $6 \cdot (4/3)^{n/2-3}$ perfect matchings.

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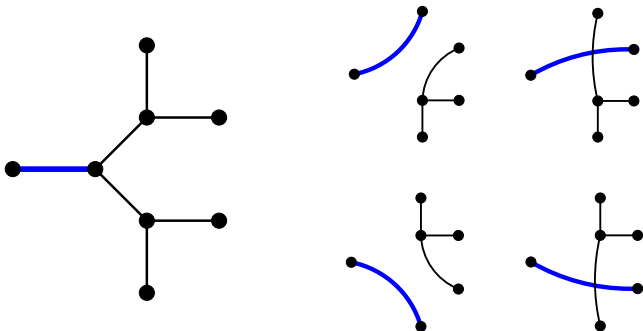
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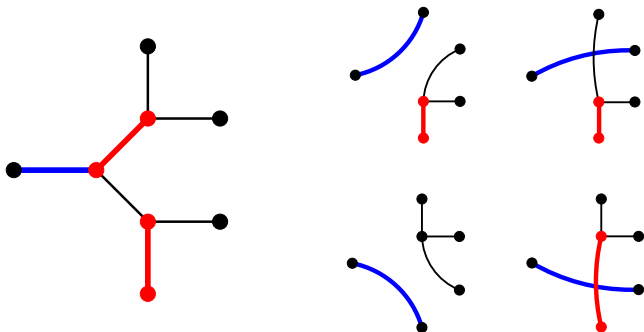
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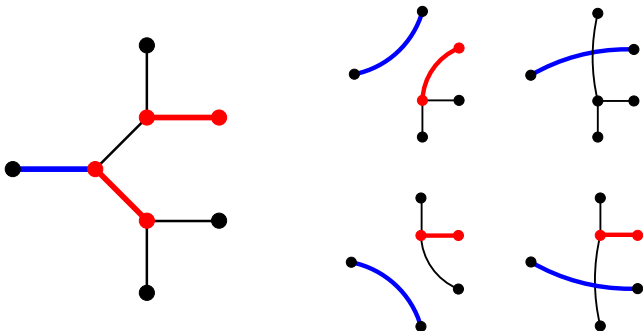
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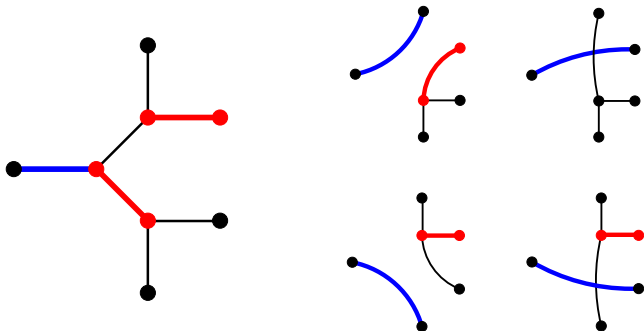
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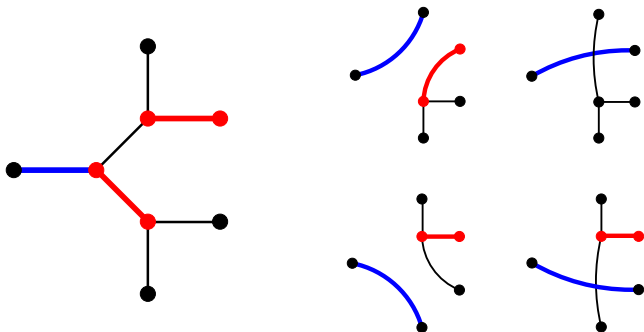


$\overline{m}(n)$ = the largest k so that every n -vertex bipartite cubic bridgeless graph has at least k perfect matchings **avoiding** any given edge.

