

ORIENTATIONS

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CONTENTS

I. ORIENTATIONS AND SYMMETRIES

II. EXISTENCE AND UNAVOIDABILITY

III. COLORINGS OF ORIENTATIONS

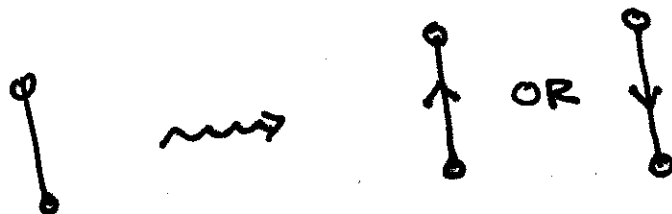
IV. HOMOMORPHISM DUALITIES

V. A PARTICULAR CASE

VI. A PASSION FOR DUALITIES

G UNDIRECTED GRAPH (V, E)

\vec{G} ORIENTATION OF G (V, \vec{E})



ANOTHER VIEW

$G = (V, E) \rightsquigarrow \vec{G} = (V, \vec{E})$

EXTENSION (LIFT)

BINARY EXTENSIONS ENCODE CLASS NP
 (N, G, KUN)

FORBIDDEN COLORED ORIENTATIONS CAPTURE NP PROBLEMS

THM (G. KUN, J.N.)

FOR EVERY NP PROBLEM P
THERE EXIST c AND A FINITE SET \mathcal{F}
OF c -EDGE COLORED ORIENTED GRAPHS
SUCH THAT THE FOLLOWING PROBLEM
IS POLYNOMIALLY EQUIVALENT TO P :

INSTANCE: A c -EDGE COLORED
UNDIRECTED GRAPH G

QUESTION: DOES THERE EXIST
AN ORIENTATION \vec{G}
SUCH THAT NO $\vec{F} \in \mathcal{F}$
IS A SUBGRAPH OF \vec{G} ?

$$\text{AUT}(G) \supseteq \text{AUT}(\vec{G})$$

FOR A COLORING (OF VERTICES) OF G



DENOTE $\text{AUT}_c(G)$ ALL THOSE
AUTOMORPHISMS OF G WHICH
PRESERVE c (AS PARTITION).
NOT LABELS

EXAMPLE

$$|\text{AUT}(\text{pentagon})| = 10$$

$$|\text{AUT}_c(\text{pentagon with coloring})| = 2$$

DENOTE BY $\text{AUT}_c(G)$ ALL THOSE
AUTOMORPHISMS OF G WHICH PRESERVE
ALL MINIMAL COLORINGS c OF G .

$$\text{AUT}_c(\text{pentagon}) = \{\text{id}\}$$

$\text{AUT}_C(G)$ IS NORMAL SUBGROUP OF
 $\text{AUT}(G)$

PROOF:

$g \in \text{AUT}_C(G)$, $f \in \text{AUT}(G)$

π A MINIMAL COLOURING (PARTITION)

PUT $\pi' = f^{-1}(\pi)$.

π' IS A MINIMAL COLOURING

g PRESERVES π' : $g(\pi') = \pi'$

AND THUS

$$f \circ g \circ f^{-1}(\pi) = \pi$$

$\text{AUT}(\vec{G})$

NEEDN'T BE A NORMAL
SUBGROUP OF
 $\text{AUT}(G)$.

SITUATION CLARIFIED IN 70ies

THM 1 (CHVÁTAL, SICHLER)

FOR EVERY GROUP Γ AND ITS NORMAL SUBGROUP Γ'

THERE EXISTS A GRAPH G SUCH THAT

$$\text{AUT}(G) \cong \Gamma \text{ AND } \text{AUT}_C(G) \cong \Gamma'.$$

THM 2 (N.)

FOR EVERY GROUP Γ AND ITS SUBGROUP Γ'

THERE EXIST A GRAPH G SUCH THAT

$$\text{AUT}(G) \cong \Gamma \text{ AND } \text{AUT}(\vec{G}) \cong \Gamma' ;$$

FOR SOME ORIENTATION.

