Chromatic Ramsey Number and Fractional Hedetniemi’s Conjecture

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Among any 6 people, there are either 3 people, any two of them know each other, or 3 people, any two of them do not know each other.

For any 2-colouring of the edges of $K_6$ with colours red and blue, there is either a red copy of $K_3$ or a blue copy of $K_3$

$$K_6 \rightarrow (K_3, K_3)$$

$F \rightarrow (G, H)$ means the following

For any 2-colouring of the edges of $F$ with colours red and blue, there is a red copy of $G$ or a blue copy of $H$. 
Theorem [Ramsey] For any graphs $G$ and $H$, there exists an $F$ such that $F \to (G, H)$.

Infinitely many graphs $F$

The Ramsey number of $(G, H)$ is

$$R(G,H) = \min\{n : K_n \to (G, H)\}$$

$$R(G) = R(G, G)$$

$$R(K_k) = R(k)$$

$$R(K_k, K_l) = R(k, l)$$

$F = K_n$ for some $n$
## Bounds for $R(k,l)$

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\[
R(G,H) = \min\{ n : K_n \rightarrow (G, H) \}
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The Ramsey number of \((G, H)\) is
\[
R(G,H) = \min\{ |V(F)| : F \rightarrow (G, H) \}
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The Size Ramsey number of \((G, H)\) is
\[
R_E(G,H) = \min\{ |E(F)| : F \rightarrow (G, H) \}
\]

The max-degree-Ramsey number of \((G, H)\) is
\[
R_{\Delta}(G,H) = \min\{ |\Delta(F)| : F \rightarrow (G, H) \}
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The chromatic Ramsey number of \((G, H)\) is
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R_\chi(G,H) = \min\{ |\chi(F)| : F \rightarrow (G, H) \}
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The chromatic Ramsey number of \((G, H)\) is
\[ R_{\chi}(G, H) = \min\{ |\chi(F)| : F \rightarrow (G, H) \} \]

\[ R_{\chi}(G) = R_{\chi}(G, G) \]

Introduced by Burr-Erdos-Lovasz in 1976

\[ R_{\chi}(G, H) \leq R(G, H) \]

\[ R_{\chi}(K_k, K_l) = R(k, l) \]
If $F$ has chromatic number $(n-1)^2$, then there is a 2 edge colouring of $F$ in which each monochromatic subgraph has chromatic number $n-1$.

\[ F \rightarrow (G, G) \]

for any $n$-chromatic $G$. 

$n = 4$
If F has chromatic number \((n-1)^2\), then there is a 2-edge colouring of F in which each monochromatic subgraph has chromatic number \(n-1\).

\[ F \rightarrow (G, G) \]

for any n-chromatic G.

Could be much larger

Observation: If \(\chi(G) = n\), then \(R_\chi(G) \geq (n-1)^2 + 1\)

Conjecture [Burr - Erdos - Lovasz, 1976]: For each n, there is a graph G with \(\chi(G) = n\) and \(R_\chi(G) = (n-1)^2 + 1\)
The conjecture is true for $n=3,4$ (Burr-Erdos-Lovasz, 1976)

The conjecture is true for $n=5$ (Z, 1992)

The conjecture is true (Z, 2010)

Conjecture [Burr - Erdos - Lovasz, 1976]: For each $n$, there is a graph $G$ with $\chi(G) = n$ and $R_\chi(G) = (n-1)^2 + 1$
Lemma [Burr - Erdos - Lovasz]

\[ R_\chi(G) \leq n \iff K_n \rightarrow \text{hom}(G) \]

For any 2 edge-colouring of \( K_n \), there is a monochromatic graph which is a homomorphic image of \( G \).
Graph homomorphism = edge preserving map
Assume $K_n \rightarrow \text{hom}(G)$

Take a huge complete $n$-partite graph $F$

Lemma [Burr - Erdos - Lovasz]

$R_{\chi}(G) \leq n \iff K_n \rightarrow \text{hom}(G)$
Assume $K_n \rightarrow \text{hom}(G)$

