

Chromatic Ramsey Number and Fractional Hedetniemi's Conjecture

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Among any 6 people, there are either 3 people, any two of them know each other, or 3 people, any two of them do not know each other.

For any 2-colouring of the edges of K_6 with colours red and blue, there is either a red copy of K_3 or a blue copy of K_3

$$K_6 \rightarrow (K_3, K_3)$$

$F \rightarrow (G, H)$ means the following

For any 2-colouring of the edges of F with colours red and blue, There is a red copy of G or a blue copy of H .

Theorem [Ramsey] For any graphs G and H , there exists an F such that $F \rightarrow (G, H)$.

Infinitely many graphs F

$F = K_n$ for some n

The Ramsey number of (G, H) is
 $R(G, H) = \min\{n : K_n \rightarrow (G, H)\}$

$$R(G) = R(G, G)$$

$$R(K_k) = R(k)$$

$$R(K_k, K_l) = R(k, l)$$

Bounds for $R(k,l)$

$k \backslash l$	3	4	5	6	7	8
3	6	9	14	18	23	28
4		18	25	35 41	49 61	56 84
5			43 49	58 87	80 143	101 216
6				102 165	113 298	169 780
7					205 540	237 1713
8						317 3583

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$$R(G, H) = \min\{ |V(F)| : F \rightarrow (G, H) \}$$

The Size Ramsey number of (G, H) is

$$R_E(G, H) = \min\{ |E(F)| : F \rightarrow (G, H) \}$$

The max - degree - Ramsey number of (G, H) is

$$R_\Delta(G, H) = \min\{ |\Delta(F)| : F \rightarrow (G, H) \}$$

The chromatic Ramsey number of (G, H) is

$$R_\chi(G, H) = \min\{ |\chi(F)| : F \rightarrow (G, H) \}$$

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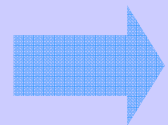
$$R_\chi(G) = R_\chi(G, G)$$

Introduced by Burr-Erdos-Lovasz in 1976

$$R_\chi(G, H) \leq R(G, H)$$

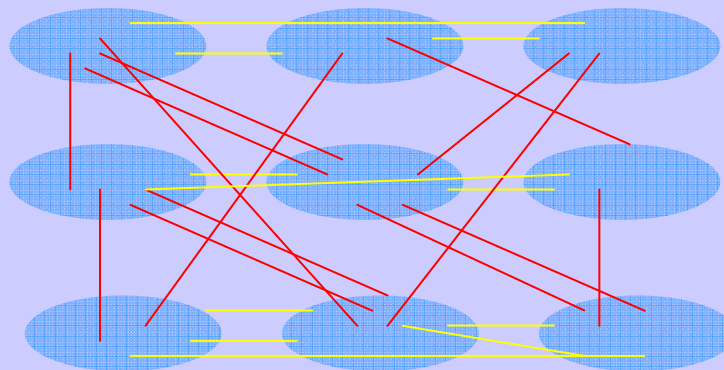
$$R_\chi(K_k, K_l) = R(k, l)$$

If F has chromatic number $(n-1)^2$, then there is a 2 edge colouring of F in which each monochromatic subgraph has chromatic number $n-1$.

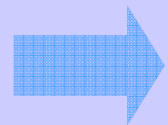


$F \rightarrow (G, G)$ for any n -chromatic G .

$n = 4$



If F has chromatic number $(n-1)^2$, then there is a 2 edge colouring of F in which each monochromatic subgraph has chromatic number $n-1$.



$F \not\Rightarrow (G, G)$ for any n -chromatic G .

Could be much larger

Observation: If $\chi(G) = n$, then $R_\chi(G) \geq (n-1)^2 + 1$

Conjecture [Burr - Erdos - Lovasz, 1976]: For each n , there is a graph G with $\chi(G) = n$ and $R_\chi(G) = (n-1)^2 + 1$

The conjecture is true for $n=3,4$ (Burr-Erdos-Lovasz, 1976)

The conjecture is true for $n=5$ (Z, 1992)

The conjecture is true (Z, 2010)

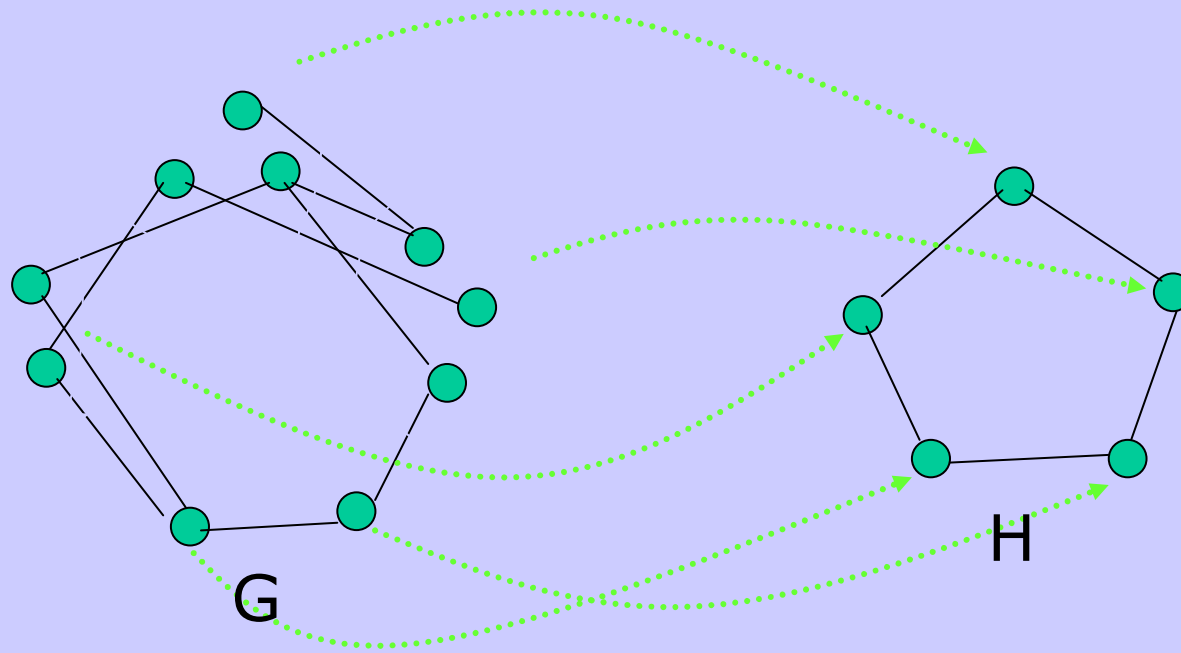
Conjecture [Burr - Erdos - Lovasz, 1976]: For each n ,
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Lemma [Burr - Erdos - Lovasz]

$$R_{\chi}(G) \leq n \Leftrightarrow K_n \rightarrow \text{hom}(G)$$

For any 2 edge-colouring of K_n , there is a monochromatic graph which is a homomorphic image of G .

Graph homomorphism = edge preserving map



Assume $K_n \rightarrow \text{hom}(G)$

Take a huge complete n-partite graph F

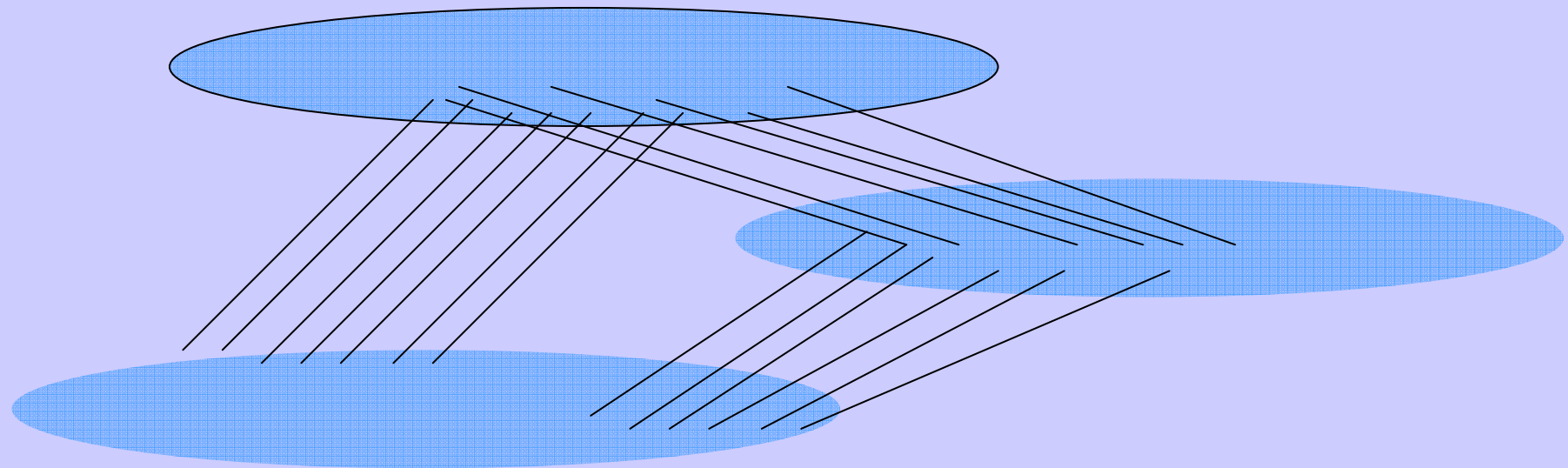
Lemma [Burr - Erdos - Lovasz]

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For any 2 edge colouring of F

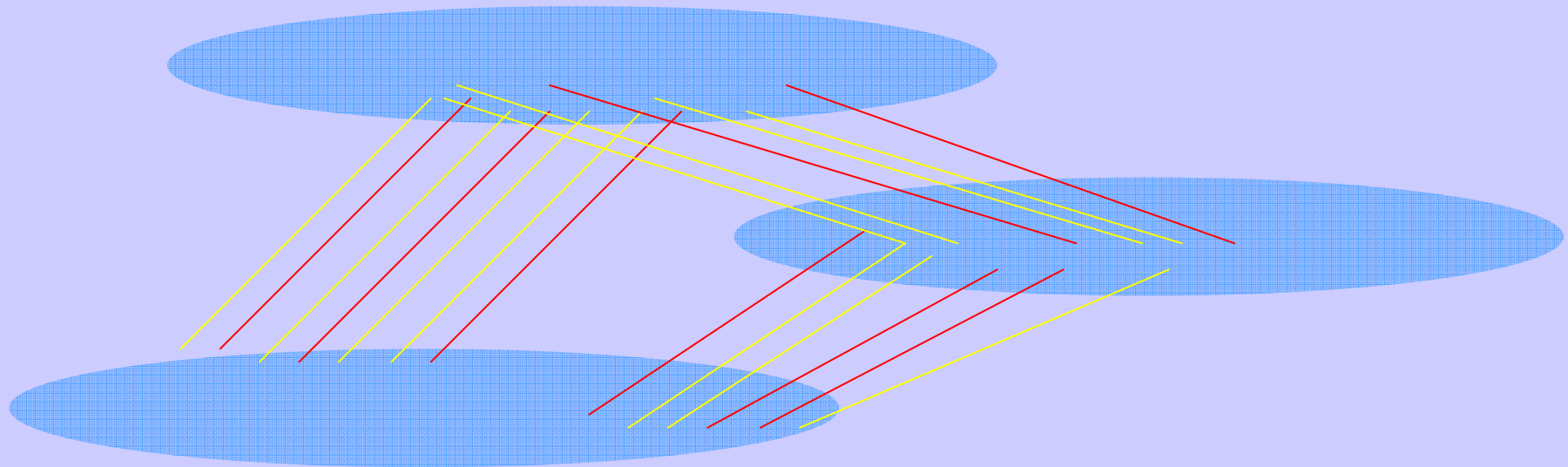


Assume $K_n \rightarrow \text{hom}(G)$

Take a huge complete n-partite graph F

For any 2 edge colouring of F

There is a large complete n-partite graph F' ,
for any two parts A, B , all the edges in $E[A, B]$ have the same colour