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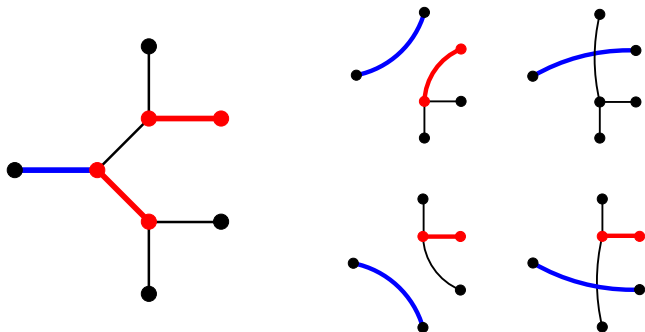


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PLANAR GRAPHS

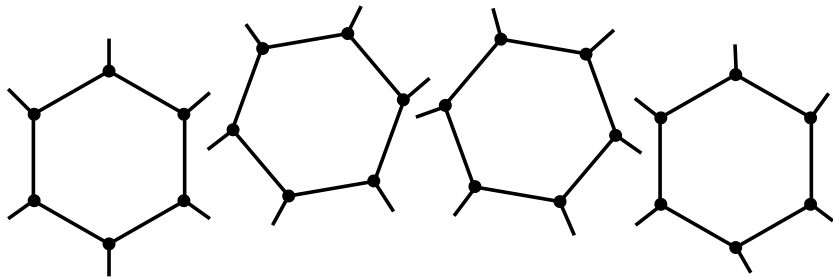
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Every planar cubic bridgeless graph with n vertices contains at least $2^{n/655978752}$ perfect matchings.

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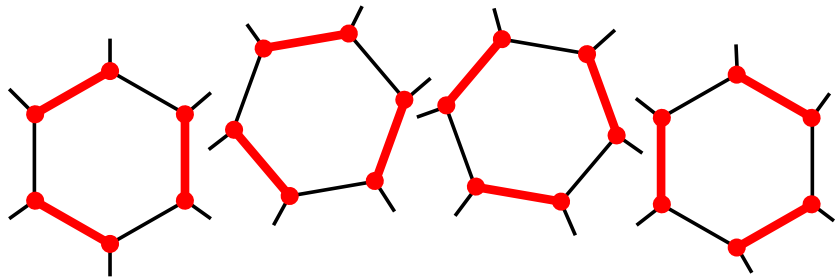
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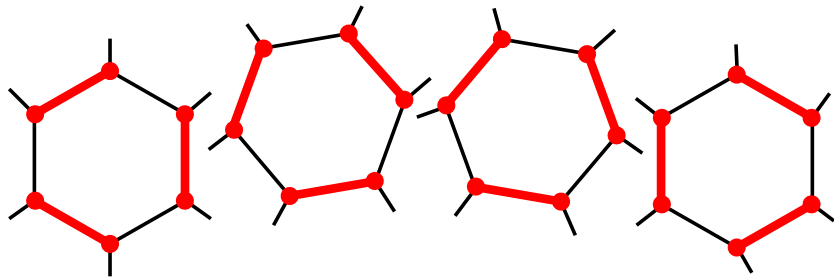
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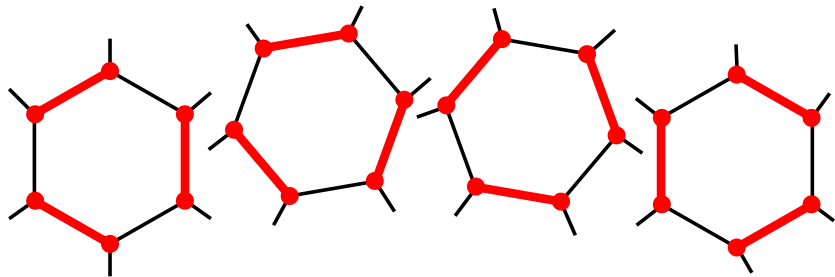
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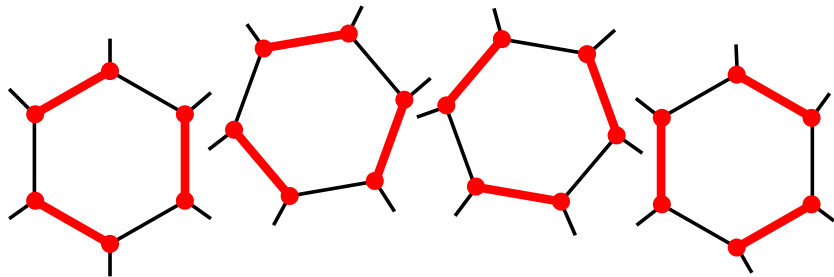
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k disjoint alternating cycles $\Rightarrow 2^k$ perfect matchings

GENERAL CASE

Let G be a cubic bridgeless graph with n vertices.

