

Edge- and vertex-partitions of graphs with bounded maximum average degree

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joint work with

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dedicated to Andre Raspaud

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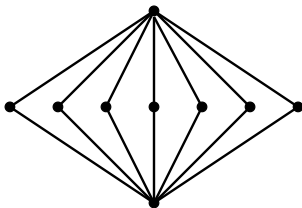


Figure: $D_1(K_{2,n}) \geq (n - 1)/2$.

Theorem 2 (He, Hou, Lih, Shao, Wang, and Zhu): Let G be a planar graph with girth g . Then

- (i) $D_1(G) \leq 4$ if $g \geq 5$;
- (ii) $D_1(G) \leq 2$ if $g \geq 7$;
- (iii) $D_1(G) \leq 1$ if $g \geq 11$;
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Theorem 3 (Bassa, Burns, Campbell, Deshpande, Farley, Halsey, Michalakis, Persson, Pylyavskyy, Rademacher, Riehl, Rios, Samuel, Tenner, Vijayasarathy, Zhao, and Kleitman): Let G be a planar graph with girth 10. Then $D_1(G) \leq 1$.

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Theorem 6 (Wang and Zhang, and independently Montassier, Ossona de Mendez, Raspaud, and Zhu): Let G be a planar graph with girth 8. Then $D_1(G) \leq 1$ and this is sharp.

Maximum average degree approach

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Theorem 7 yields that $F(d) \geq 4 - \frac{8d+12}{d^2+6d+6}$.

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Theorem 9 (Montassier, Ossona de Mendez, Raspaud, and Zhu): Let G be a graph with $2|E(H)| \leq 3|V(H)| - 3$ for every $H \subseteq G$. Then G decomposes into a forest and a linear forest, and this is sharp.

Edge decomposition results

Theorem 10 (Kim, A.K., West, Wu, and Zhu):

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Comment: Theorem 10 does not imply Theorem 5.

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A graph G is (improperly) (d_1, d_2) -colorable, if $V(G)$ can be partitioned into V_1 and V_2 so that $\Delta(G(V_1)) \leq d_1$ and $\Delta(G(V_2)) \leq d_2$.

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Theorem 13 (Glevov and Zambalaeva): Every planar graph of girth at least 16 is $(0, 1)$ -colorable.

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Let $F(j, k)$ denote the supremum over positive x such that each graph G with $Mad(G) < x$ is (j, k) -colorable.

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Corollary: Every planar graph of girth at least 14 is $(0, 1)$ -colorable.

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Proved even for list version.

Andre again

Theorem 16 (Borodin, Ivanova, Montassier, Ochem, and Raspaud): $3 - \frac{2}{k+2} \leq F(0, k) \leq 3 - \frac{1}{k+1}$. Furthermore,

$$\frac{7}{3} \leq F(0, 1) \leq \frac{17}{7}$$

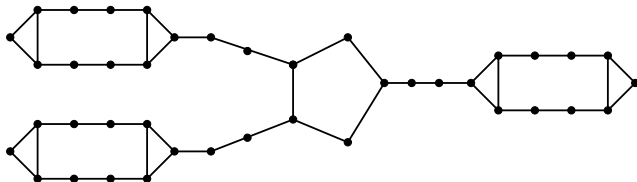
and there exist planar non-(0, 1)-colorable graphs with girth 7.

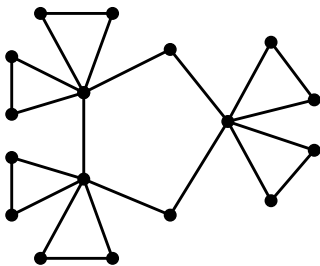
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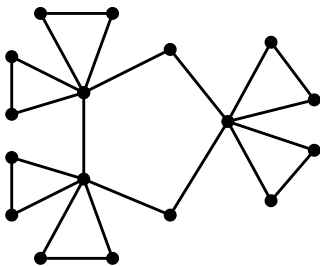
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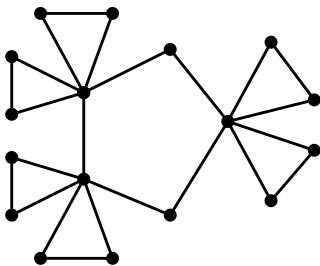


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Theorem 17 (Borodin, Ivanova, Montassier, and Raspaud): If $k \geq 2$, then $\frac{10k+22}{3k+9} \leq F(1, k) \leq \frac{14k}{4k+1}$.



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Theorem 18 (Borodin, Ivanova, Montassier, and Raspaud): Bounds on $F(j, k)$. (Hard to state).

Our results

Theorem 19 (Borodin and A. K.): $F(0, 1) = \frac{12}{5}$. Moreover, if

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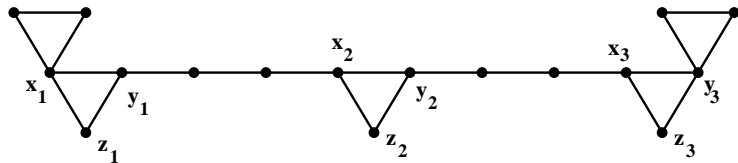
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Theorem 20 (O. B. and A. K.): If $k \geq 2$, then

$$F(0, k) = 3 - \frac{1}{k+1} = \frac{3k+2}{k+1}.$$

Moreover, if

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Question: What is the minimum g such that each planar graph of girth g is $(0, 1)$ -colorable?