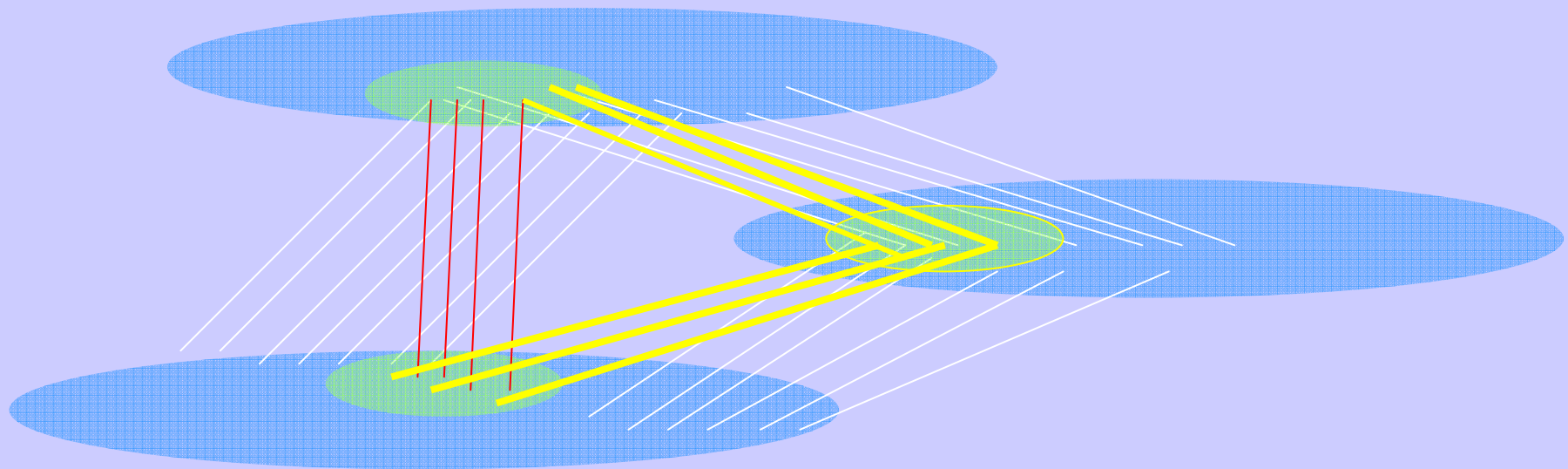


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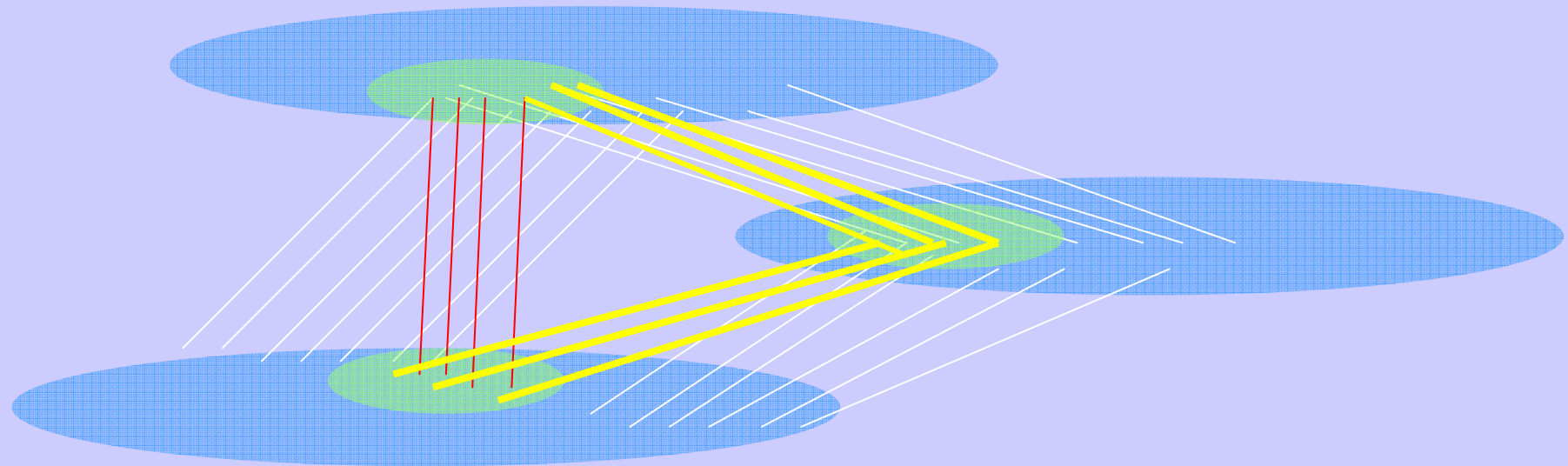
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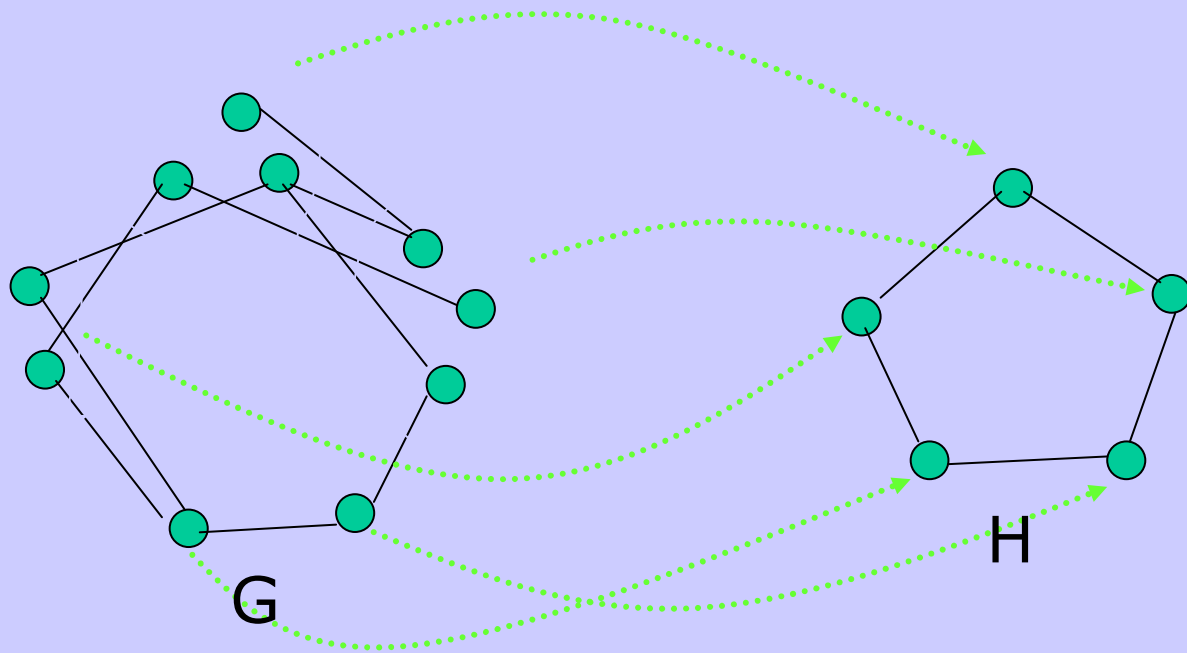
For any 2 edge colouring of F

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This defines a 2 edge colouring of K_n

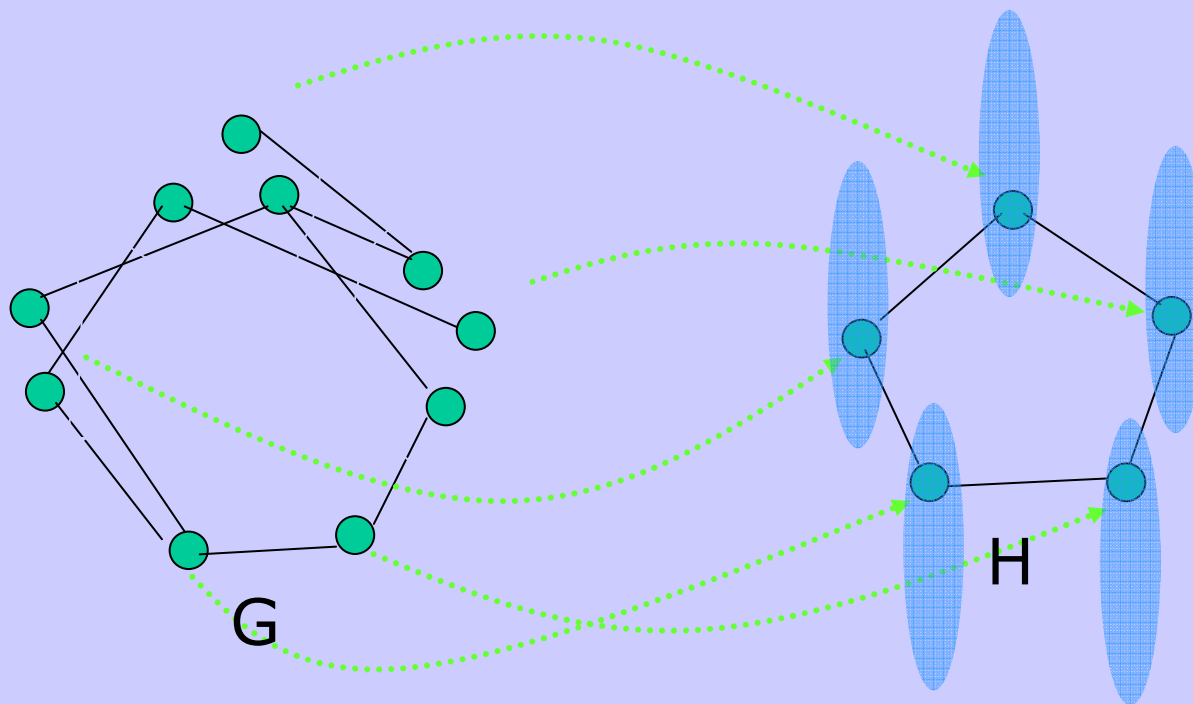


There is a monochromatic graph H in K_n ,
which is a homomorphic image of G



There is a monochromatic graph H in K_n ,
which is a homomorphic image of G

Pull H back to the complete multipartite graph to find a monochromatic copy of G



Lemma [Burr - Erdos - Lovasz]

$$R_\chi(G) \leq n \iff K_n \rightarrow \text{hom}(G)$$

To prove Burr-Erdos-Lovasz conjecture for n , we need to construct an n -chromatic graph G , so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of G .

The construction of G is easy:

Take all 2 edge colourings of $K_{(n-1)^2+1}$

c_1, c_2, \dots, c_m

For each 2 edge colouring c_i of $K_{(n-1)^2+1}$, one of the monochromatic subgraph, say G_i , has chromatic number at least n .

Conjecture [Burr - Erdos - Lovasz, 1976]: For each n , there is a graph G with $\chi(G) = n$ and $R_\chi(G) = (n-1)^2 + 1$

To prove this conjecture for n , we need to construct an n -chromatic graph G , so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of G .

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$$G = G_1 \times G_2 \times \dots \times G_m$$

$G \times H$

— H

|
G

×
G × H

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Each G_i is a homomorphic image of G



Conjecture [Hedetniemi, 1966]:

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

To prove this conjecture for n , we need to construct an n -chromatic graph G , so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of G .



If Hedetniemi's conjecture is true, then

Burr-Erdos-Lovasz conjecture is true.

Conjecture [Hedetniemi, 1966]:

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

Fractional Hedetniemi's conjecture

Conjecture [Z, 2002]:

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$$

To prove this conjecture for n , we need to construct an n -chromatic graph G , so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of G .

If Hedetniemi's conjecture is true, then

Burr-Erdos-Lovasz conjecture is true.

Observation [Claude Tardif] If fractional Hedetniemi's conjecture is true, then Burr-Erdos-Lovasz conjecture is true.

To prove this conjecture for n , we need to construct an n -chromatic graph G , so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of G .



The construction of G is easy:

Take all 2 edge colourings of $K_{(n-1)^2+1}$

$$c_1, c_2, \dots, c_m$$

For each 2 edge colouring c_i of $K_{(n-1)^2+1}$, one of the monochromatic subgraph, say G_i , has fractional chromatic number $> n-1$

$$G = G_1 \times G_2 \times \dots \times G_m$$

$$\chi_f(G) > n-1$$

$$\chi(G) \geq \chi_f(G) > n-1$$

Each G_i is a homomorphic image of G

A fractional colouring of G is a mapping f which assigns to each independent set U of G a nonnegative weight $f(U)$ so that for any vertex v of G , $\sum_{v \in U} f(U) \geq 1$.

The minimum total weight of a fractional colouring of G is the fractional chromatic number of G , and is denoted by

$$\chi_f(G)$$

Conjecture [Hedetniemi, 1966]:

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

Fractional Hedetniemi's conjecture

Theorem [Z, 2010] :

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$$

A fractional clique of G is a mapping f which assigns to each vertex v a nonnegative weight $f(v)$ so that for any independent set U of G , $f(U) = \sum_{v \in U} f(v) \leq 1$.

The maximum total weight of a fractional clique of G is the fractional clique number of G , and is denoted by

$$\omega_f(G)$$

The fractional chromatic number of G is obtained by solving a linear programming problem

The fractional clique number of G is obtained by solving its dual problem

$$\chi_f(G) = \omega_f(G)$$

Fractional Hedetniemi's conjecture is true

Theorem [Z, 2010] :

$$\omega_f(G \times H) = \min\{\omega_f(G), \omega_f(H)\}$$

Proof sketch :

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$$

$$\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\} \rightarrow \text{Easy!}$$

$$\chi_f(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\} \rightarrow \text{Difficult!}$$

$$\omega_f(G \times H) \geq \min\{\omega_f(G), \omega_f(H)\}$$

suffices to construct a fractional clique of $G \times H$
with total weight $\min\{\omega_f(G), \omega_f(H)\}$

$g : V(G) \rightarrow [0,1]$, a maximum fractional clique of G

$h : V(H) \rightarrow [0,1]$, a maximum fractional clique of H

$\varphi : V(G \times H) \rightarrow [0,1]$, defined as

$$\varphi(x,y) = \frac{g(x)h(y)}{\max\{\omega_f(G), \omega_f(H)\}}$$

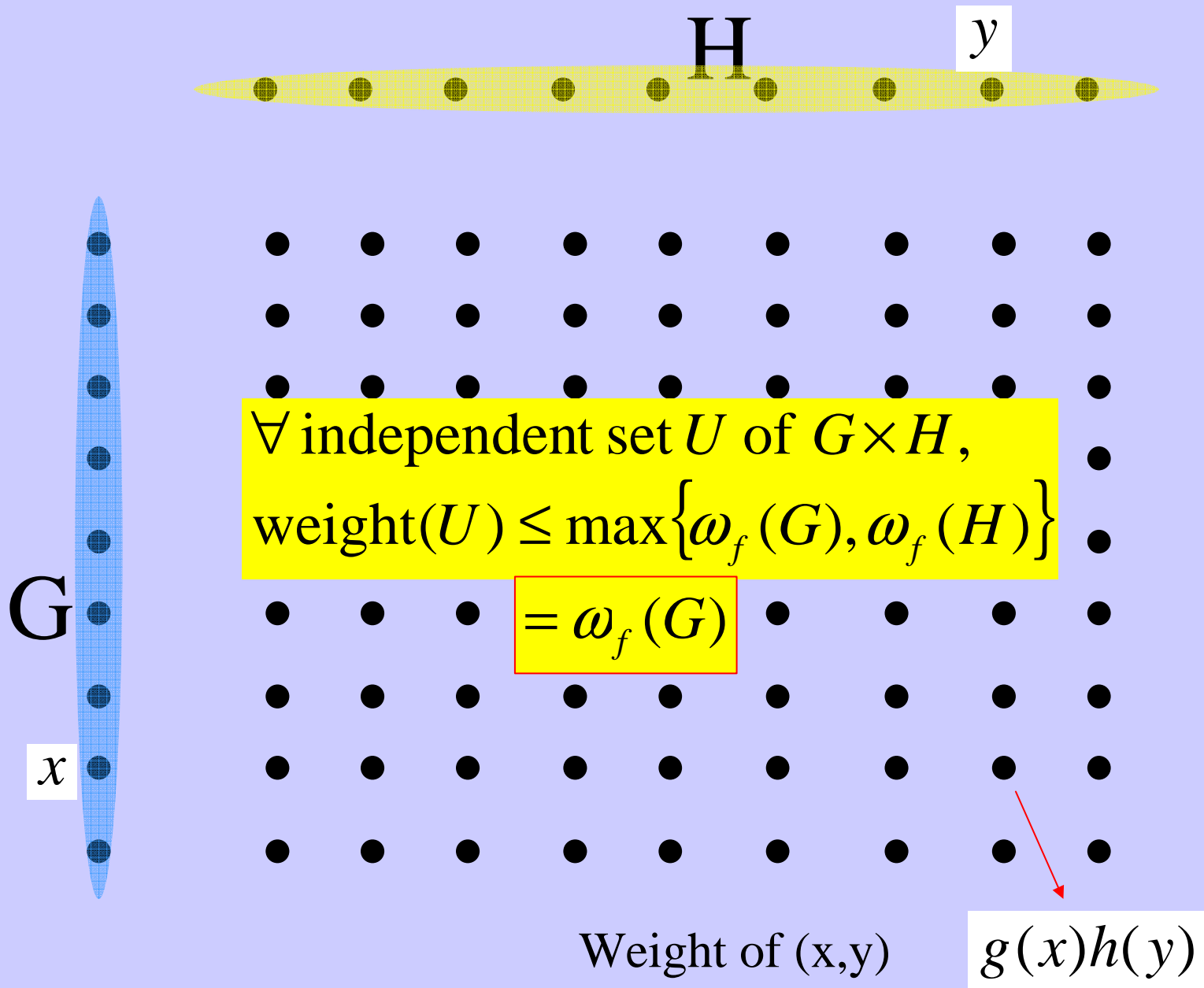
is a fractional clique of $G \times H$ Difficult!

with total weight $\min\{\omega_f(G), \omega_f(H)\}$ Easy!