(IN FACT WE CAN DEMAND:

$$\{ \text{AUT}(\vec{G}) \mid \vec{G} \text{ ORIENTATION} \} = \{ \Gamma' \mid \Gamma' \text{ SUBGROUP OF } \Gamma \} .)$$

FOR WHICH AUTOMORPHISM ~~ES~~ CLOSED COLOURINGS **THM 1** HOLDS ?

- EDGE COLOURINGS, $\leq k$ COLOURINGS,
- χ_p COLOURINGS, ACYCLIC, ...

$\text{AUT}(G)$ vs $\text{AUT}(\vec{G})$

INTERESTING

$G(\vec{G})$ IS CRITICALLY ASYMMETRIC

IF $\text{AUT}(G)$ (OR $\text{AUT}(\vec{G})$) = $\{\text{id}\}$

WHILE $\text{AUT}(G-x)$ (OR $\text{AUT}(\vec{G}-x)$) $\neq \{\text{id}\}$.

EXAMPLE



CONJ.

THERE IS NO CRITICALLY ASYMMETRIC
 \vec{G} .

TRUE FOR TREES

FOR ACYCLIC ORIENTATIONS
(WOJCYK)

II. EXISTENCE & UNAVOIDABILITY

ORDERING LEMMA (N., RÖDL)

FOR EVERY GRAPH G THERE EXISTS GRAPH H :
AND $\ell \geq 3$

① FOR EVERY ACYCLIC ORIENTATION \vec{G}
AND FOR EVERY ORIENTATION \vec{H}
THERE EXISTS IN \vec{H} AN INDUCED
SUBGRAPH \vec{G}' , $\vec{G}' \cong \vec{G}$.

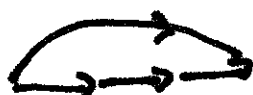
② EVERY CIRCUIT OF LENGTH $\leq \ell$
IN H BELONGS TO A COPY OF G .

NON TRIVIAL EVEN FOR $G = P_\ell$



GIVES NON-HASSE
DIAGRAMS OF HIGH
GIRTH

DECISION PROBLEM NPC
FOR VERY SIMPLE GRAPHS



\vec{H} ORDERING PROBLEM

INSTANCE: UNDIRECTED G

QUESTION: IS THERE A COPY OF \vec{H}
IN EVERY (ACYCLIC)
ORIENTATION \vec{G} OF G ?

"LEFT SIDE" PROBLEM



III COLORINGS OF ORIENTATIONS (VIA HOMOMORPHISMS)

$G \rightarrow H$ - EXISTENCE OF A
HOMOMORPHISM
- H -COLORING OF G

H -COLORING ALMOST ALWAYS NPC
($H \neq \bullet, !$)
(HELL, N.)

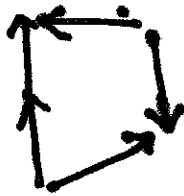
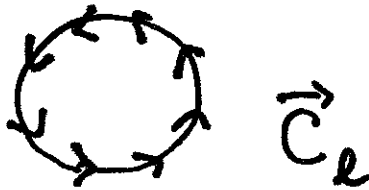
$\vec{G} \rightarrow \vec{H}$ - EXISTENCE OF A
HOMOMORPHISM PRESERVING
ARCS

\vec{H} -COLORING IN MANY INSTANCES \vec{H}
POLYNOMIAL

SOME POLYNOMIAL INSTANCES

$$\vec{H} = \text{---} \rightarrow \rightarrow \rightarrow \vec{P}_3, \vec{P}_k$$

ARBITRARY ORIENTED PATH
(HELL, ZHU)



CONS. \vec{H} -COLORING PROBLEM

ALWAYS EITHER P OR NPC

(DICHOTOMY CONS.)

