

Rainbow Connection and minimum degree

Ingo Schiermeyer

Institut für Diskrete Mathematik und Algebra
Technische Universität Bergakademie Freiberg



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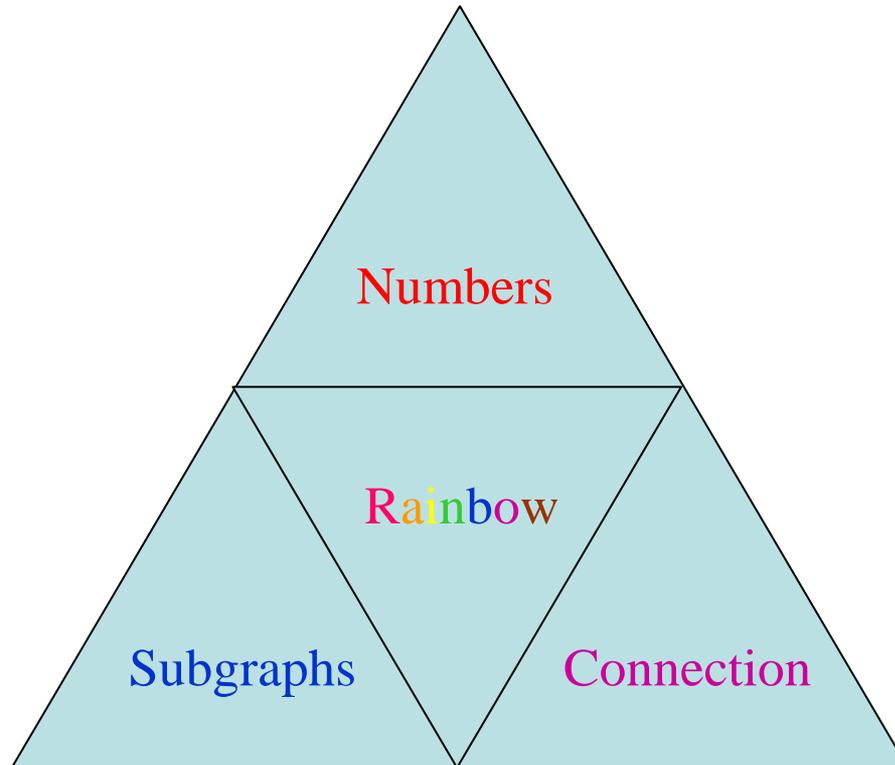
Rainbow Connection and minimum degree

Professeur provocateur

Institut für Diskrete Mathematik und Algebra
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Rainbow Colourings



Rainbow connection

An edge-coloured graph G is called **rainbow** connected if any two vertices are connected by a path whose edges have distinct colours. This concept of rainbow connection in graphs was recently introduced by Chartrand, Johns, McKeon, and Zhang.

The **rainbow** connection number of a connected graph G , denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected.

Rainbow connection

Rainbow connection has an interesting application for the secure transfer of classified information between agencies. While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties.

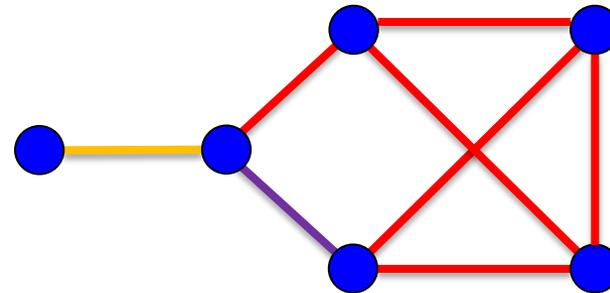
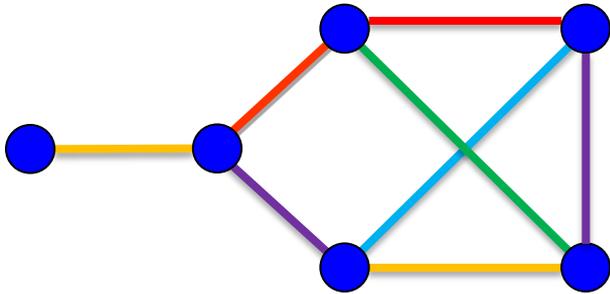
This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated).

Rainbow connection

An immediate **question** arises:

What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

Rainbow connection



Rainbow connection

- The computational complexity has been studied by Chakraborty, Fischer, Matsliah and Yuster (STACS 2009) and by Le and Tuza (2009).
- The computation of $rc(G)$ is NP-hard.
- It is NP-complete to decide if $rc(G) = 2$.
- It is NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow-connected.