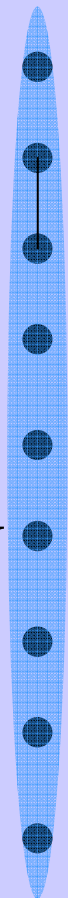
\forall independent set U of $G \times H$,

$$\sum_{(x,y) \in U} g(x)h(y) \leq \max\{\omega_f(G), \omega_f(H)\}$$

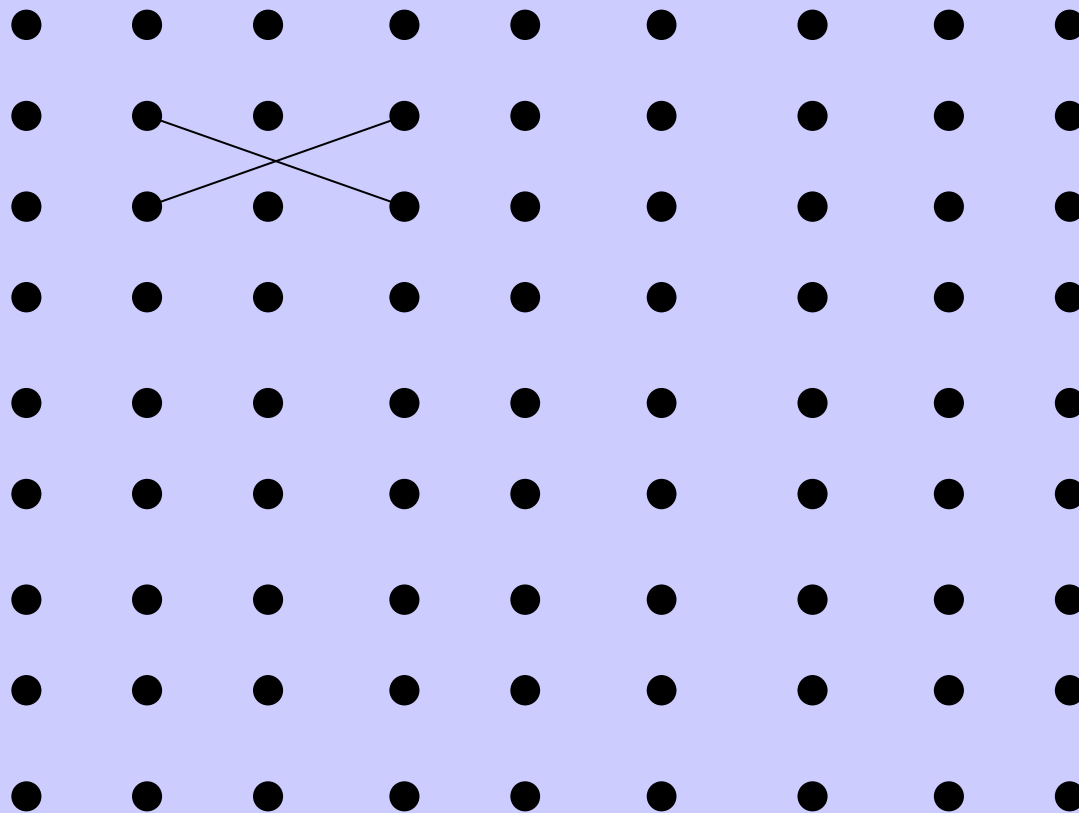
Assume $\omega_f(G) \geq \omega_f(H)$



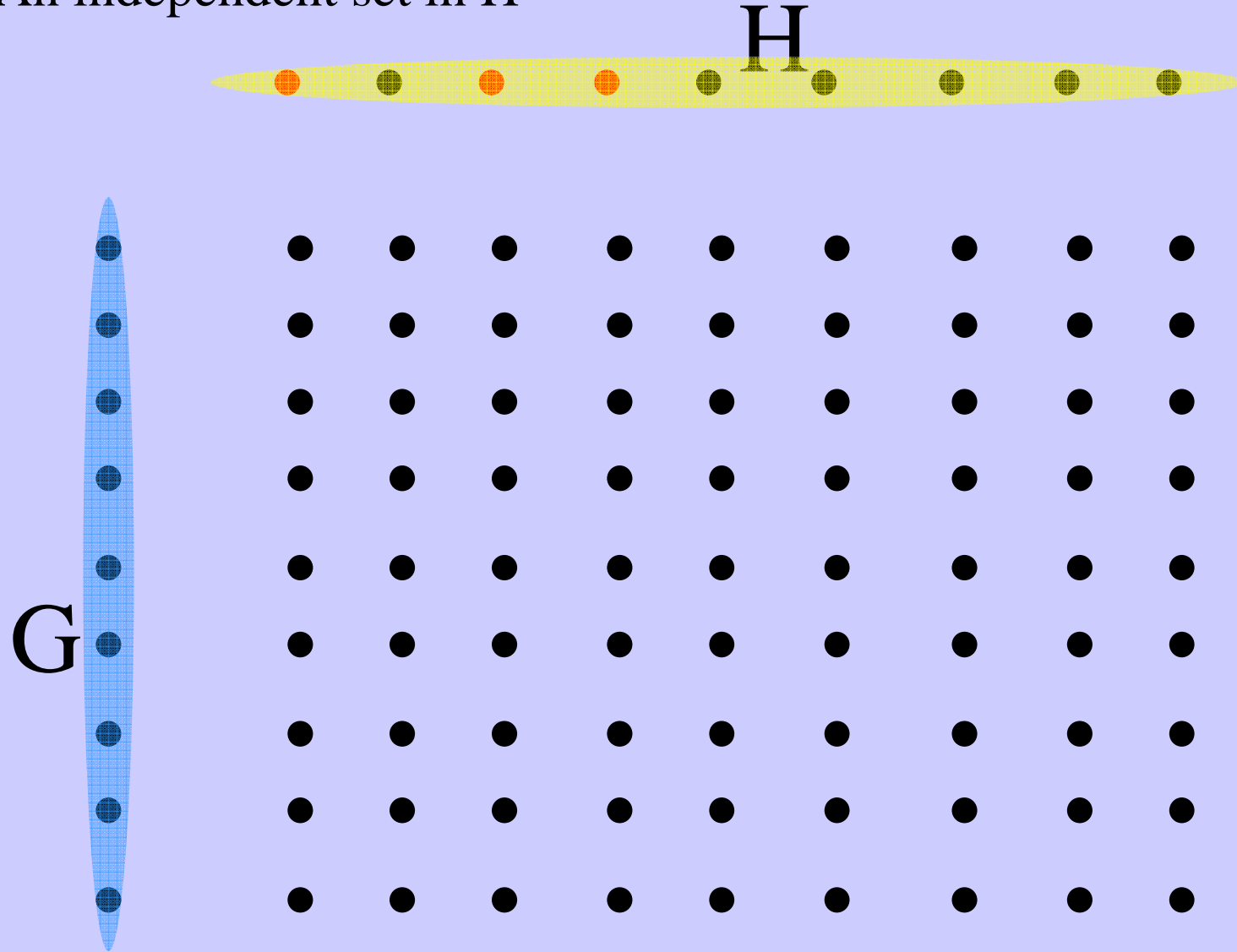
G

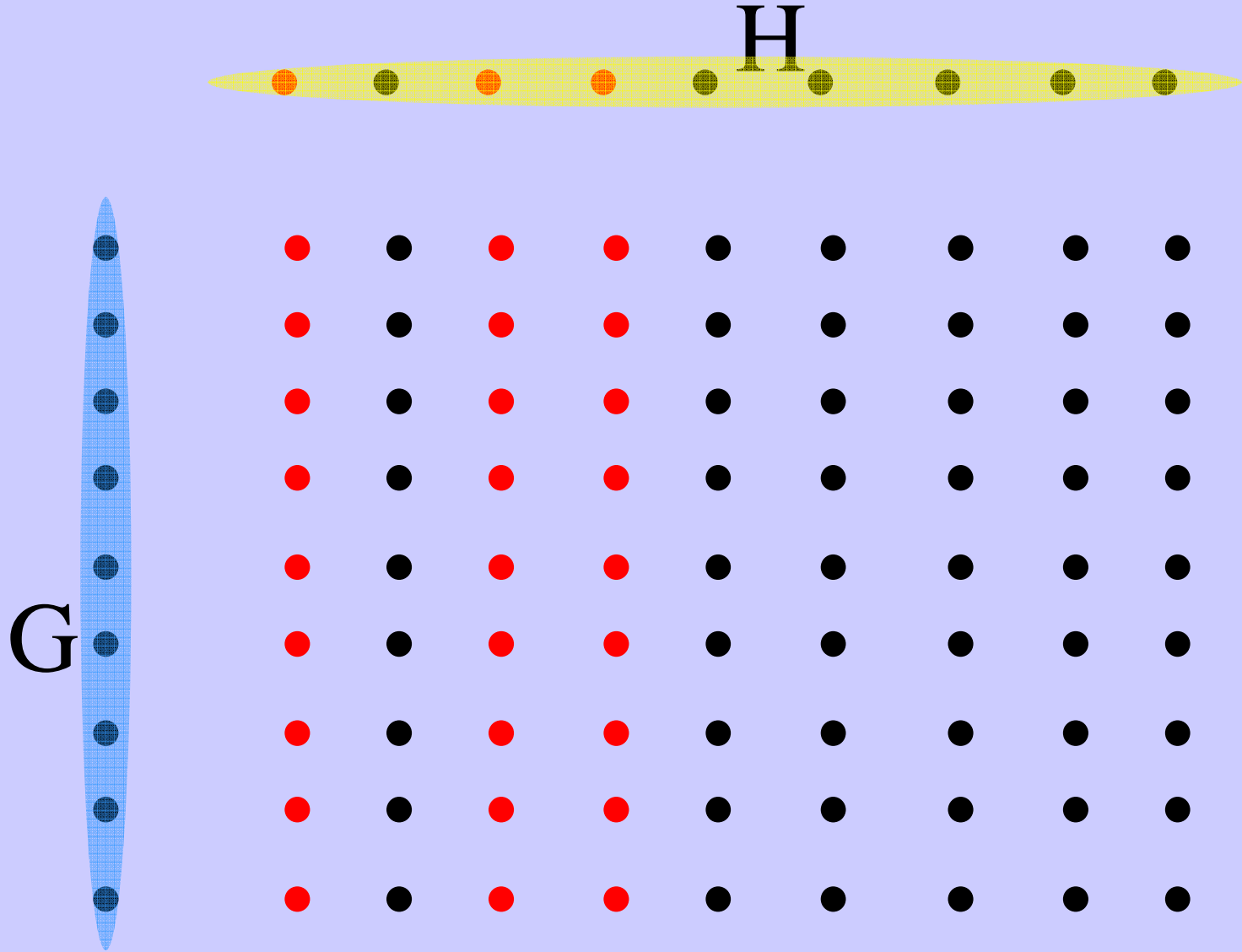


H



An independent set in H





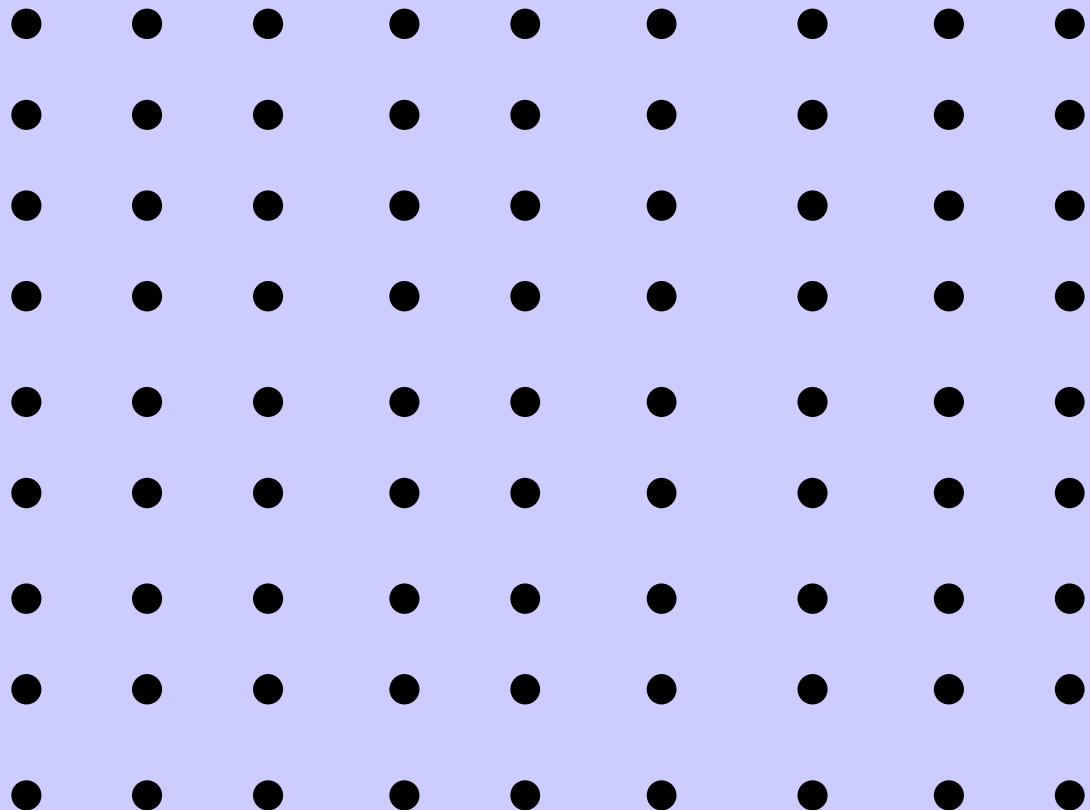
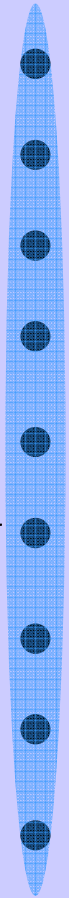
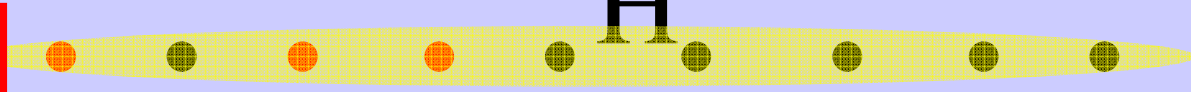
An independent set U in $G \times H$