CONS H-COLORING PROBLEM

EITHER P OR NPC FOR RELATIONAL
STRUCTURES

(DICHOTOMY CONS.)
FEDER-VARDI

(MANY POLYNOMIAL INSTANCES FROM)

IV. HOMOMORPHISM DUALITIES

A FINITE HOMOMORPHISM DUALITY IS FOLLOWING STATEMENT

FOR EVERY GRAPH G

$$F \dashrightarrow G \quad \text{IFF} \quad G \rightarrow D$$

$$\mathcal{F} = \{F_1, \dots, F_t\}$$

$$F_i \dashrightarrow G \quad \text{IFF} \quad G \rightarrow D$$

$i = 1, \dots, t$

$$D = D_F = D_{\mathcal{F}} \quad \text{DUAL OF } F, \mathcal{F}$$

FOR UNDIRECTED GRAPHS
NO FINITE DUALITIES

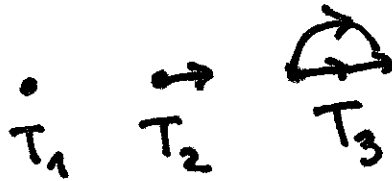
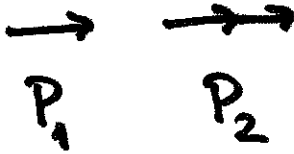
(EXCEPT: $F = ! \quad D_F = \bullet$)



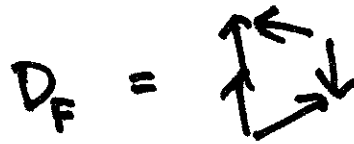
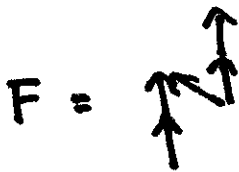
FOR ORIENTED GRAPHS MANY DUALITIES

$$F = \vec{P}_k$$

$$D_F = \vec{T}_k$$



GALLAI, HASSE, ROY, VITAVER



F ORIENTED
TREE

D_F EXISTS

\mathcal{F} FINITE SET
OF ORIENTED
TREES

$D_{\mathcal{F}}$ EXISTS

NO OTHER FINITE
DUALITIES
KOMA'REK ; N, TARDIF

OTHER POLYNOMIAL INSTANCES FROM RESTRICTED DUALITIES

\mathcal{C} A CLASS OF GRAPHS (PLANAR,
BOUNDED DEGREE)

\mathcal{C} -RESTRICTED FINITE DUALITY

FOR EVERY $G \in \mathcal{C}$

$F \rightarrow G$ IFF $G \rightarrow D$

$F \rightarrow D$

$\mathcal{F} = \{F_1, \dots, F_t\}$

$F_i \rightarrow G$ IFF $G \rightarrow D$

$i=1, \dots, t$ $F_i \rightarrow D$



FOR MANY CLASSES OF UNDIRECTED
GRAPHS MANY RESTRICTED DUALITIES

EXAMPLES

$$\mathcal{C} = \{G \mid \Delta(G) \leq d\}$$

$\mathcal{F} = F_1, \dots, F_t$ CONNECTED

$D_{\mathcal{F}}$ EXISTS

HÄGGKVIST, HELL

$$\mathcal{C} = \{G \mid G \text{ PLANAR}\}$$

\mathcal{F} FINITE SET

$D_{\mathcal{F}}$ EXISTS

(N., P. OSSONA DE MENDEZ)

ALL RESTRICTED DUALITIES

WHICH CLASSES \mathcal{C} HAVE ALL RESTRICTED DUALITIES?

CLASSES WITH BOUNDED EXPANSION HAVE A.R.D.

VIA ORIENTATIONS CHARACTERIZED BY THIS PROPERTY:

THM (N., P. OSSONA DE MENDEZ)
2010

Let \mathcal{C} be a class of undirected graphs.

$\vec{\mathcal{C}}$ class of all orientations,

$\vec{\mathcal{C}}^a$ class of all acyclic orientations
of graphs in \mathcal{C} .



