

# Vertex transitive cop-win graphs . . . and more

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based on work with

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## The game

A **cop** chases a **robber** on an undirected *reflexive* graph  $G = (V, E)$ . The cop wins if she occupies the same vertex as the robber. Otherwise the robber wins.

## The rules

A series of *rounds* each consisting of the cop's move followed by the robber's move.

A *move* takes the player from the vertex currently occupied to some adjacent vertex, except at round 0.

Round zero: the cop chooses a vertex, then the robber chooses a vertex.

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## Assumption to begin with

All graphs are finite.

## Questions, problems

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## Answers

Of the sample questions above, only two have answers, one of them only partial.

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## Observation

A regular cop-win graph is complete.

## Gavenčiak

proved that the cop needs at most  $n - 3$  rounds to catch the robber on a cop-win graph with  $n$  vertices and characterized the graphs that reach the bound.

## Aigner and Fromme

defined cop number  $cn(G)$  (also called *search number*  $sn(G)$ ) of a graph  $G$ , and showed that it is at most 3 for planar graphs.

## Schroeder

showed that the cop number of a graph  $G$  is at most  $\lfloor \frac{3}{2}g(G) \rfloor$  and conjectures that it is at most  $g(G) + 3$ , with  $g(G)$  being the genus of  $G$ .

## Theorem (Nowakowski & Winkler, Quilliot)

*A graph  $G = (V, E)$  is cop-win if and only if its vertices can be ordered  $V = \{v_1, \dots, v_n\}$  so that for every  $1 \leq i < n$  there is a  $i < j \leq n$  such that  $N_i[v_i] \subseteq N_i[v_j]$ .*

Here  $N_i[x] = N[x] \cap \{v_i, v_{i+1}, \dots, v_n\}$  and  $N[u]$  is the *closed* neighbourhood (but, of course, since  $G$  is reflexive,  $N[u] = N(u)$ ).

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Consider the robber's move at the penultimate round of the game. He is at  $u$ , the cop at  $v$  and at the next round he will be caught, So wherever he goes, the cop can reach him. Translation:  $N[u] \subseteq N[v]$ . Thus clearly  $G - u$  is a retract of  $G$ . □

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These two lemmas prove the theorem.

Let  $G$  be a graph and let  $C$  be a cycle in  $G$ .

1. A *bridge* of  $C$  is a shortest path in  $G$  between two vertices in  $C$  whose distance in  $G$  is strictly smaller than their distance on  $C$ . If a bridge is an edge, it is called a *chord*.
2. The graph  $G$  is *chordal* if each cycle of length at least four has a chord.
3. The graph  $G$  is *bridged* if each cycle of length at least four has a bridge.
4. A vertex of  $G$  is *simplicial* if its neighbourhood induces a complete graph.
5. A vertex  $u$  of  $G$  is *isometric* if the distances between the vertices of  $G \setminus \{u\}$  are the same as those between corresponding vertices in  $G$ .

## Theorem

*A graph  $G$  is chordal if and only if its vertices can be ordered  $v_1, \dots, v_n$  so that  $v_i$  is simplicial in the graph induced by  $\{v_i, \dots, v_n\}$ .*

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So...

Bridged graphs are cop-win.

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*A graph is bridged if and only if it is cop-win and has no induced cycles of length 4 or 5.*

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## Observation

*A ray (one-way infinite path) is chordal, satisfies the dismantling condition, but is not cop-win.*

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And it is worse. . .

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So which infinite graphs are cop-win?

## Trivial examples

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But there is a characterization.

Given a graph  $G = (V, E)$  (finite or infinite), define relations  $\leq_\alpha$  on its vertex set inductively for all ordinals  $\alpha$  as follows.

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- ▶  $u \leq_\alpha v$  for  $\alpha > 0$  if for every  $z \in N[u]$  there is a  $w \in N[v]$  and  $\beta < \alpha$  such that  $z \leq_\beta w$

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Observe that  $\leq_\alpha \subseteq \leq_\beta$  for  $\alpha < \beta$ . As there are no more than  $|V(G)|$  different such relations for infinite graphs and only a finite number for finite ones, there is a least  $\alpha$  such that  $\leq_\alpha = \leq_{\alpha+1}$ . Let  $\preceq$  be this  $\leq_\alpha$ .

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### Theorem (Nowakowski, Winkler)

*A graph  $G$  is cop-win if and only if the relation  $\preceq$  is trivial, that is, if and only if  $u \preceq v$  for all  $u, v \in G$ .*

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Note, however, that this characterization leads to an algorithm to decide if  $k$  cops can catch  $\ell$  robbers on a finite graph that can be directed or not, have loops at some but not all vertices, and where the moves of the players can be *constrained*. The algorithm is polynomial in the number of vertices of the graph, provided  $k$  and  $\ell$  are fixed.

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In contrast, we have seen that the only finite regular complete graphs are complete.

What we shall do is explain how to construct a vertex transitive graph of cardinality  $\kappa$  from any graph of cardinality  $\kappa$  and use the knowledge of the existence of  $2^\kappa$  non-isomorphic trees of cardinality  $\kappa$  to get the required graphs.