Take a huge complete $n$-partite graph $F$

For any 2 edge colouring of $F$
Assume $Kn \rightarrow \text{hom}(G)$

Take a huge complete $n$-partite graph $F$

For any two edge colouring of $F$

There is a large complete $n$-partite graph $F'$, for any two parts $A,B$, all the edges in $E[A,B]$ have the same colour
Assume $K_n \rightarrow \text{hom}(G)$

Take a huge complete n-partite graph $F$

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There is a large complete n-partite graph $F'$, for any two parts $A, B$, all the edges in $E[A, B]$ have the same colour.
Assume $Kn \rightarrow \text{hom}(G)$

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For any 2 edge colouring of $F$

There is a large complete $n$-partite graph $F'$, for any two parts $A,B$, all the edges in $E[A,B]$ have the same colour

This defines a 2 edge colouring of $Kn$
There is a monochromatic graph $H$ in $K_n$, which is a homomorphic image of $G$
There is a monochromatic graph $H$ in $K_n$, which is a homomorphic image of $G$

Pull $H$ back to the complete multipartite graph to find a monochromatic copy of $G$

Lemma [Burr - Erdos - Lovasz]

$R_{\chi}(G) \leq n \leftrightarrow K_n \rightarrow \text{hom}(G)$
To prove Burr-Erdos-Lovasz conjecture for $n$, we need to construct an $n$-chromatic graph $G$, so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of $G$.

The construction of $G$ is easy:

Take all 2 edge colourings of $K_{(n-1)^2+1}$

$c_1, c_2, \cdots, c_m$

For each 2 edge colouring $c_i$ of $K_{(n-1)^2+1}$, one of the monochromatic subgraph, say $G_i$, has chromatic number at least $n$.

Conjecture [Burr - Erdos - Lovasz, 1976]: For each $n$, there is a graph $G$ with $\chi(G) = n$ and $R_\chi(G) = (n-1)^2 + 1$
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$G = G_1 \times G_2 \times \cdots \times G_m$
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$G = G_1 \times G_2 \times \cdots \times G_m$

Each $G_i$ is a homomorphic image of $G$
Conjecture [Hedetniemi, 1966]:

\[ \chi(G \times H) = \min\{\chi(G), \chi(H)\} \]
To prove this conjecture for $n$, we need to construct an $n$-chromatic graph $G$, so that any 2 edge colouring of has a monochromatic subgraph which is a homomorphic image of $G$.

If Hedetniemi’s conjecture is true, then

Burr-Erdos-Lovasz conjecture is true.
Conjecture [Hedetniemi, 1966]:
\[ \chi(G \times H) = \min \{ \chi(G), \chi(H) \} \]

Fractional Hedetniemi’s conjecture

Conjecture [Z, 2002]:
\[ \chi_f(G \times H) = \min \{ \chi_f(G), \chi_f(H) \} \]
To prove this conjecture for $n$, we need to construct an $n$-chromatic graph $G$, so that any 2 edge colouring of $K_{(n-1)^2+1}$ has a monochromatic subgraph which is a homomorphic image of $G$.

If Hedetniemi’s conjecture is true, then

Burr-Erdos-Lovasz conjecture is true.

Observation [Claude Tardif] If fractional Hedetniemi’s conjecture is true, then Burr-Erdos-Lovasz conjecture is true.
To prove this conjecture for $n$, we need to construct an $n$-chromatic graph $G$, so that any 2 edge colouring of $K_{(n-1)^2 + 1}$ has a monochromatic subgraph which is a homomorphic image of $G$.

The construction of $G$ is easy:

Take all 2 edge colourings of $K_{(n-1)^2 + 1}$

$c_1, c_2, \ldots, c_m$

For each 2 edge colouring $c_i$ of $K_{(n-1)^2 + 1}$, one of the monochromatic subgraph, say $G_i$, has fractional chromatic number $> n-1$

$G = G_1 \times G_2 \times \cdots \times G_m$  $\chi_f(G) > n - 1$  $\chi(G) \geq \chi_f(G) > n - 1$

Each $G_i$ is a homomorphic image of $G$
A fractional colouring of $G$ is a mapping $f$ which assigns to each independent set $U$ of $G$ a nonnegative weight $f(U)$ so that for any vertex $v$ of $G$, $\sum_{v \in U} f(U) \geq 1$.