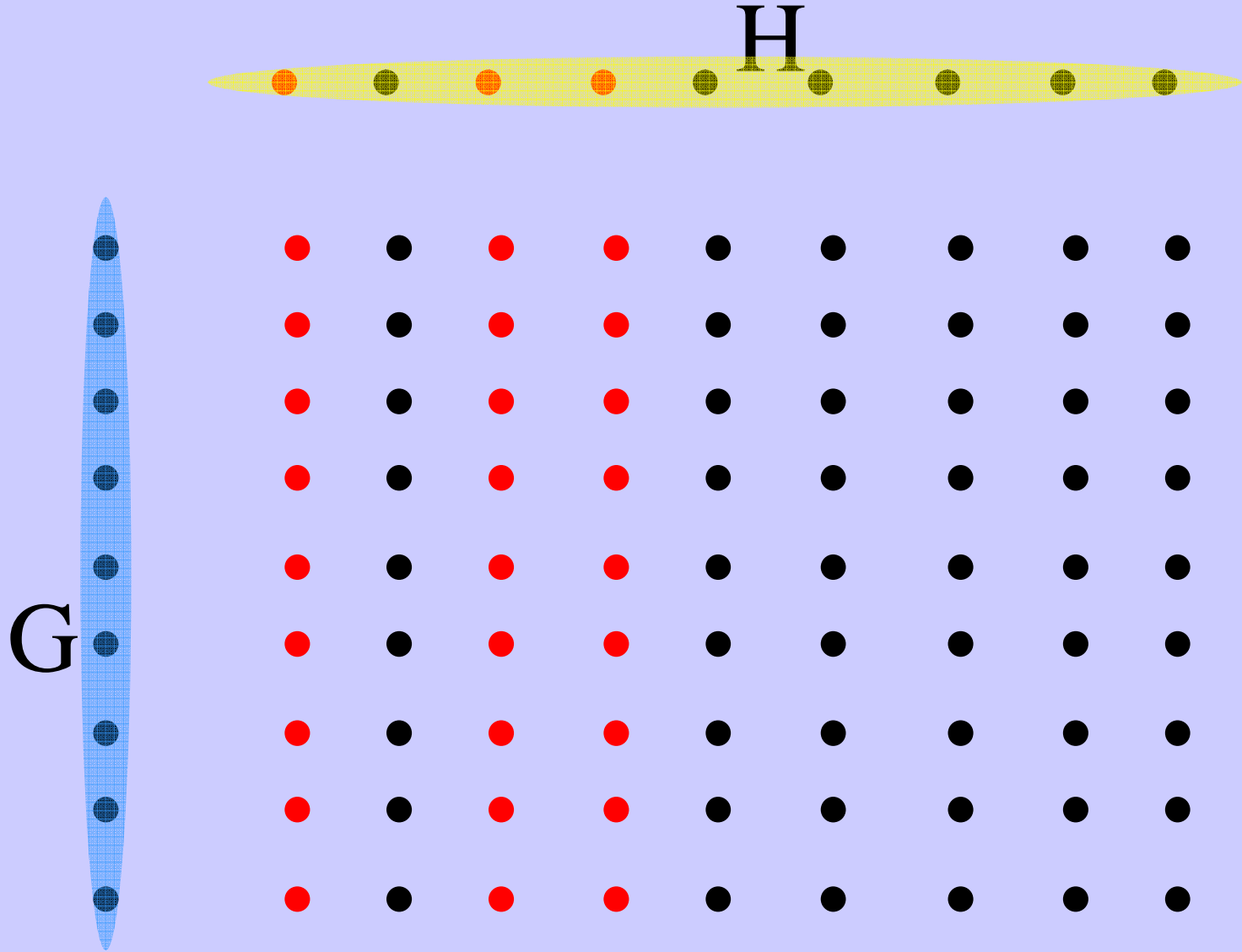
An independent set in H

of weight α

H

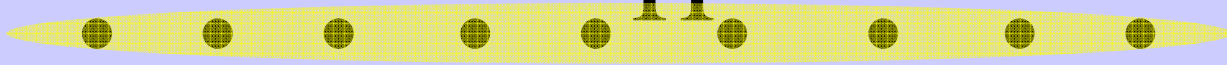
G





An independent set U in $G \times H$ of weight $\alpha \cdot \omega_f(G)$

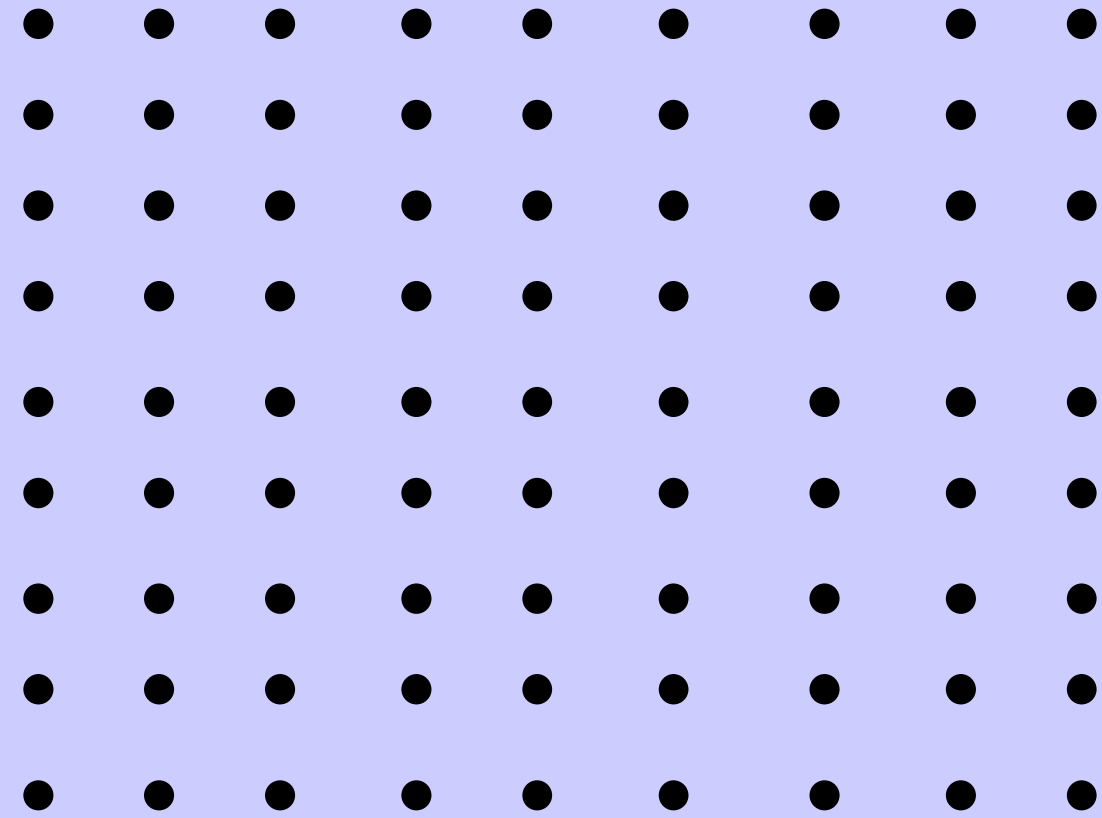
H

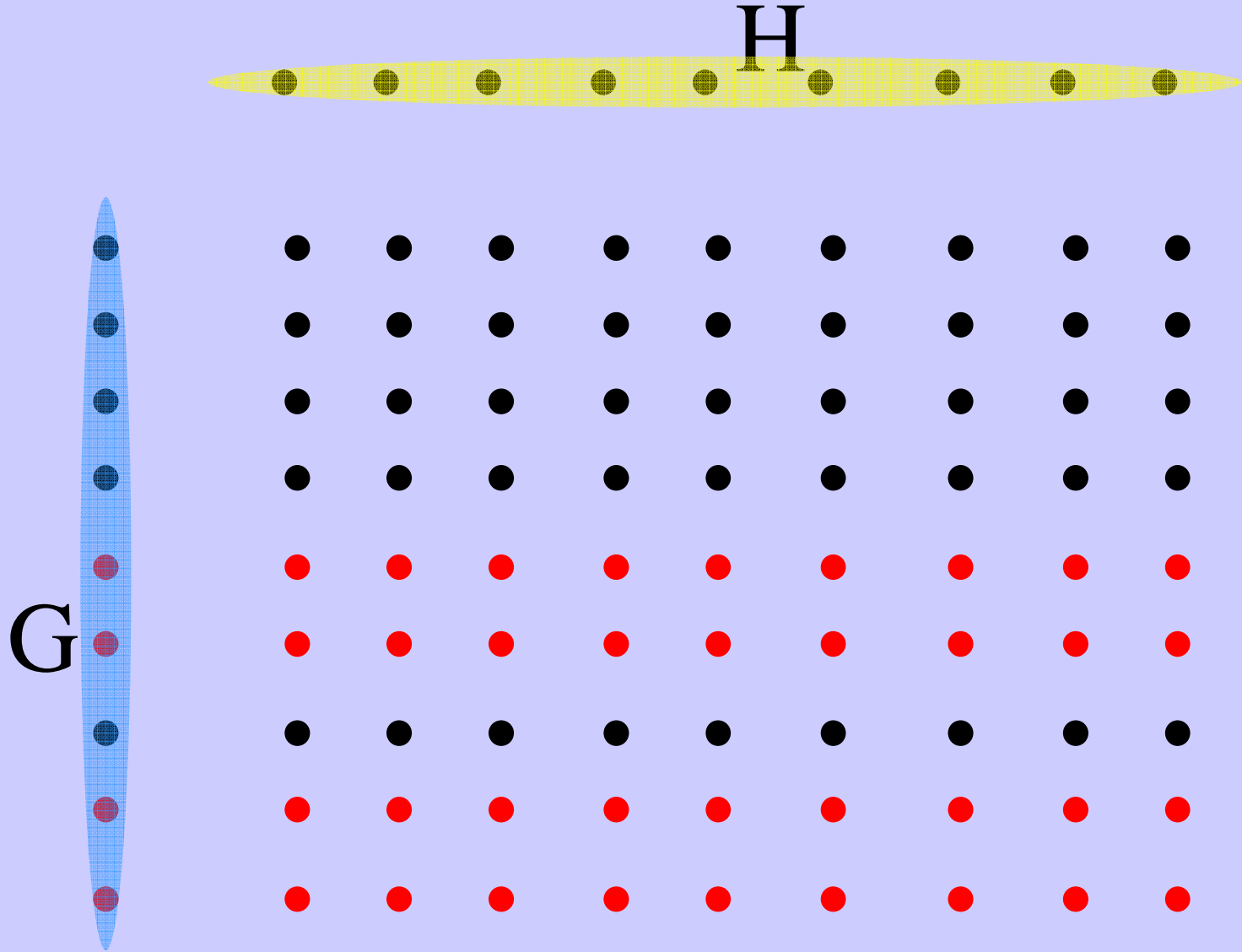


An independent
set in G

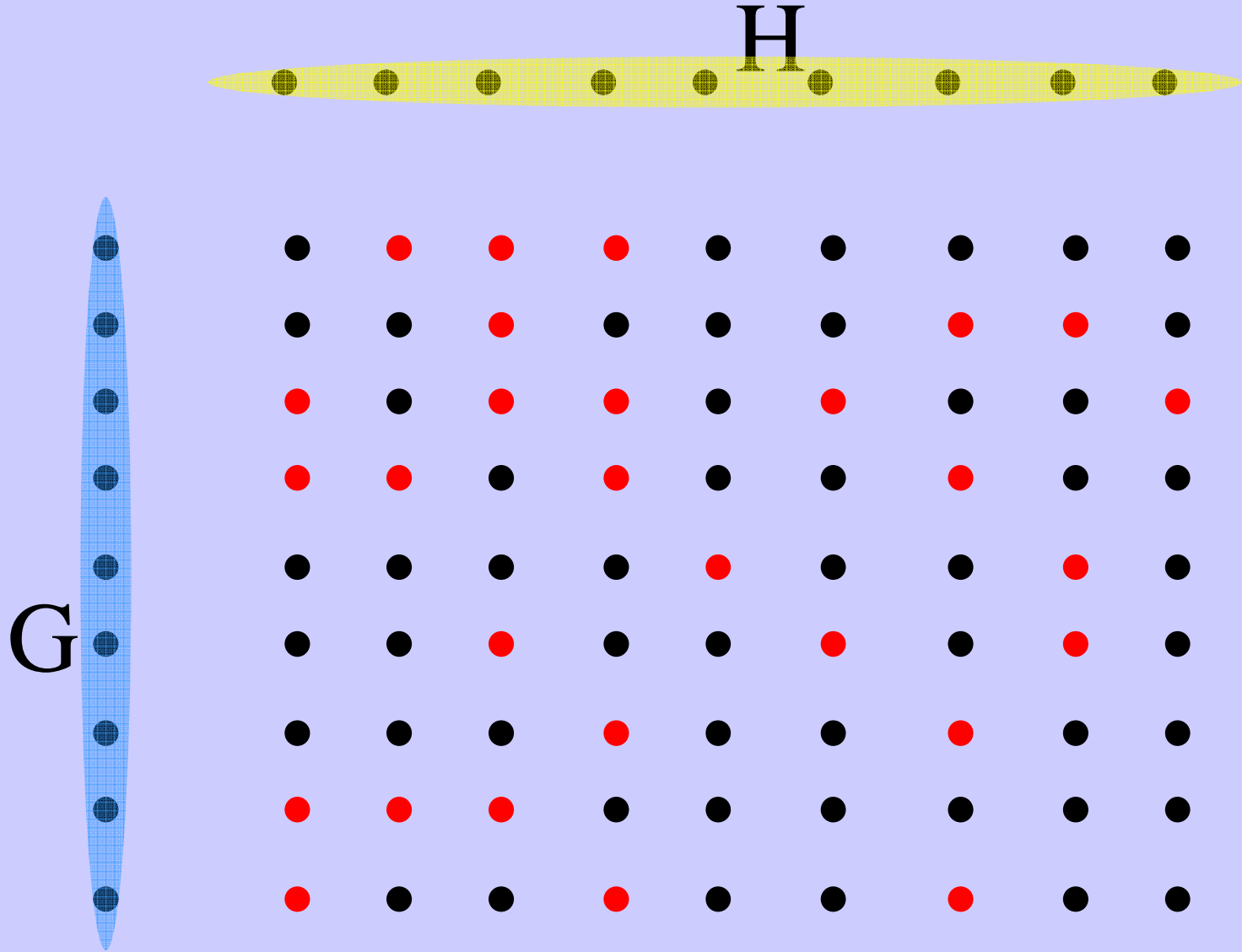
of weight β

G

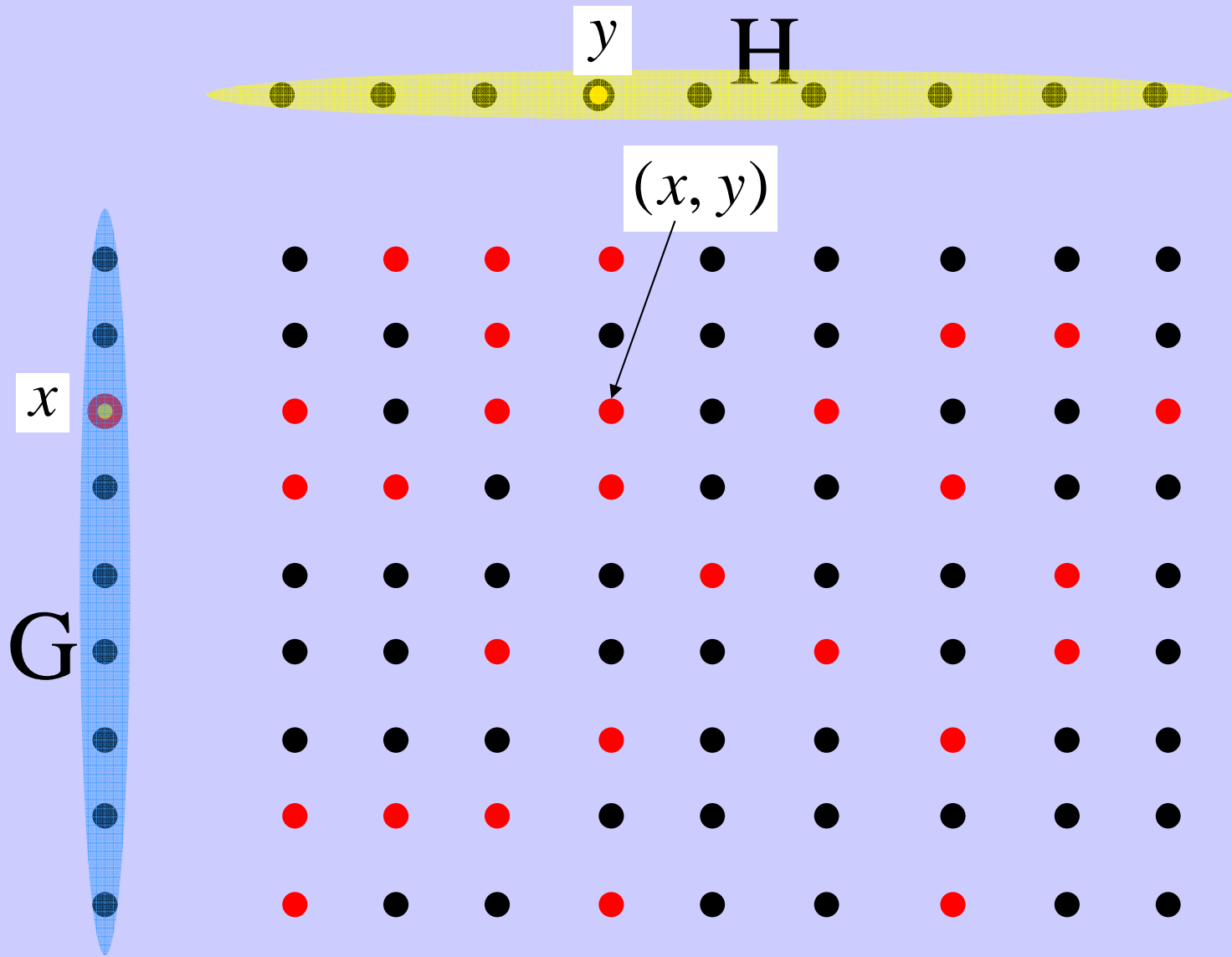




An independent set U in $G \times H$ of weight $\beta \cdot \omega_f(H)$