- $m(G) \geq n/4 + 2$ (Edmonds, Lovász & Pulleyblank, Naddef 1982)
- $m(G) \geq n/2$ (Kráľ', Sereni & Stiebitz 2008)
- $m(G) \geq 3n/4 - 10$ (E., Kráľ', Škoda & Škrekovski 2008)

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A graph G is **cyclically k -edge-connected**, denoted CkC , if at least k edges need to be removed from G in order to disconnect two cycles.

A SUPERLINEAR BOUND

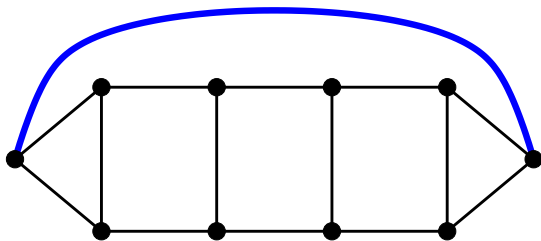
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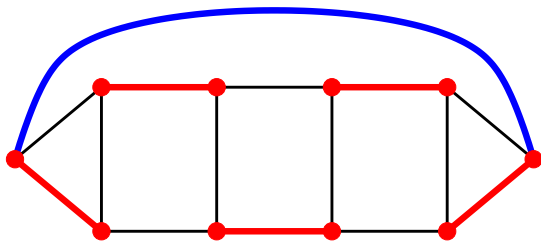
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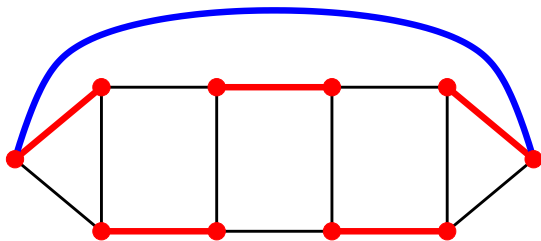
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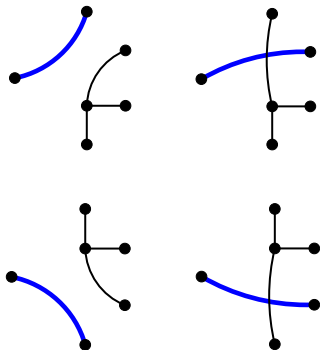
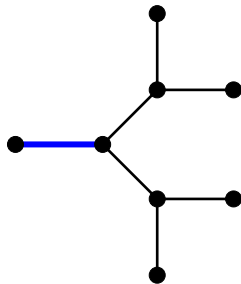
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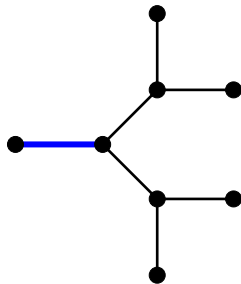
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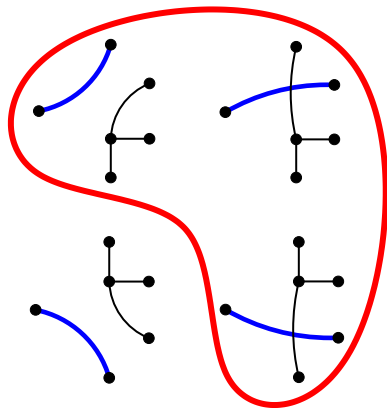
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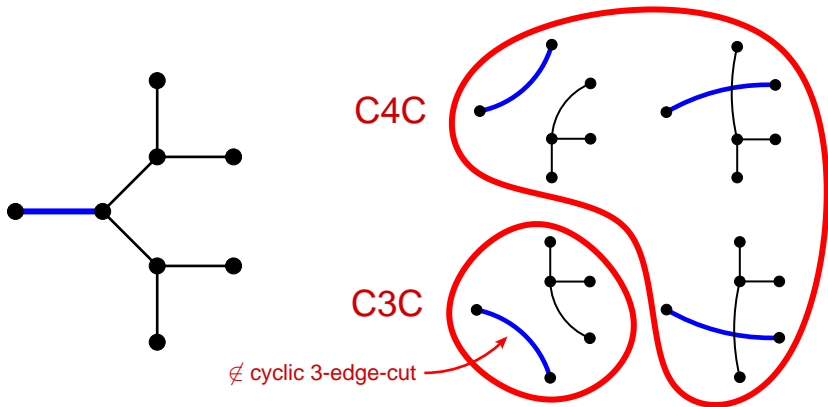
C4C