Rainbow connection

Results:

$$1 \leq \text{rc}(G) \leq n - 1$$

$\text{rc}(G) = 1$ if and only if G is a clique

$\text{rc}(G) = n - 1$ if and only if G is a tree

$$\text{rc}(G) \geq \text{diam}(G)$$

$$\text{rc}(C_k) = \left\lceil \frac{k}{2} \right\rceil \text{ for } k \geq 4$$

Rainbow connection

- There are graphs with minimum degree 2 and with $rc(G) = n - 3$.

- **Theorem** (Krivelevich and Yuster, 2010)

If G is a connected graph then $rc(G) < \frac{20n}{\delta}$.

- **Theorem** (Caro et al. 2008)

If G is a connected graph with minimum degree $\delta(G)$ then $rc(G) \leq n - \delta$.

Rainbow connection

Theorem (Caro et al. 2008)

If G is a connected graph with minimum degree

$$\delta(G) \geq 3 \text{ then } rc(G) < \frac{5n}{6}.$$

Conjecture

If G is a connected graph with minimum degree

$$\delta(G) \geq 3 \text{ then } rc(G) < \frac{3n}{4}.$$

Conjecture became **Theorem**

IWOCA 2009

Rainbow connection



G_{4k+10} is 3 - regular, has order $n = 4k + 10$

and $rc^*(G_{4k+10}) = rc(G_{4k+10})$

$$= \text{diam}(G_{4k+10}) = \frac{3n - 10}{4}.$$

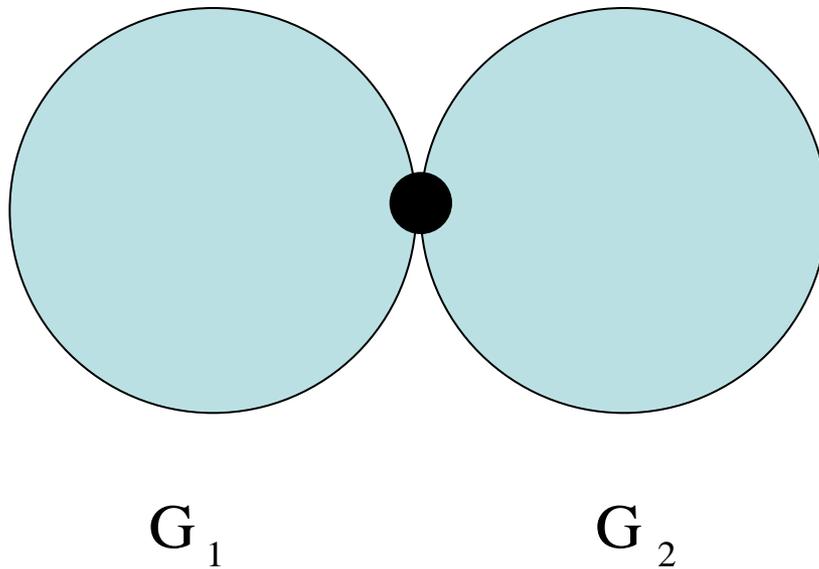
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Proposition

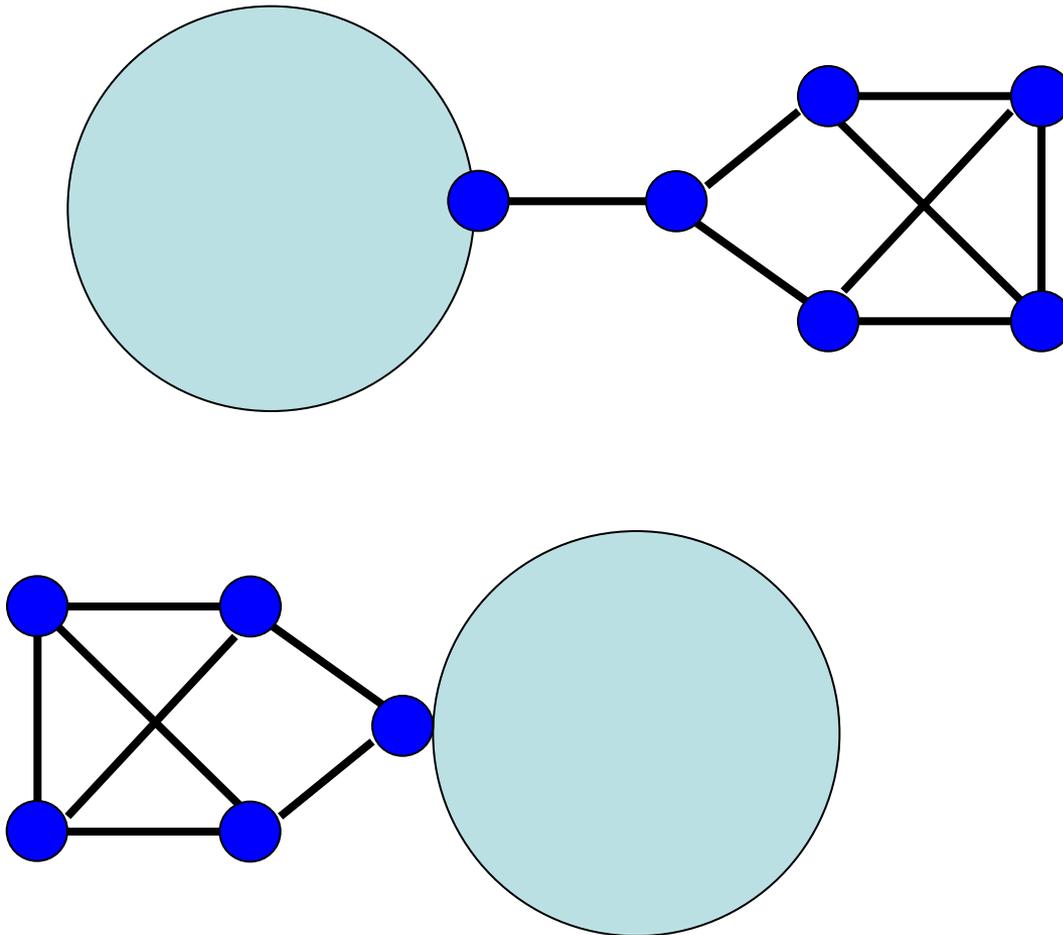
Let G be a 2-connected graph with n vertices and degree sequence $2 \leq d_1 \leq d_2 \leq \dots \leq d_n$.