① \mathcal{C} has bounded expansion;

② $\vec{\mathcal{C}}$ has all $\vec{\mathcal{C}}$ -restricted dualities;

③ for every p there exist $D_p \in \vec{\mathcal{C}}$
without (directed) circuits $\leq p$

such that $G \rightarrow D_p$

for every $G \in \vec{\mathcal{C}}^a$

(i.e. \vec{C}_p has $\vec{\mathcal{C}}^a$ -restricted dual).

ALL R.D. BY MEANS OF A **SPECIAL**
CASE.

POWER OF ORIENTATIONS

V A PARTICULAR CASE:

DUALITIES FOR SIMPLE, EVEN FINITE,
~~CLASSES~~ SETS

\mathcal{C} A CLASS WITH ALL RESTRICTED DUALITIES
 FIX F AND DENOTE

$$\boxed{m(F, \mathcal{C})} = \min \{ |D| : D \text{ DUAL FOR } F \text{ RESTRICTED TO } \mathcal{C} \}$$

REMARK

IF \mathcal{C} IS FINITE, THEN \mathcal{C} HAS A.R.D. :

FOR F WE PUT

$$D_F = \sum \{ G : G \in \mathcal{C}, F \rightarrow G \} .$$

THIS MOTIVATES:

$$\boxed{h(F, \mathcal{C})} = \min \{ \text{max SIZE OF A COMPONENT OF } D_F \}$$

$m(F, \mathcal{C})$ INTENSIVELY STUDIED

— UNIVERSAL TOURNAMENTS
($F = \phi$, \mathcal{C} TOURNAMENTS)

— MINIMAL SIZE OF DUALS

(F, \mathcal{C} ORIENTED GRAPHS)

$$2^{|F|/\log|F|} \leq m(F, \mathcal{C}) \leq 2^{|F|\log|F|}$$

(N., TARDIF)

ALMOST ALL ORIENTED PATHS
HAVE EXPONENTIAL DUAL

(KUN, TARDIF)

A PARTICULAR CASE:

$$\vec{\mathcal{C}} = \{ \vec{G} : G \text{ PLANAR} \}$$

$$F = \text{Diagram of a cycle with two vertices and two directed edges forming a loop}$$

THEN

THM (RASPAUD, SOPENA)

$$m(F, \vec{\mathcal{C}}) \leq 80, n(F, \vec{\mathcal{C}}) \leq 80$$

FORMULATED IN LOCAL FORM:

A VERY PARTICULAR CASE:

$$\vec{\mathcal{C}} = \{ \text{ALL ORIENTATIONS OF } G \}$$

$$F = \text{Diagram of a cycle with two vertices and two directed edges forming a loop}, G \text{ CONNECTED}$$

$$n(F, \vec{\mathcal{C}}) = \chi_0(G) = \vec{\chi}(G)$$

ORIENTED CHROMATIC NUMBER

$$m(F, \vec{\mathcal{C}}) = \chi_0^+(G) = \text{UPPER ORIENTED CHROMATIC NUMBER}$$

ALTERNATIVE DEFINITION

$$\chi_0(\vec{G}) = \min \{ |\vec{D}| : \vec{G} \rightarrow \vec{D} \}$$

$$\chi_0(G) = \max_{\vec{G}} \chi_0(\vec{G})$$

$$\chi_0^+(G) = \min \{ |\vec{D}| : \vec{G} \rightarrow \vec{D} \text{ FOR EVERY ORIENTATION} \}$$

MANY PARTICULAR CASES SOLVED

FOR $\mathcal{F} = \{ \mathcal{Q}, \dots \}$

STRONGER NOTION (SOPENA, VIGNAL)

PROBLEM

$$17 \leq \chi_0(\text{PLANAR}) \leq 80$$

VI PASSION FOR DUALITIES

COLORING \longleftrightarrow NOWHERE ZERO FLOW

ORIENTED COLORING \longleftrightarrow ANTISYMMETRIC FLOW
(N., RASPAUD)



