

# Square-free graphs are multiplicative

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## EXTENDED ABSTRACT

Graph homomorphism is an ubiquitous notion in graph theory with a variety of applications, and a first step to understanding more general constraints given by relational structures, see e.g. the monograph of Hell and Nešetřil [5]. We write  $\mu : G \rightarrow H$  if  $\mu$  is a homomorphism from the graph  $G$  to  $H$ , or simply  $G \rightarrow H$  if such a homomorphism exists. For example, a graph  $G$  is  $k$ -colorable iff it has a homomorphism to the complete graph on  $k$  vertices,  $G \rightarrow K_k$ .

The tensor product  $G \times H$  is a natural operation arising in this context, defined as having a vertex for every pair  $(g, h) \in V(G) \times V(H)$ , and an edge between  $(g, h)$  and  $(g', h')$  whenever  $gg' \in E(G)$  and  $hh' \in E(H)$ . It coincides with the so called categorical product in the category of graphs, with homomorphisms as arrows. In particular, for graphs  $K, G, H$  we have that  $K \rightarrow G \times H$  if and only if  $K \rightarrow G$  and  $K \rightarrow H$ .

Hedetniemi [4] conjectured the following for the chromatic number of a product of graphs:

**Conjecture 1** *For any two graphs  $G, H$ ,*

$$\chi(G \times H) = \min(\chi(G), \chi(H))$$

This seemingly innocent statement withstood 50 years of research, and is only known to hold in very special cases. Seeing colorings as homomorphism, we get the following equivalent statement: for any graphs  $G, H$  and any integer  $k$ ,  $G \times H \rightarrow K_k \Leftrightarrow G \rightarrow K_k$  or  $H \rightarrow K_k$ .

The same motivations lead to the more general question of which graphs  $K$  satisfy that  $G \times H \rightarrow K \Leftrightarrow G \rightarrow K$  or  $H \rightarrow K$  (for all graphs  $G, H$ ). Such graphs are called **multiplicative**. Hedetniemi's conjecture is then the statement that all cliques  $K_k$  are multiplicative.

There are many equivalent definitions. To give one more example, the tensor product and disjoint union define a distributive lattice, in which multiplicative graphs are precisely the *meet-irreducible* elements, crucial for understanding this lattice structure. We refer to the surveys of Zhu [11], Sauer [8], and Tardif [10] for a variety of results on Hedetniemi's conjecture and multiplicative graphs.

## Results

Previously, the only graphs shown to be multiplicative were:  $K_1$  and  $K_2$  (trivially),  $K_3$  by El-Zahar and Sauer [2], a generalization to odd cycles  $C_{2i+1}$  by Häggkvist et al. [3], and, much later, a further generalization to all circular cliques  $K_{p/q}$  with  $\frac{p}{q} < 4$  by Tardif [9].

We use topology to show the following, giving in particular the first examples of multiplicative graphs with arbitrarily high chromatic number.

**Theorem 2** *Every square-free graph is multiplicative.*

*Square-free* graphs, that is, graphs without the 4-cycle  $C_4$  as a subgraph (whether induced or not), were previously considered by Delhommé and Sauer [1], who showed them to be multiplicative if we limit ourselves to products of graphs that both contain a triangle. Our proofs, however, share little with those in [1], instead formalizing and extending the topological intuitions implicitly used in [2] and [3]. We use the same methods to give a different proof for circular cliques, which subsumes all previously known cases.

**Theorem 3** *Every circular clique  $K_{p/q}$  with  $\frac{p}{q} < 4$  is multiplicative.*

Our general approach relies on assigning a topological space to every graph and using topological invariants to show or improve properties of a given coloring of a product of graphs. While topology underlies most of the ideas, the formal proofs only need to use relatively simple, discrete, algebraic descriptions of the invariants.

The topological spaces we use could be equivalently defined with the so called *box complex* – one of the basic objects studied in *topological combinatorics*; we refer to Matoušek’s book [7] for a gentle, but in-depth introduction to this field. The box complex is usually used to give lower bounds on the chromatic number of a graph, as in Lovász’ famous proof of Kneser’s conjecture [6]; however, we use it for square-free graphs, which is precisely the case where the topological lower bounds are almost trivial. This suggests there are still unexplored connections of topology with combinatorics and hints at many avenues for generalizations.

## References

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