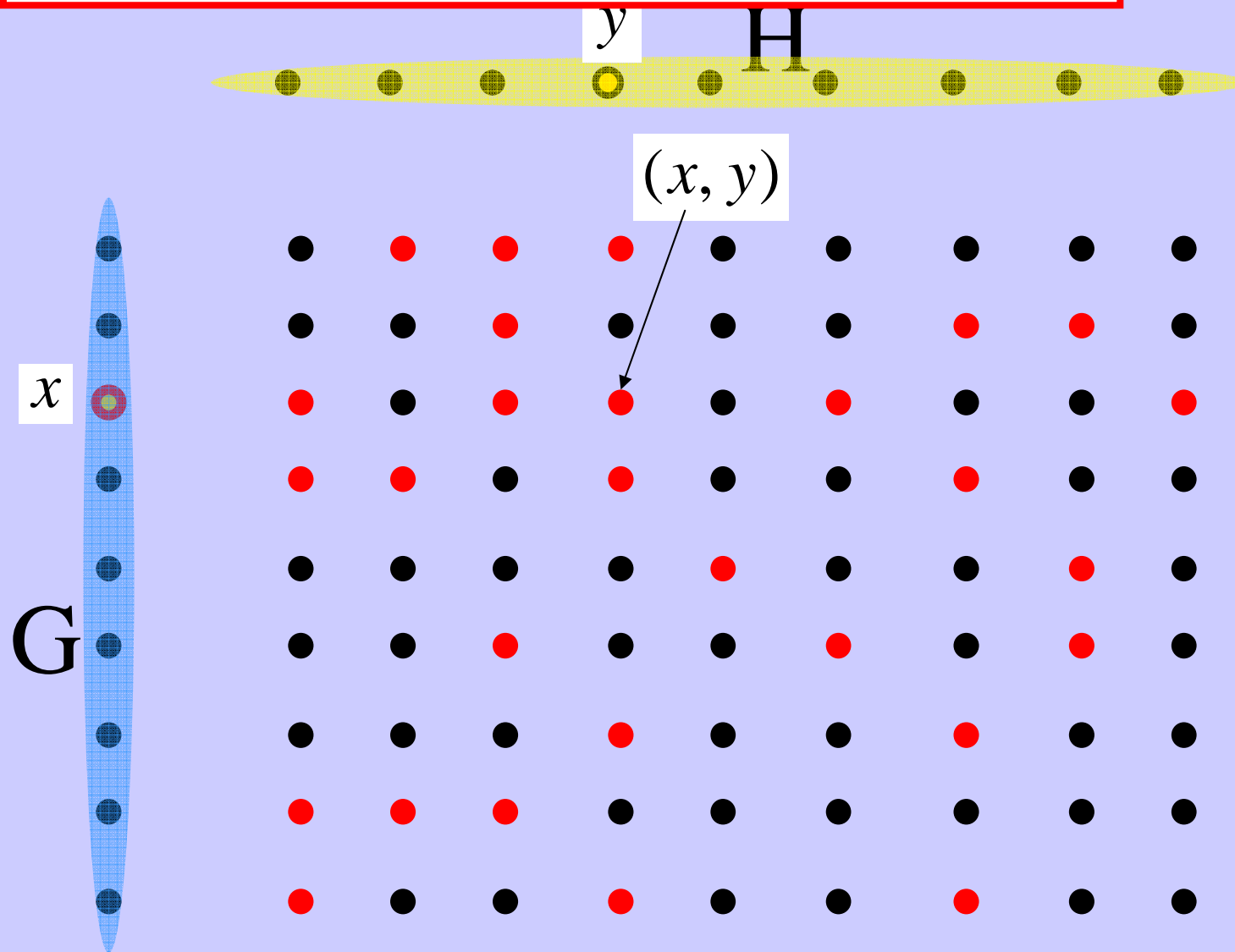


An independent set U in $G \times H$



An independent set U in $G \times H$

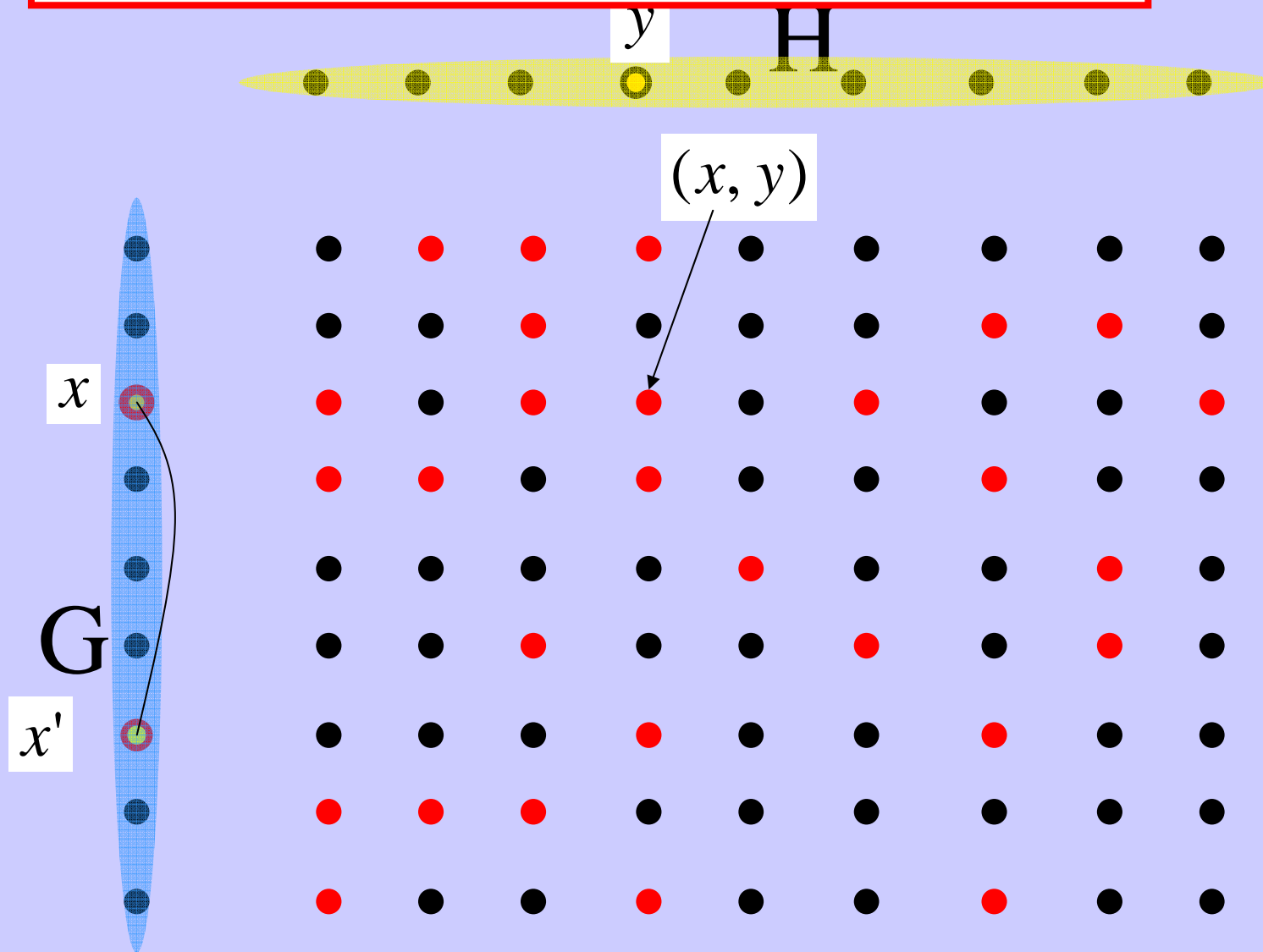
$$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.$$



An independent set U in $G \times H$

$$U = A \cup B$$

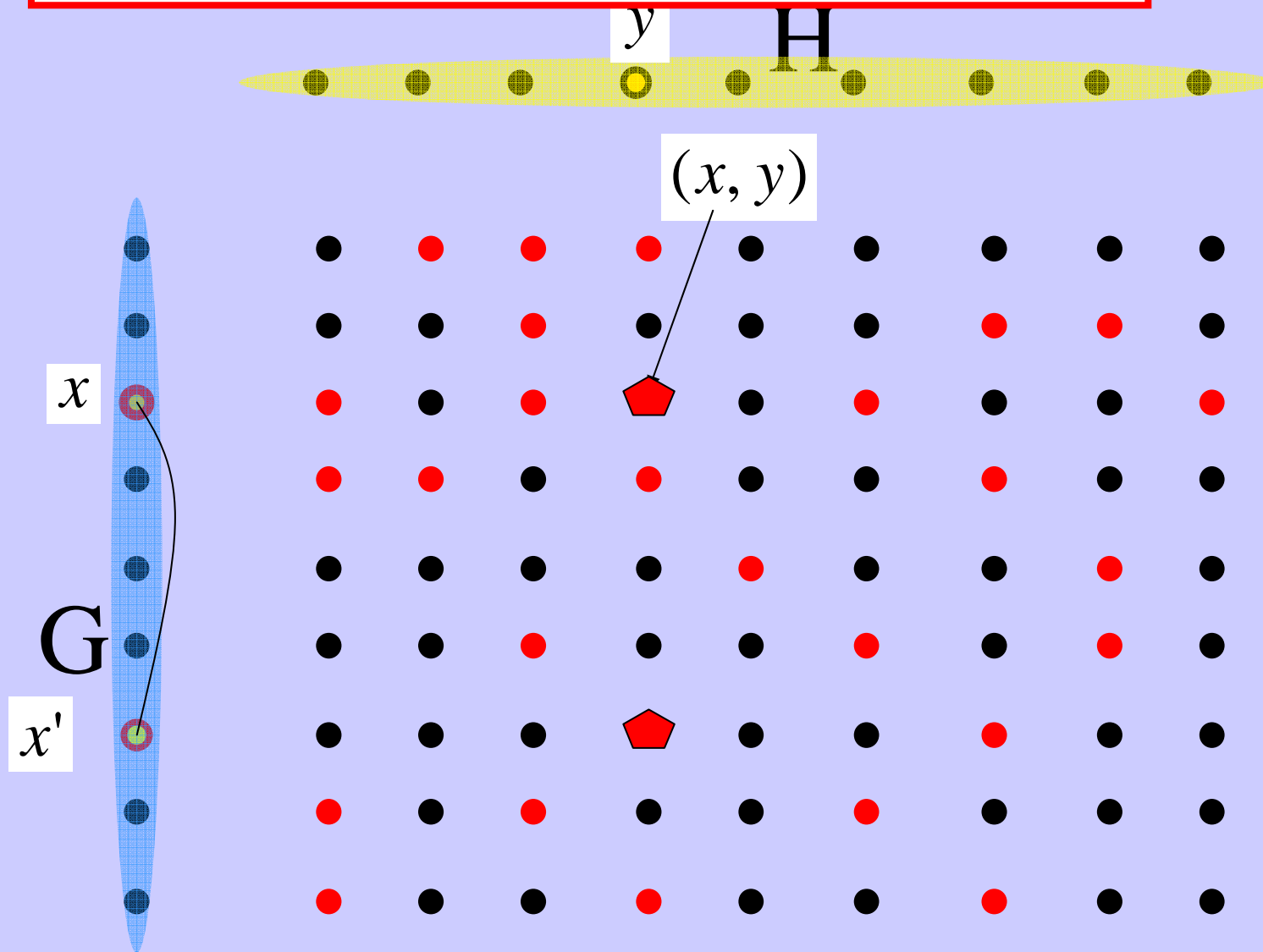
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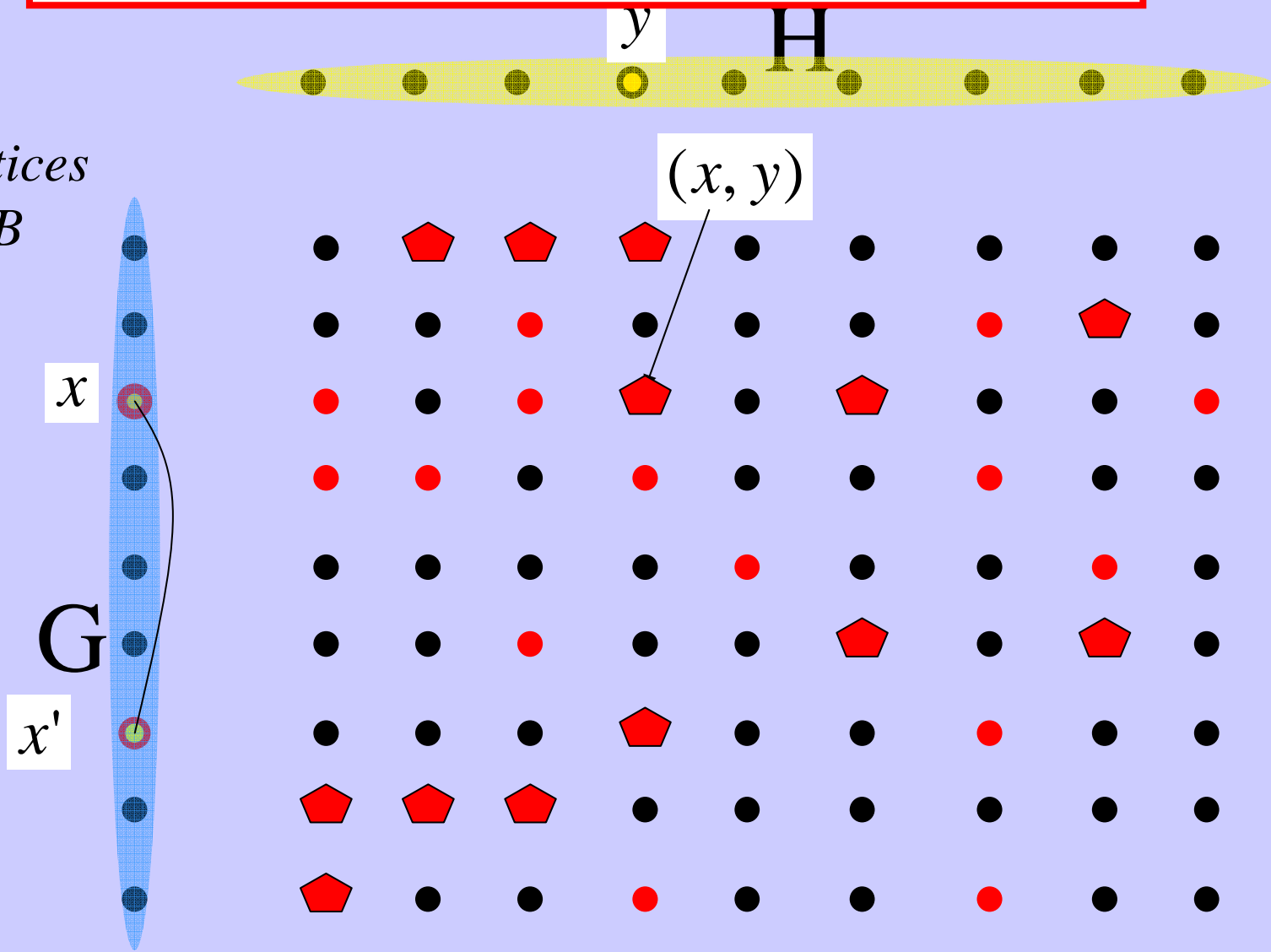
An independent set U in $G \times H$