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Theorem

If G is a cubic bridgeless graph on n vertices, then

- 1 $m^*(G) \geq 2^{n/3761}$, or
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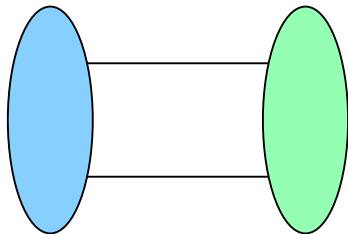
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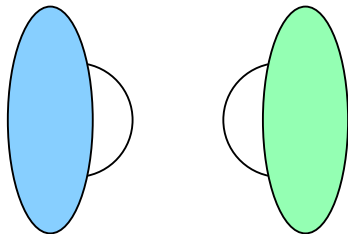


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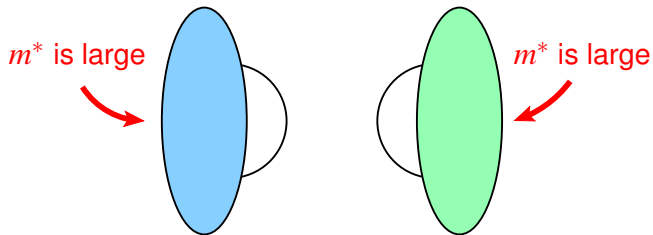


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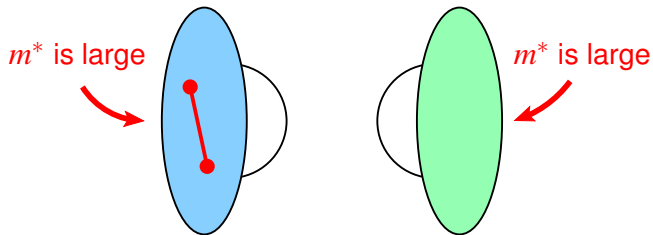


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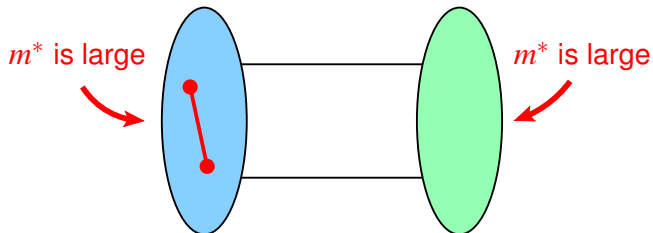


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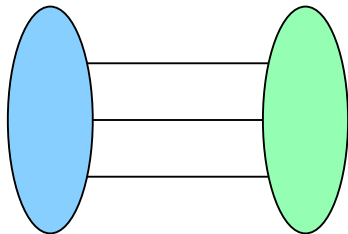


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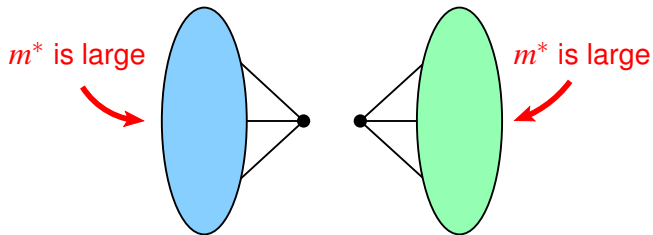