The minimum total weight of a fractional colouring of $G$ is the fractional chromatic number of $G$, and is denoted by $\chi_f(G)$. 
Conjecture [Hedetniemi, 1966]:
\[ \chi(G \times H) = \min\{\chi(G), \chi(H)\} \]

Fractional Hedetniemi’s conjecture

Theorem [Z, 2010]:
\[ \chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\} \]
A fractional clique of $G$ is a mapping $f$ which assigns to each vertex $v$ a nonnegative weight $f(v)$ so that for any independent set $U$ of $G$, $f(U) = \sum_{v \in U} f(v) \leq 1$.

The maximum total weight of a fractional clique of $G$ is the fractional clique number of $G$, and is denoted by $\omega_f(G)$. 
The fractional chromatic number of $G$ is obtained by solving a linear programming problem.

The fractional clique number of $G$ is obtained by solving its dual problem.

\[ \chi_f(G) = \omega_f(G) \]
Fractional Hedetniemi’s conjecture is true

**Theorem [Z, 2010]**:

\[ \omega_f(G \times H) = \min \{ \omega_f(G), \omega_f(H) \} \]

**Proof sketch**:

\[ \chi_f(G \times H) = \min \{ \chi_f(G), \chi_f(H) \} \]

\[ \chi_f(G \times H) \leq \min \{ \chi_f(G), \chi_f(H) \} \rightarrow \text{Easy!} \]

\[ \chi_f(G \times H) \geq \min \{ \chi_f(G), \chi_f(H) \} \rightarrow \text{Difficult!} \]

\[ \omega_f(G \times H) \geq \min \{ \omega_f(G), \omega_f(H) \} \]

suffices to construct a fractional clique of \( G \times H \) with total weight \( \min \{ \omega_f(G), \omega_f(H) \} \)
\( g : V(G) \rightarrow [0,1], \) a maximum fractional clique of \( G \)

\( h : V(H) \rightarrow [0,1], \) a maximum fractional clique of \( H \)

\( \varphi : V(G \times H) \rightarrow [0,1], \) defined as

\[
\varphi(x,y) = \frac{g(x)h(y)}{\max\{\omega_f(G), \omega_f(H)\}}
\]

is a fractional clique of \( G \times H \)

with total weight \( \min\{\omega_f(G), \omega_f(H)\} \)

\( \forall \) independent set \( U \) of \( G \times H \),

\[
\sum_{(x,y) \in U} g(x)h(y) \leq \max\{\omega_f(G), \omega_f(H)\}
\]

Assume \( \omega_f(G) \geq \omega_f(H) \)
∀ independent set $U$ of $G \times H$, 
weight($U$) ≤ max\{$\omega_f(G)$, $\omega_f(H)$\} 
= $\omega_f(G)$

Weight of $(x,y)$ 
$g(x)h(y)$
An independent set in $H$
An independent set $U$ in $G \times H$
An independent set in $H$ of weight $\alpha$
An independent set $U$ in $G \times H$ of weight $\alpha \cdot \omega_f(G)$.
An independent set in $G$ of weight $\beta$. 
An independent set $U$ in $G \times H$ of weight $\beta \cdot \omega_f(H)$
An independent set $U$ in $G \times H$
An independent set $U$ in $G \times H$
An independent set $U$ in $G \times H$.

$$U = A \cup B$$

$$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.$$
An independent set $U$ in $G \times H$.