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vertices
in B



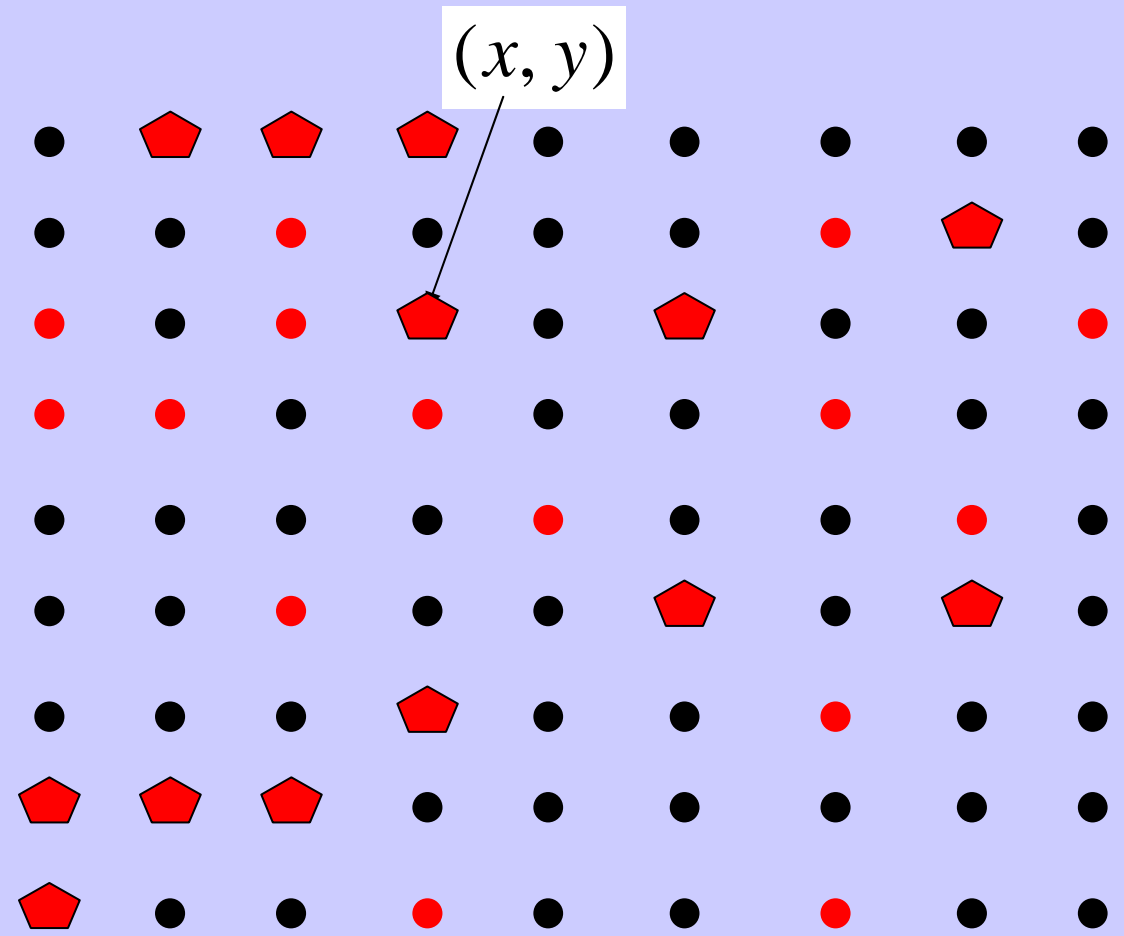
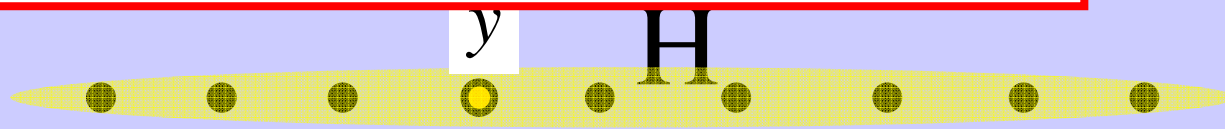
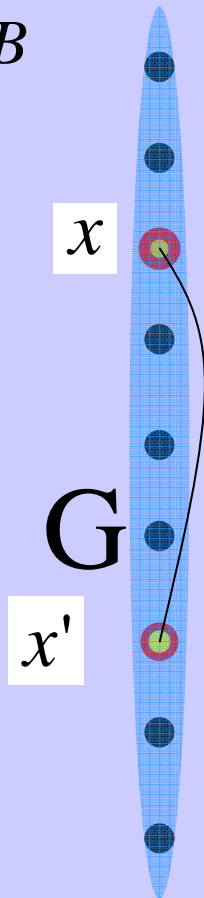
An independent set U in $G \times H$

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vertices
in B



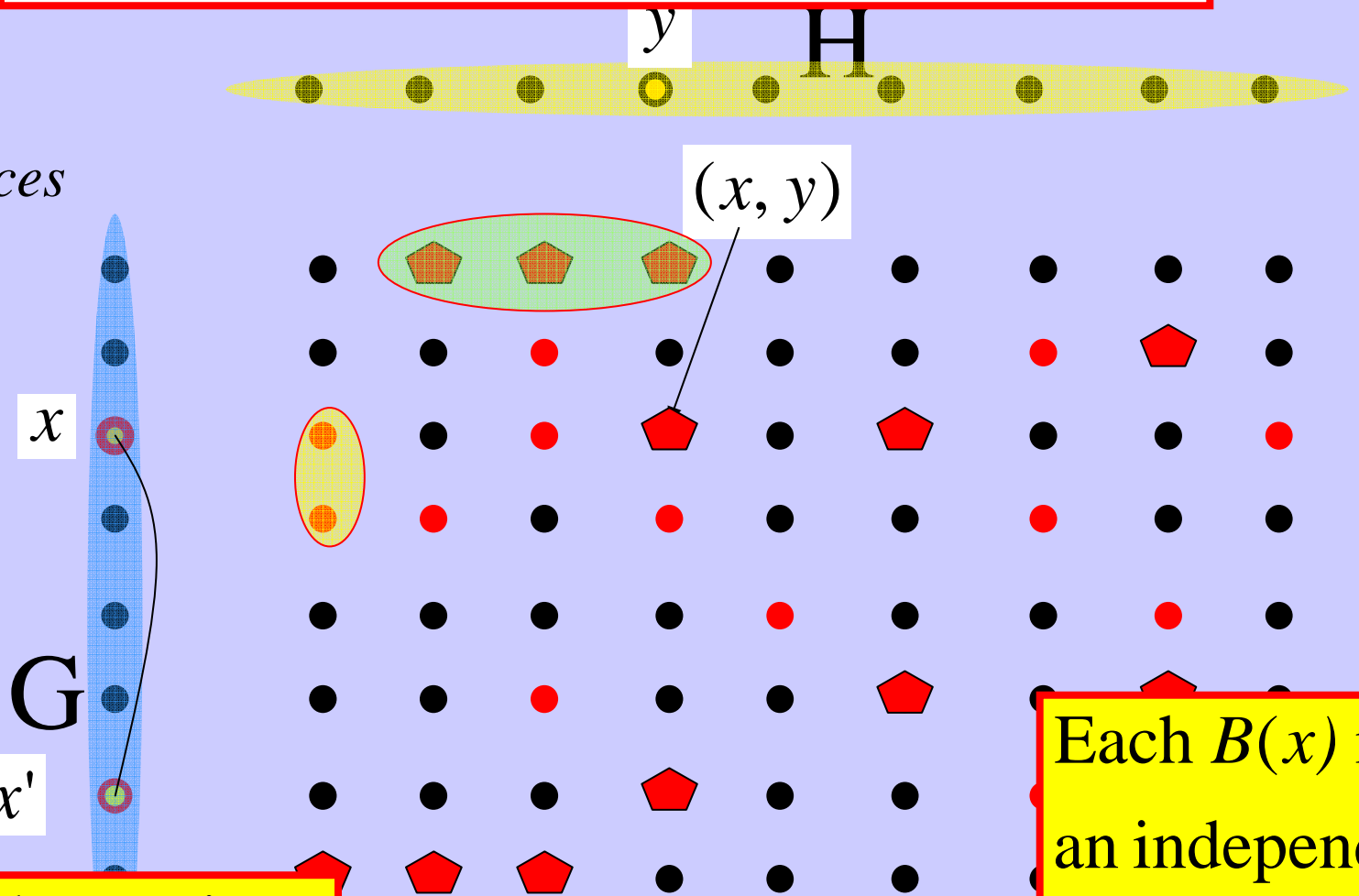
$$A(y) = \{x : (x, y) \in A\}$$

$$U = A \cup B$$

$$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.$$



vertices
in B



Each $A(y)$ is
an independent
set of G .

$$B(x) = \{y : (x, y) \in A\}$$

$$A(y) = \{x : (x, y) \in A\}$$

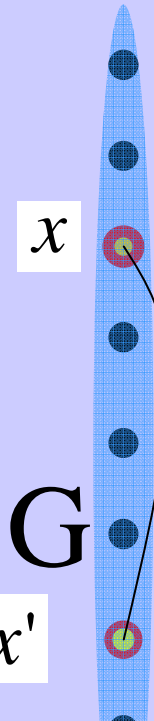
Each $B(x)$ is
an independent
set of H .

$$U = A \cup B$$

$$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.$$



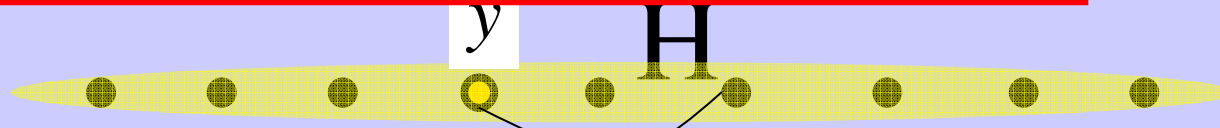
vertices
in B



G

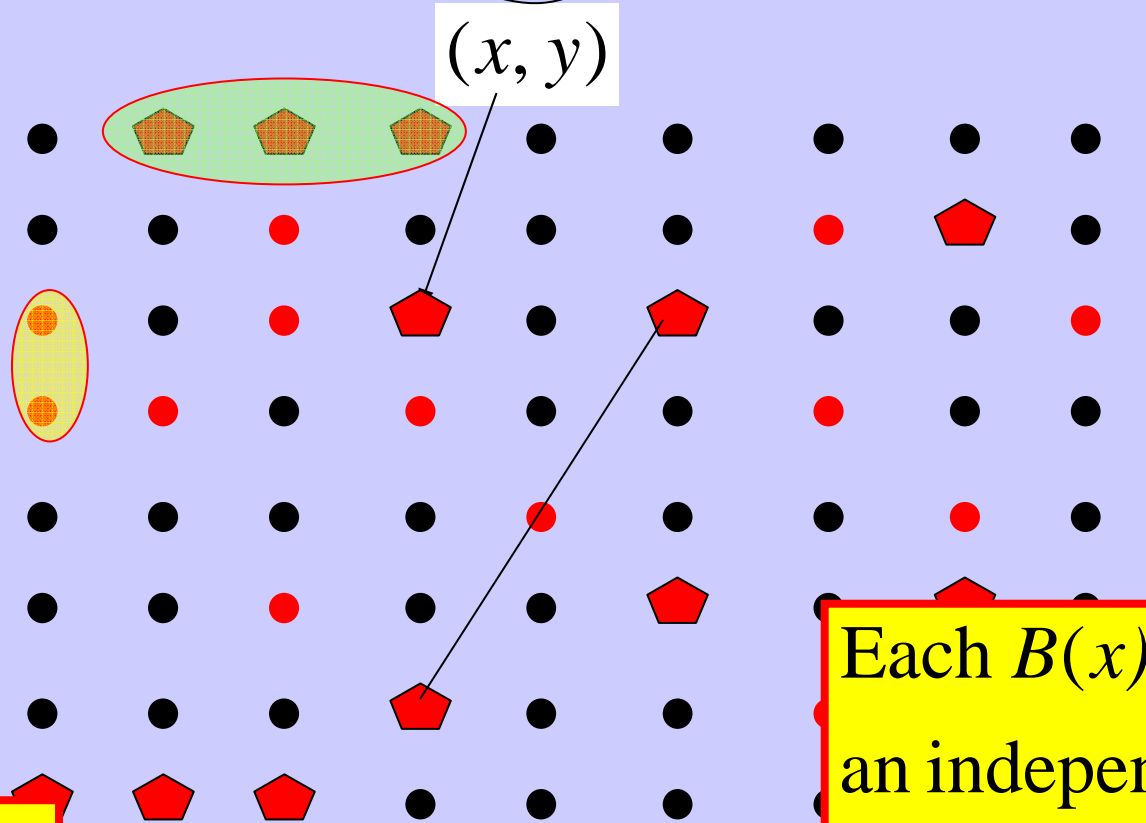
x'

x



H

y



(x, y)