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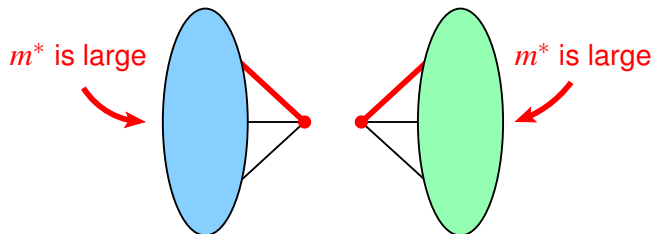


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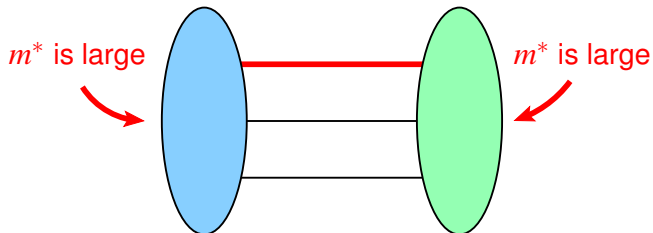


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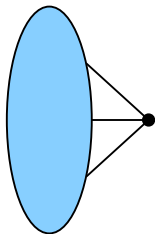
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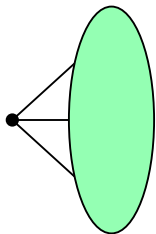
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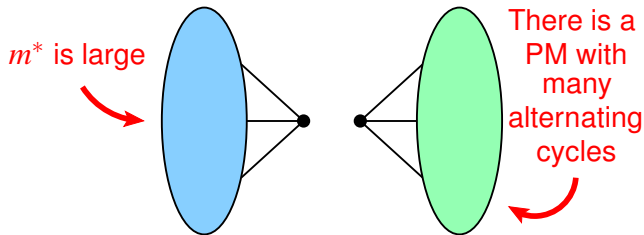


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In order to find a perfect matching with many disjoint alternating cycles, we will find a **a probability distribution on the perfect matchings** and a collection \mathcal{X} of disjoint sets of vertices, so that the subgraph induced by each set $X \in \mathcal{X}$ contains an alternating cycle with probability at least $\frac{1}{3}$.