Γ Abelian group

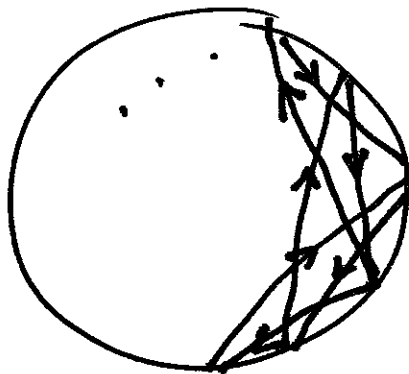
$$c: V(\vec{G}) \rightarrow \Gamma$$

Γ -STRONG ORIENTED COLORING

IF FOR EVERY PAIR (x, y) AND $(x', y') \in E(\vec{G})$

$$c(y) - c(x) \neq c(y') - c(x')$$

(i.e. c IS A HOMOMORPHISM INTO
ORIENTED CAYLEY GRAPH)



Γ -ANTISYMMETRIC FLOW ON \vec{G}

IS A FLOW $f: E(\vec{G}) \rightarrow \Gamma$

SATISFYING $f(e) \neq -f(e')$

FOR ANY TWO ARCS $e, e' \in E(\vec{G})$

(THUS IT IS NZF).

Γ -STRONG
ORIENTED
COLORING

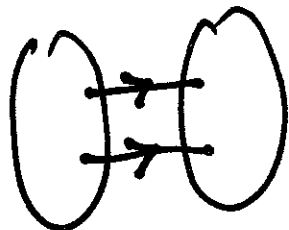


Γ -ANTISYMMETRIC
FLOW

NZF EXISTS FOR GRAPHS WITHOUT
A BRIDGE
(AND ANY ORIENTATION)

ANTISYMMETRIC EXIST FOR GRAPHS WITHOUT
A BRIDGE

AND ORIENTED 2-CUT



DOES EVERY ORIENTATION OF A
3-CONNECTED GRAPH HAVE
BOUNDED Γ -ANTIFLOW
(OR INTEGER ANTIFLOW)
OF BOUNDED SIZE

?

NO 5-FLOW CONJECTURE
LOWER BOUND 17 BY DUALITY.

THM

THERE EXISTS Γ SUCH THAT EVERY
 ORIENTED GRAPH \vec{G} WITHOUT DIRECTED
 EDGE CUT OF SIZE 2
 HAS AN ANTI SYMMETRIC Γ -FLOW.

DE VOS, JOHNSON, SEYMOUR

$$\Gamma = \mathbb{Z}_2^8 \cdot \mathbb{Z}_3^{17}$$

DE VOS, N., RASPAUD

$$\Gamma = \mathbb{Z}_2^6 \cdot \mathbb{Z}_3^9$$

DVOŘÁK, KAISER, KRÁL, SEBENI

$$\Gamma = \mathbb{Z}_2^3 \cdot \mathbb{Z}_3^9$$

LOWER BOUND 17

A STRONGER STATEMENT:

THERE EXISTS $B \subseteq \Gamma$, $B \cap -B = \emptyset$,

SUCH THAT EVERY ORIENTED GRAPH WITHOUT OR. 2-CUT HAS B -FLOW.

DEF

B IS k -FREE IF ALL PARTIAL SUMS

$\sum_{i=1}^{k'} x_i$ ARE NON-ZERO, $k' \leq k$.

1 - FREE

$$x_i \neq 0$$

2 - FREE

$$x_i + x_j \neq 0$$

3 - FREE

$$x_i + x_j + x_k \neq 0$$

PROBLEM

IS IT TRUE THAT FOR EVERY k
THERE EXISTS $n(k)$ AND $B \subseteq \Gamma$

SUCH THAT :

1. B IS k -FREE
2. EVERY $n(k)$ CONNECTED
GRAPH \vec{G} HAS A B -FLOW.

IN PRIMAL (PLANAR) CASE TRUE

(ŠÁMAL, P. OSSONA DE MENDEZ, N.)

THANK YOU FOR YOUR ATTENTION