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Each $B(x)$ is
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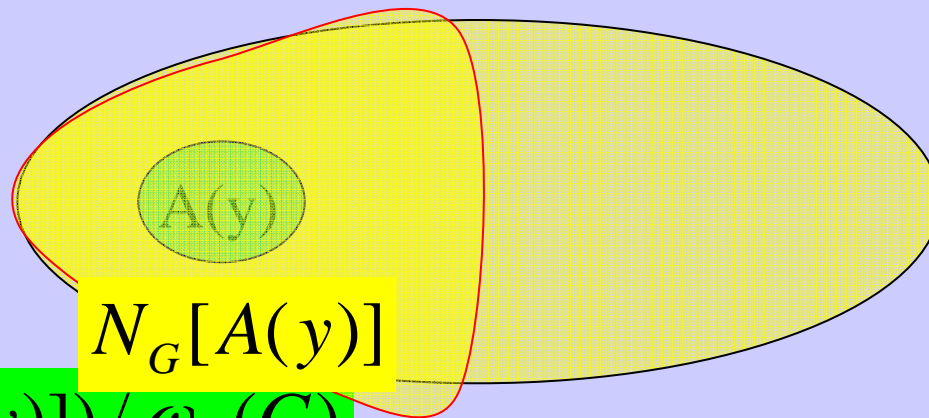
$$U = A \cup B$$

\forall independent set U of $G \times H$,

$$\sum_{(x,y) \in U} g(x)h(y) \leq \omega_f(G)$$

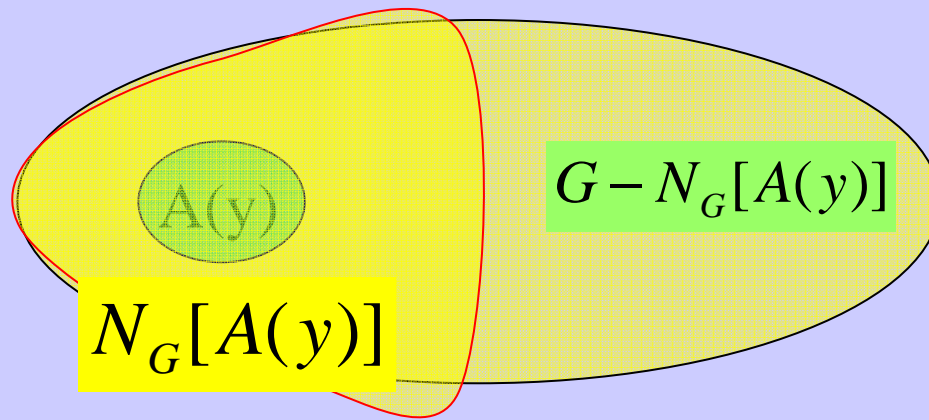
$$\sum_{(x,y) \in U} g(x)h(y) = \sum_{(x,y) \in A} g(x)h(y) + \sum_{(x,y) \in B} g(x)h(y)$$

G



$$g(A(y)) \leq g(N_G[A(y)]) / \omega_f(G)$$

G



$N_G[A(y)]$

$G - N_G[A(y)]$

for any independent set Y'
in $G - N_G[A(y)]$, $A(y) \cup Y'$
is independent

$$g(A(y)) + g(Y') \leq 1$$

$$g(Y') \leq 1 - g(A(y))$$

$$g'(x) = \frac{g(x)}{1 - g(A(y))} \text{ is a}$$

fractional clique of $G - N_G[A(y)]$

$$\sum_{x \in V(G) - N_G[A(y)]} \frac{g(x)}{1 - g(A(y))} \leq \omega_f(G)$$

$$\omega_f(G) - g(N_G[A(y)]) \leq \omega_f(G) - \omega_f(G)g(A(y))$$

$$\omega_f(G) \cdot g(A(y)) \leq g(N_G[A(y)])$$

\forall independent set U of $G \times H$,

$$\sum_{(x,y) \in U} g(x)h(y) \leq \omega_f(G)$$

$$\begin{aligned} \sum_{(x,y) \in U} g(x)h(y) &= \sum_{(x,y) \in A} g(x)h(y) + \sum_{(x,y) \in B} g(x)h(y) \\ &= \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x)) \end{aligned}$$

$$\leq \frac{1}{\omega_f(G)} \sum_{y \in V(H)} g(N_G[A(y)])h(y)$$

$$+ \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x)$$

$$g(A(y)) \leq g(N_G[A(y)]) / \omega_f(G)$$

$$\begin{aligned} \sum_{(x,y) \in U} g(x)h(y) &= \sum_{(x,y) \in A} g(x)h(y) + \sum_{(x,y) \in B} g(x)h(y) \\ &= \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x)) \end{aligned}$$

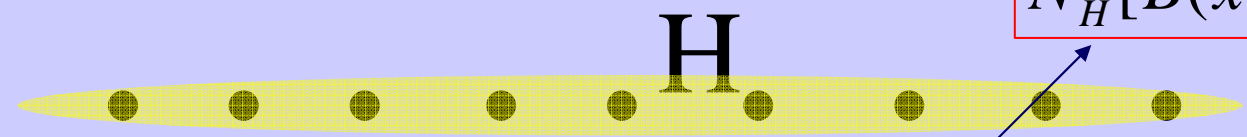
$$\leq \frac{1}{\omega_f(H)} \sum_{y \in V(H)} g(N_G[A(y)])h(y)$$

$$G \quad + \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x)$$



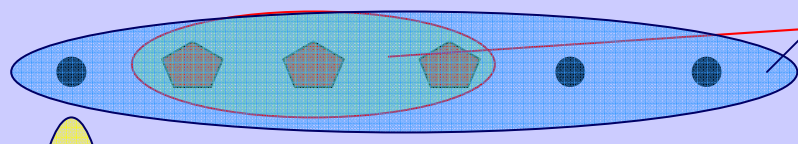
vertices
in B

G

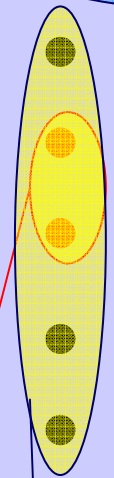
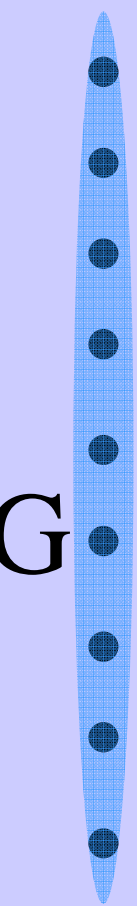


H

$N_H[B(x)]$

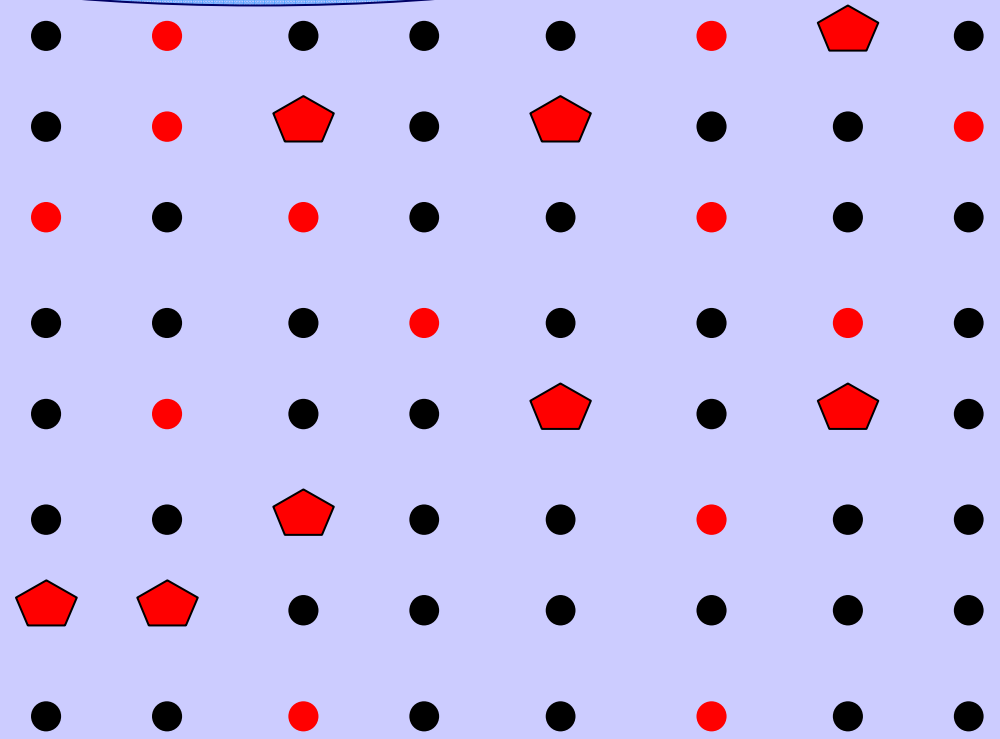


$B(x)$



$A(y)$

$N_G[A(y)]$

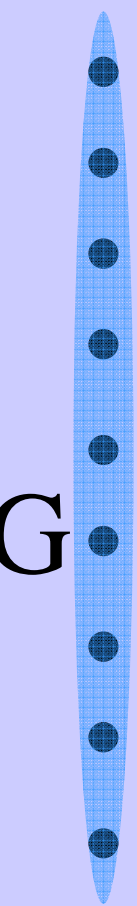
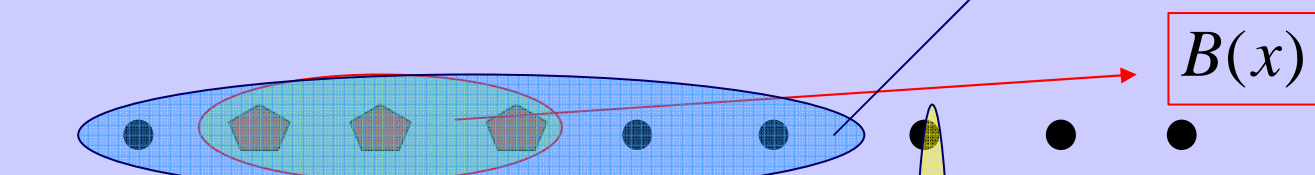
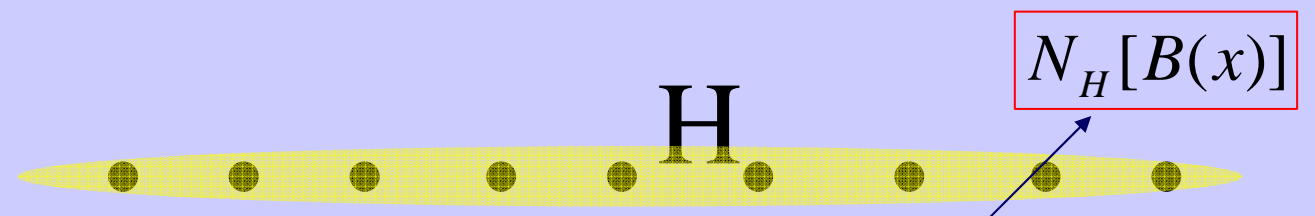


$U = A \cup B$



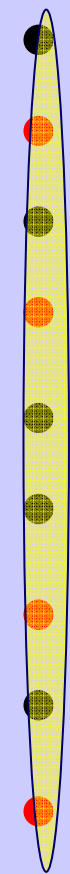
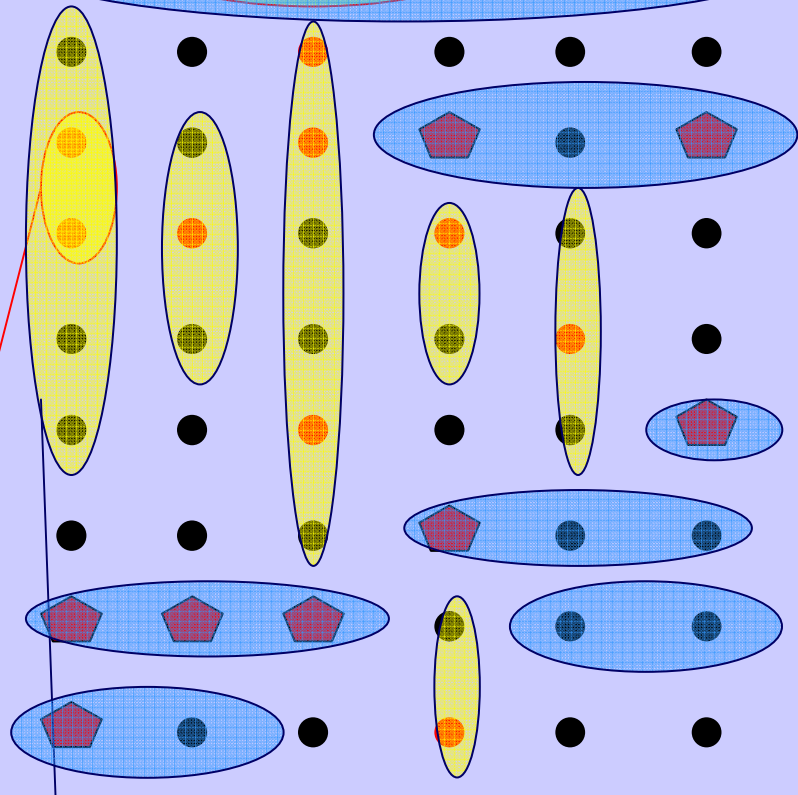
vertices
in B