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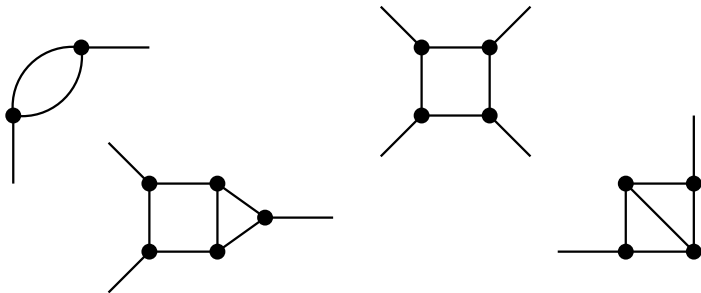
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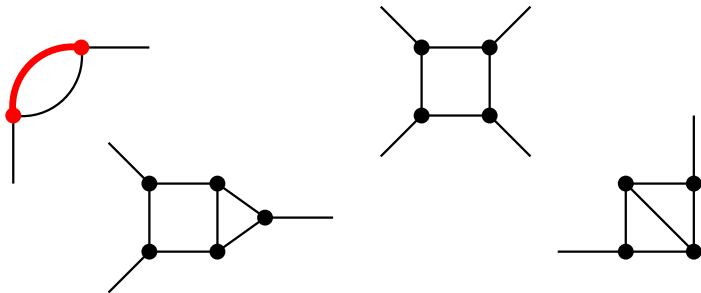
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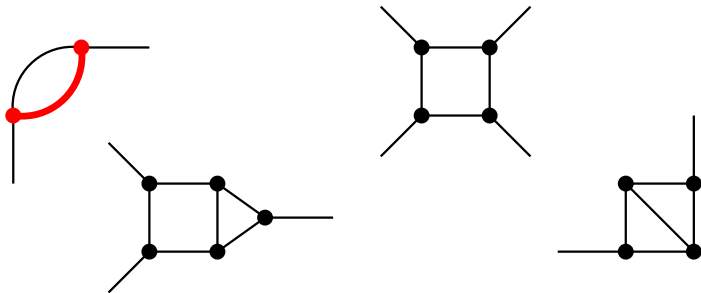
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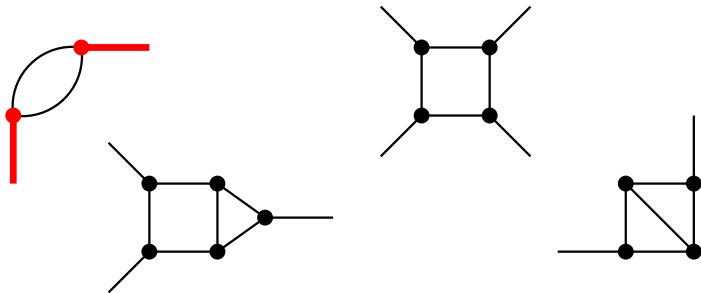
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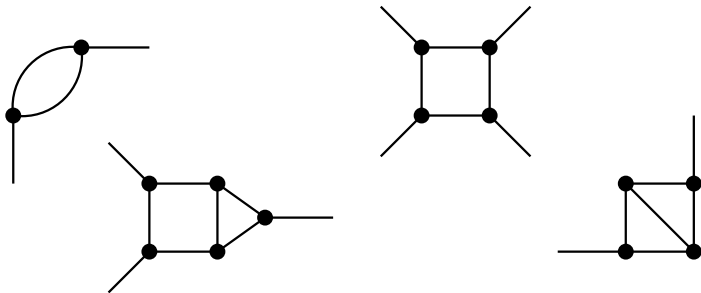
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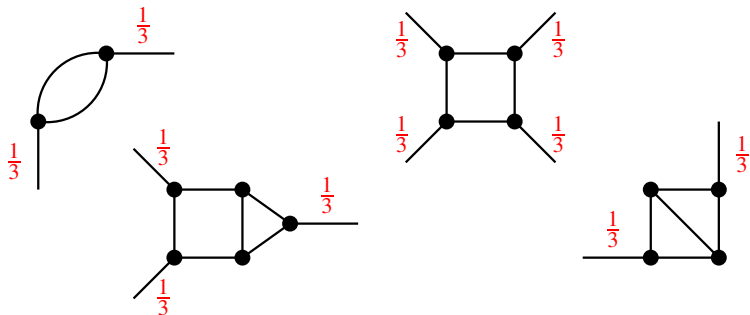
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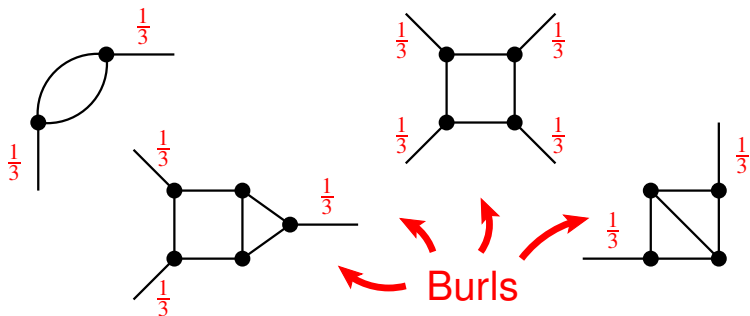
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ALTERNATING SETS

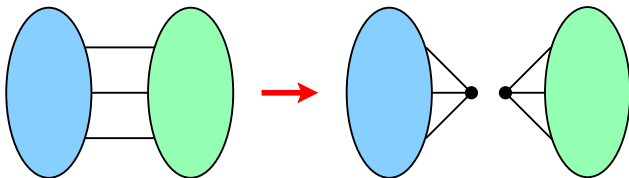
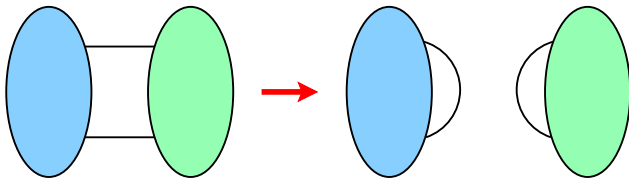
In order to find a perfect matching with many disjoint alternating cycles, we will find a **a probability distribution on the perfect matchings** and a collection \mathcal{X} of disjoint sets of vertices, so that the subgraph induced by each set $X \in \mathcal{X}$ contains an alternating cycle with probability at least $\frac{1}{3}$.