The existence of the trees is well known in logic and, in fact, they are not difficult to construct. To make them into cop-win graphs, just add a universal vertex to each (this makes it into a *pointed tree*). Thus the only real work we need to do is that involved in proving

1. The construction that turns graphs into vertex transitive graphs preserves the non-isomorphisms.
2. There is a way to turn a graph into a vertex transitive graph of the same cardinality.

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## Reminder

Let  $G_i = (V_i, E_i)$ ,  $i = 0, 1$  be two graphs. The *strong* product of  $G_0$  and  $G_1$  is the graph  $G_0 \boxtimes G_1 = (V_0 \times V_1, E)$  with  $E = \{(u, x), (v, y)\} : \text{either } u = v, [x, y] \in E_1, \text{ or } [u, v] \in E_0, x = y, \text{ or } [u, v] \in E_0 \text{ and } [x, y] \in E_1\}$ .

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This generalizes.

Let  $I$  be an index set. The *strong product* of a set  $\{G_i : i \in I\}$  of graphs is the graph  $\boxtimes_{i \in I} G_i$  defined by

$$V(\boxtimes_{i \in I} G_i) = \{f : I \rightarrow \bigcup_{i \in I} V(G_i) : f(i) \in V(G_i) \text{ for all } i \in I\},$$
$$E(\boxtimes_{i \in I} G_i) = \{fg : \text{for all } i \in I, f(i) = g(i) \text{ or } f(i)g(i) \in E(G_i)\}.$$

This, however, is not what we need – a strong product of an infinite number of connected graphs could be disconnected, even if all the component graphs are finite. For an example, take  $I = \mathbb{N}$  and  $G_i$  a path  $x_1^i \dots x_{i+1}^i$ . The vertex  $f$  such that  $f(i) = x_{i+1}^i$  is not reachable by a finite path from the vertex  $g$  with  $g(i) = x_1^i$  in the product.

So we modify...

Fix a vertex  $f \in \boxtimes_{i \in I} G_i$ . Define the *weak strong product* of  $\{G_i : i \in I\}$  with *base*  $f$  as the subgraph  $\boxtimes_{i \in I}^f G_i$  of  $\boxtimes_{i \in I} G_i$  induced by the set of all  $g \in V(\boxtimes_{i \in I} G_i)$  such that  $\{i \in I : g(i) \neq f(i)\}$  is finite.

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Observe that  $\boxtimes_{i \in I}^f G_i$  is connected, and if  $|I| \leq \kappa$  and  $|V(G_i)| \leq \kappa$  for each  $i \in I$ , then  $|V(\boxtimes_{i \in I}^f G_i)| \leq \kappa$ .

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Let  $\kappa$  be a cardinal, and let  $H$  be a graph of order  $\kappa$ . Let  $I = \kappa \times V(H)$  and define  $f : I \rightarrow V(H)$  by  $f(\beta, v) = v$ . The power  $H_f^I$  of  $H$  with base  $f$  will be called the *canonical power* of  $H$  and will be denoted by  $H^H$ .

## Lemma

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## Proof (outline)

The key realization is that the sets  $f^{-1}(v)$  partition the set  $\kappa \times V$  and all have the same cardinality  $\kappa$ , and that the same is true for any  $g \in V(H^H)$ . For a fixed  $g \in V(H^H)$ , this allows us to define a bijections  $\phi_v : g^{-1}(v) \longrightarrow f^{-1}(v)$  and use these to define a bijection  $\phi : I \longrightarrow I$  by

$$\phi(\beta, v) = \phi_{g(\beta, v)}(\beta, v).$$

Thus  $g(\beta, v) = f(\phi(\beta, v))$ .

This in turn leads to the definition of a function

$\psi_g : V(H^H) \longrightarrow V(H^H)$  which maps our fixed  $g \in V(H^H)$  to  $f$  and turns out to be an automorphism of  $H^H$ :

$$\psi_g(h) = \hat{h} \text{ such that } \hat{h}(\beta, \nu) = h(\phi(\beta, \nu)).$$

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Since any  $g \in V(H^H)$  can be mapped to the base vertex (function)  $f$  by an automorphism of  $H^H$ , the graph is transitive.  $\square$

As a corollary, we have.

### Corollary

*For each  $k \geq 1$  and each infinite cardinal  $\kappa$  there are  $2^\kappa$  non-isomorphic vertex transitive  $k$ -cop-win graphs of cardinality  $\kappa$ .*

# Open problems

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- ▶ Can anything intelligent be said about the structure of infinite cop-win graphs?
- ▶ Find a bound on the length of the game on a given cop-win graph in terms of other parameters, if possible.

## References

G. Hahn, F. Laviolette, N. Sauer, R. E. Woodrow, On cop-win graphs, *Discrete Mathematics*. **258** (2002) 27–41.

A. Bonato, G. Hahn, C. Tardif, Large classes of infinite  $k$ -cop-win graphs, *J. Graph Theory*. **65** (2010), 334 – 342.

A. Bonato, G. Hahn, C. Wang, The cop density of a graph, *Contributions to Discrete Mathematics* **2** (2007) 133–144.

A. Bonato, P. Golovach, G. Hahn, J. Kratochvíl, The capture time of a graph, in print, *Discrete Mathematics* **309** (2009), 5588 – 5595.

M. Chastand, F. Laviolette, N. Polat, On constructible graphs, infinite bridged graphs and weakly cop-win graphs, *Discrete Mathematics* **224** (2000), 61 – 78.

# Thank you