$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.$
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$U = A \cup B$
An independent set $U$ in $G \times H$
\[(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.\]

\[A(y) = \{x : (x, y) \in A\} \quad U = A \cup B\]
\[(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U.\]

Each \(B(x)\) is an independent set of \(H\).

\[B(x) = \{y : (x, y) \in A\}\]

\[A(y) = \{x : (x, y) \in A\}\]

\[U = A \cup B\]
Each $A(y)$ is an independent set of $G$.

$B(x) = \{y : (x, y) \in A\}$

$A(y) = \{x : (x, y) \in A\}$

Each $B(x)$ is an independent set of $H$.

$U = A \cup B$

$(x, y) \in B \iff \exists x', (x, x') \in E(G), (x', y) \in U$.
∀ independent set $U$ of $G \times H$,

\[ \sum_{(x,y) \in U} g(x)h(y) \leq \omega_f(G) \]

\[ \sum_{(x,y) \in U} g(x)h(y) = \sum_{(x,y) \in A} g(x)h(y) + \sum_{(x,y) \in B} g(x)h(y) \]

\[ g(A(y)) \leq g(N_G[A(y)]) / \omega_f(G) \]
for any independent set \( Y' \) in \( G - N_G[A(y)] \), \( A(y) \cup Y' \) is independent

\[
g'(x) = \frac{g(x)}{1 - g(A(y))} \text{ is a fractional clique of } G - N_G[A(y)]
\]

\[
\sum_{x \in V(G) - N_G[A(y)]} \frac{g(x)}{1 - g(A(y))} \leq \omega_f(G)
\]

\[
\omega_f(G) - g(N_G[A(y)]) \leq \omega_f(G) - \omega_f(G) \cdot g(A(y))
\]

\[
\omega_f(G) \cdot g(A(y)) \leq g(N_G[A(y)])
\]
\[ \forall \text{ independent set } U \text{ of } G \times H, \]
\[ \sum_{(x, y) \in U} g(x)h(y) \leq \omega_f(G) \]

\[ \sum_{(x, y) \in U} g(x)h(y) = \sum_{(x, y) \in A} g(x)h(y) + \sum_{(x, y) \in B} g(x)h(y) \]
\[ = \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x)) \]
\[ \leq \frac{1}{\omega_f(G)} \sum_{y \in V(H)} g(N_G[A(y)])h(y) \]
\[ + \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x) \]

\[ g(A(y)) \leq g(N_G[A(y)]) / \omega_f(G) \]
\[
\sum_{(x, y) \in U} g(x)h(y) = \sum_{(x, y) \in A} g(x)h(y) + \sum_{(x, y) \in B} g(x)h(y)
\]

\[
= \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x))
\]

\[
\leq \frac{1}{\omega_f(H)} \sum_{y \in V(H)} g(N_G[A(y)])h(y)
\]

\[
+ \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x)
\]
vertices in $B$

$G$

$H$

$N_H[B(x)]$

$B(x)$

$A(y)$

$N_G[A(y)]$

$U = A \cup B$
The yellow area and the blue area do not intersect!
The yellow area and the blue area do not intersect!
vertices in $B$

$A(y)$ $N_G[A(y)]$ $B(x)$ $U = A \cup B$

The yellow area and the blue area do not intersect!
vertices in $B$
\[
\sum_{(x, y) \in U} g(x)h(y) = \sum_{(x, y) \in A} g(x)h(y) + \sum_{(x, y) \in B} g(x)h(y)
\]

\[
= \sum_{y \in V(H)} g(A(y))h(y) + \sum_{x \in V(G)} g(x)h(B(x))
\]

\[
\leq \frac{1}{\omega_f(H)} \sum_{y \in V(H)} g(N_G[A(y)])h(y)
\]

\[
+ \frac{1}{\omega_f(H)} \sum_{y \in V(H)} h(N_H[B(x)])g(x)
\]

\[
= \frac{1}{\omega_f(H)} \left( \sum_{\text{yellow}} g(x)h(y) + \sum_{\text{blue}} g(x)h(y) \right)
\]

\[
\leq \frac{1}{\omega_f(H)} \sum_{x \in V(G), y \in V(H)} g(x)h(y)
\]