G

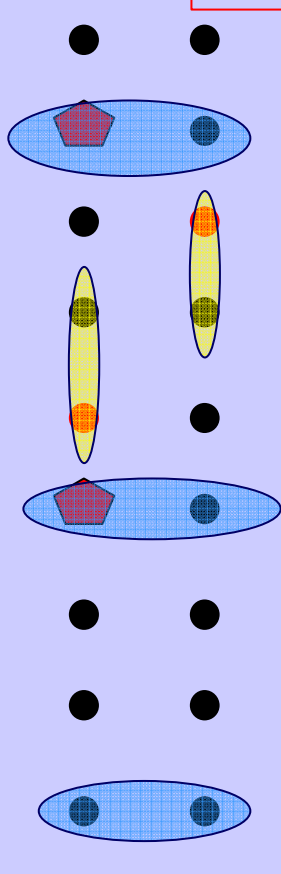


$A(y)$

$N_G[A(y)]$



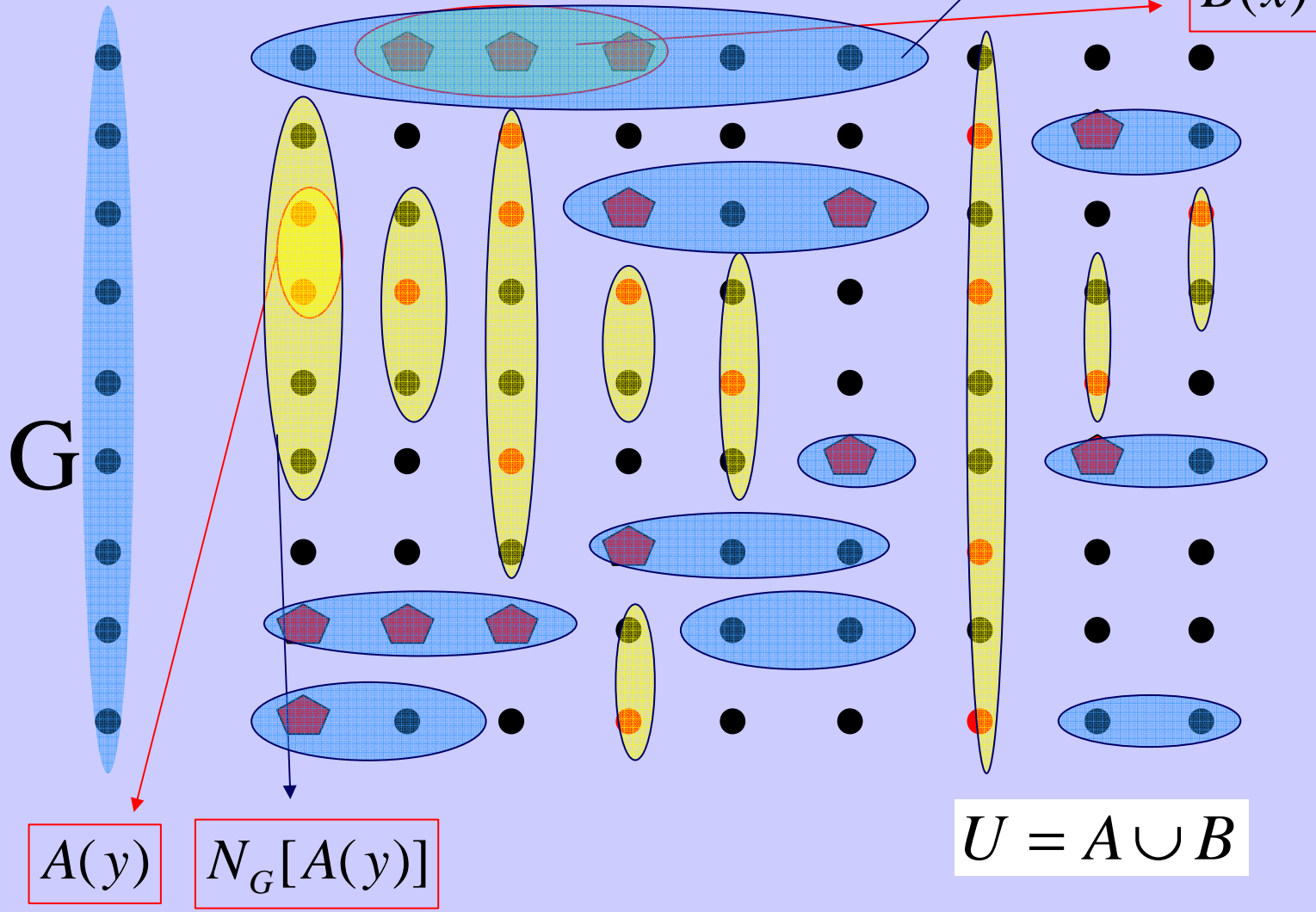
$U = A \cup B$



The yellow area and the blue area do not intersect!



vertices
in B



$A(y)$

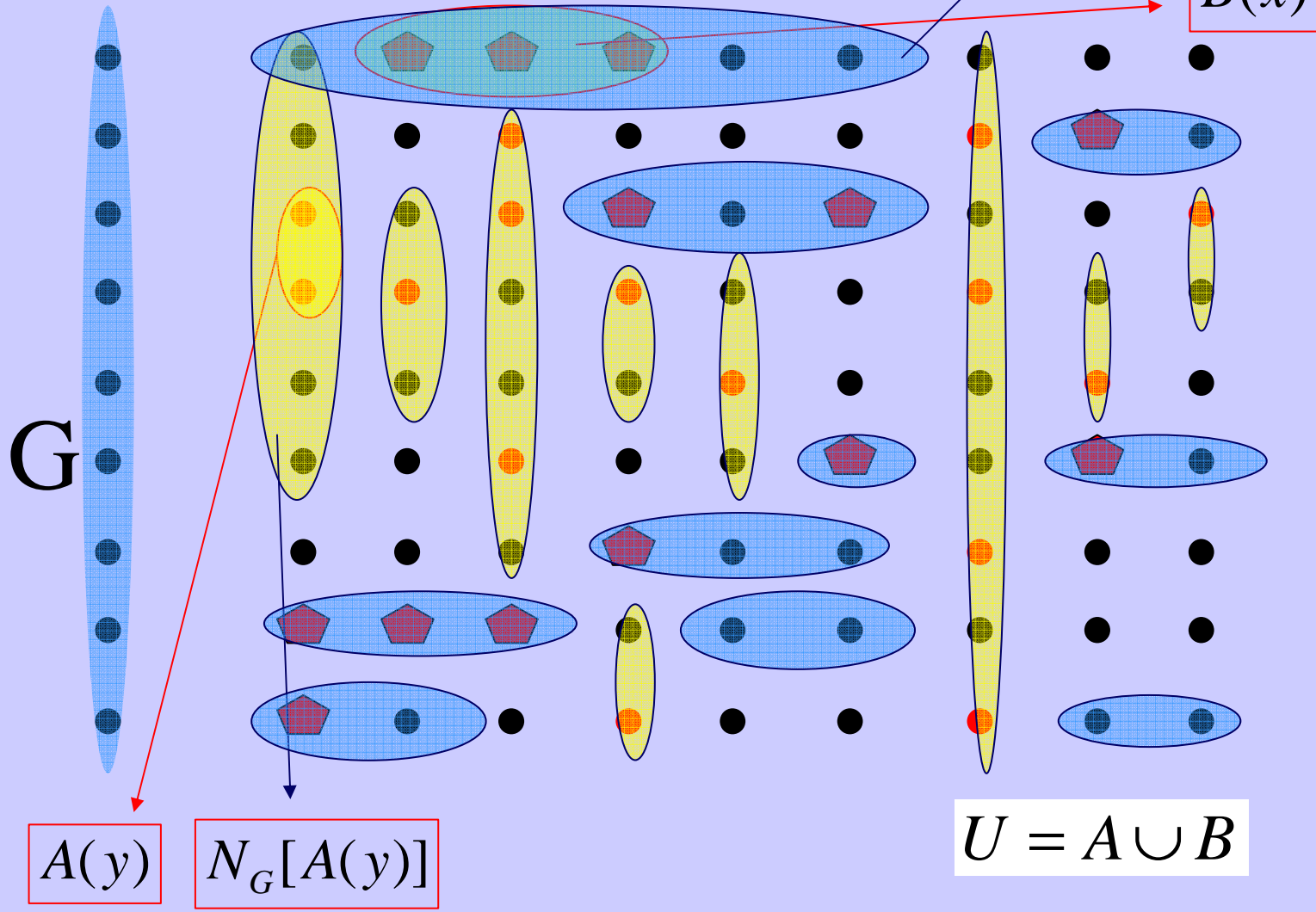
$N_G[A(y)]$

$U = A \cup B$

The yellow area and the blue area do not intersect!



vertices
in B



$A(y)$

$N_G[A(y)]$

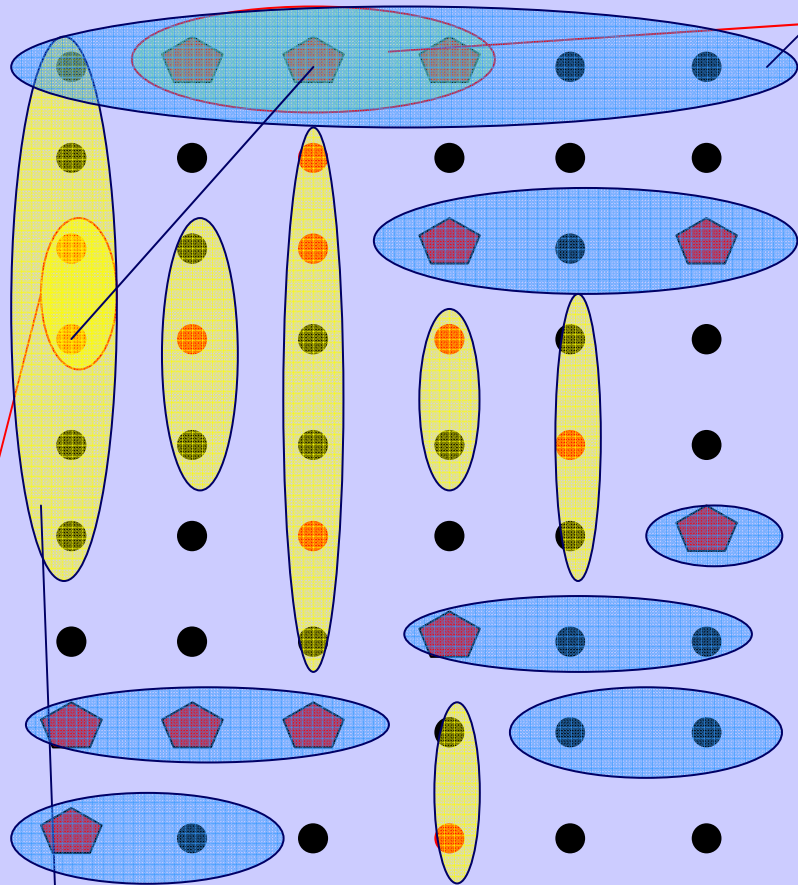
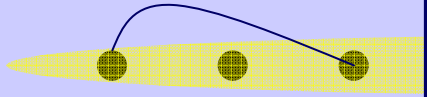
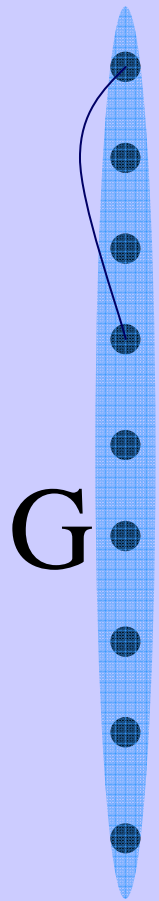
$U = A \cup B$

$B(x)$

The yellow area and the blue area do not intersect!



vertices
in B



$B(x)$

$A(y)$

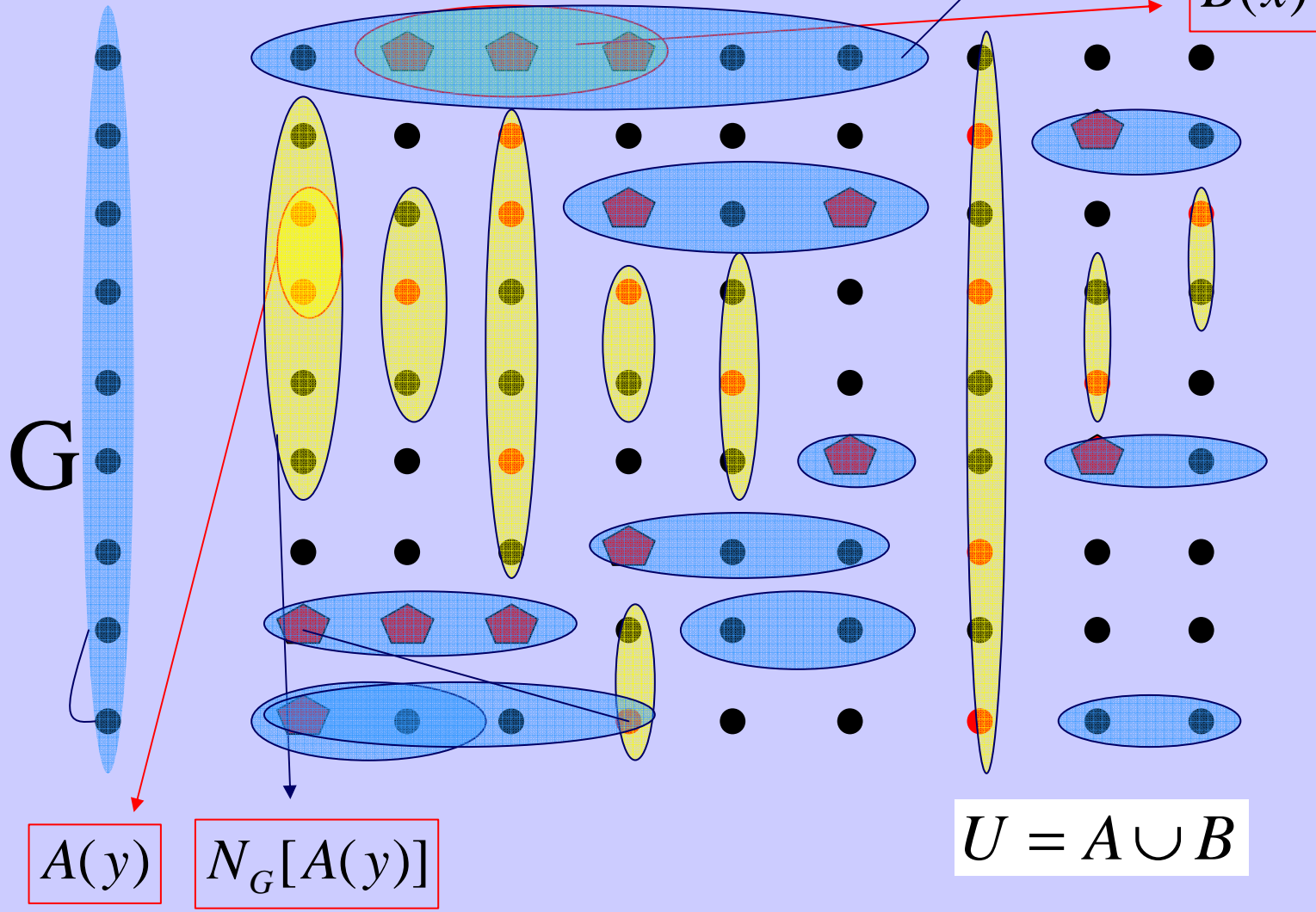
$N_G[A(y)]$

$U = A \cup B$

The yellow area and the blue area do not intersect!



vertices
in B



$A(y)$

$N_G[A(y)]$

$U = A \cup B$

$$\begin{aligned} \sum_{(x,y) \in U} g(x)h(y) &= \sum_{(x,y) \in A} g(x)h(y) + \sum_{(x,y) \in B} g(x)h(y) \\ &= \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x)) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\omega_f(H)} \sum_{y \in V(H)} g(N_G[A(y)])h(y) \\ &\quad + \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x) \end{aligned}$$

$$= \frac{1}{\omega_f(H)} \left(\sum_{\text{yellow}} g(x)h(y) + \sum_{\text{blue}} g(x)h(y) \right)$$

$$\leq \frac{1}{\omega_f(H)} \sum_{x \in V(G), y \in V(H)} g(x)h(y)$$



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