If $d_3 \geq 3$, then $rc(G) \leq \frac{3n - 4}{4}$.

Rainbow connection



Rainbow connection



Rainbow connection

- Theorem (Krivelevich and Yuster, 2010)

If G is a connected graph then $rc(G) < \frac{20n}{\delta}$.

- Theorem (S., 2010)

If G is a connected graph with minimum degree

$\delta(G)$ then $rc(G) \leq \frac{4}{\delta(G)+1}n + 4$.

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Problem

For every $k \geq 2$ find a minimal constant c_k with $0 < c_k \leq 1$ such that $rc(G) \leq c_k \cdot n$ for all graphs G with minimum degree $\delta(G) \geq k$. Is it true that

$$c_k = \frac{3}{k+1} \text{ for all } k \geq 2 ?$$

This is true for $k = 2, 3$.

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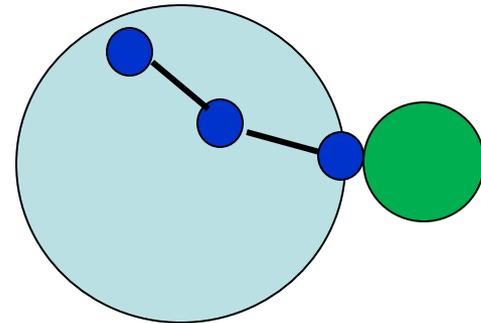
Problem

For every k , $1 \leq k \leq n - 1$, compute and minimize a function $f(n, k)$ with the following property:

If $|E(G)| \geq f(n, k)$, then $rc(G) \leq k$.

Proposition

$$f(n, k) \geq \binom{n - k + 1}{2} + (k - 1).$$



Rainbow connection

Proposition

$$f(n, n - 1) = n - 1,$$

$$f(n, n - 2) = n,$$

$$f(n, 2) = \binom{n - 1}{2} + 1,$$

$$f(n, 1) = \binom{n}{2}.$$

Rainbow connection

A graph G is called **rainbow** k -connected, if there is an edge-colouring of G with k colours such that G is rainbow connected.

For an integer $n \geq 3$ and $1 \leq k \leq n-1$ let $t(n,k)$ denote the minimum size of a graph G of order n , which is rainbow k -connected.

Rainbow connection

Theorem 2010

$$t(n,1) = \binom{n}{2},$$

$$t(n,2) \leq \lfloor \log_2 n \rfloor (n+1) - 2^{\lfloor \log_2 n \rfloor + 1} + 2,$$

$$t(n,3) \leq 2n - 5,$$

$$t(n,k) \leq n-1 + \left\lceil \frac{n-2}{k-2} \right\rceil \text{ for } 4 \leq k \leq \frac{n-1}{2},$$

$$t(n,k) = n \text{ for } \frac{n}{2} \leq k \leq n-2,$$

$$t(n,n-1) = n-1.$$

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Open Problem

Is $t(n,2) = \lfloor \log_2 n \rfloor (n+1) - 2^{\lfloor \log_2 n \rfloor + 1} + c$?

Theorem Kostochka, Łuczak, Simonyi and Sopena, 1997

$t(n,2) \leq \lceil \log_2 n \rceil (n - \lceil \log_2 n \rceil)$ for n large.

Rainbow Colourings



Mercie beaucoup

Thank you very much!