\Rightarrow there must be a perfect matching with a least disjoint $|\mathcal{X}|/3$ alternating cycles.



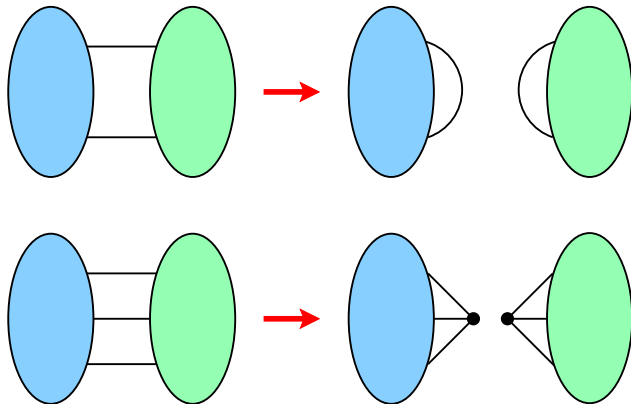
FINDING BURLS WHEN THERE IS NO CORE

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In this case, we can find a linear number of burls

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Otherwise (if there is a core), we prove by induction that $\exists \alpha, \beta, \gamma > 0$:

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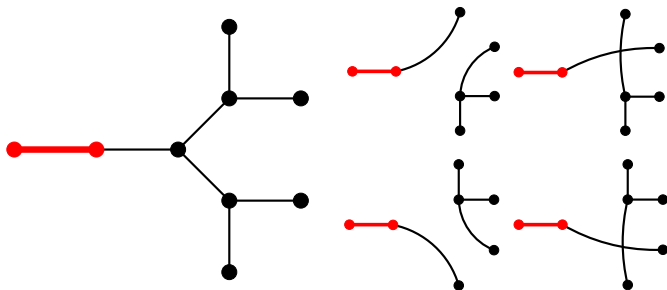
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We look at the four splittings
- 2 G has a small edge-cut

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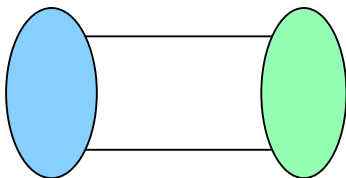
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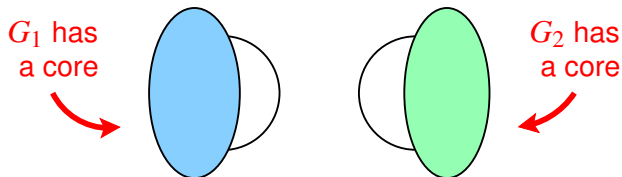


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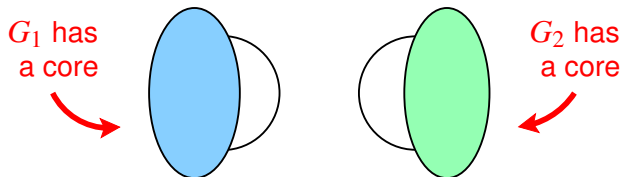


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$$m^*(G) \geq m^*(G_1) \cdot m^*(G_2)$$

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We apply induction on both sides

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 - 2 There is a unique core.

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We apply induction on both sides
 - 2 There is a unique core.
We contract everything except the core and apply induction

CONCLUSION

Theorem (E., Kardoš, King, Král' & Norin 2010)

Every cubic bridgeless graph on n vertices has at least $2^{n/3761}$ perfect matchings.

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- 1 $m^*(G) \geq 2^{n/3761}$, or
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Theorem

Every k -regular $(k - 1)$ -edge-connected graph on n vertices has at least $2^{n/22026}$ perfect matchings.