Local Tait colourings of cubic graphs

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In honour of André Raspaud

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- **Part I:** Local Tait colourings
  - generalisation of 3-edge-colourings
  - relation to Steiner triple systems and point-line configurations
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  - Fano colourings
  - abelian colourings
  - integer k-colourings
  - colourings by symmetric configurations
- **Conclusion**
  - Hierarchy of conjectures related to local Tait colourings

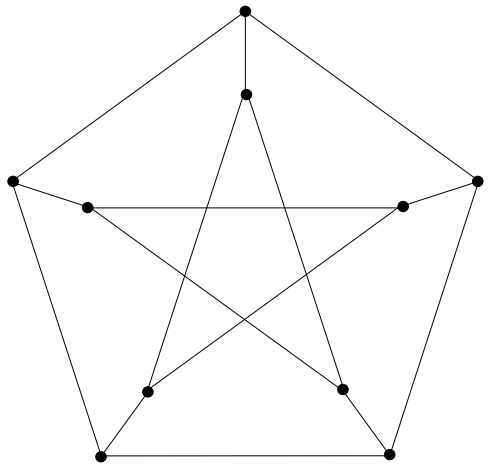
Credits

D. Archdeacon, H.-L. Fu, M. Grannell, T. Griggs, F. Holroyd, M. Knor, D. Král', E. Máčajová, O. Pangrác, A. Pór, A. Raspaud, J.-S. Sereni, & M. S.

Edge-colourings of cubic graphs

Edge-colouring of a graph ... assignment of colours to its edges such that adjacent edges receive distinct colours

- First studied by P. G. Tait (1880)
- J. Petersen (1898) ... four colours suffice to colour every cubic (= trivalent) graph



Edge-colourings of cubic graphs

Chromatic index $\chi'(G)$ ... minimum # of colours needed to colour G

Class 1 ... graphs with $\chi' = 3$ (Tait-colourable graphs)
Class 2 ... graphs with $\chi' = 4$

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Well-known:

- colouring by 4 colours ... *easy*
- deciding whether $\chi' = 3$ or 4 ... *difficult* ... Holyer (1981)
- almost all cubic graphs are Class 1 ... Wormald (1992)

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Non-trivial graphs with $\chi' = 4$ (called *snarks*) closely related to

- Cycle-Double Cover Conj., Fulkerson's Conj., 5-Flow Conj., etc.

Generalisation of 3-edge-colouring

Local Tait colourings

- allow an arbitrary number of colours
- global condition on # of colours $\longrightarrow$ local condition:

(L) *any two colours meeting at a vertex always determine the same the third colour*

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(L) *any two colours meeting at a vertex always determine the same the third colour*

Condition (L) trivially satisfied by usual 3-edge-colourings

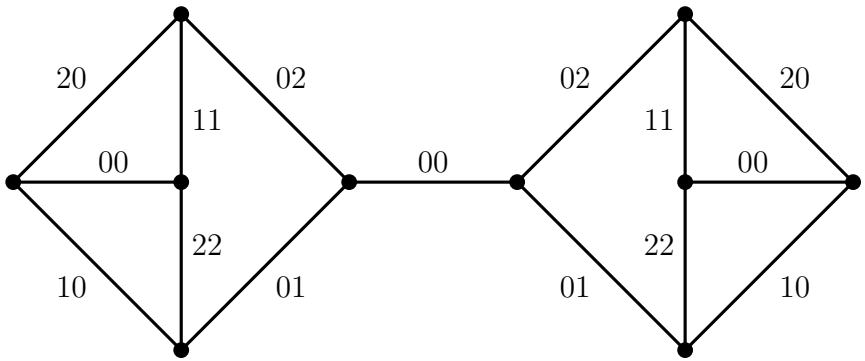
$\Rightarrow$ "local" Tait colourings generalise "global" ones

Local Tait colourings: example

Consider a proper edge-colouring $\phi : E(G) \to \mathbb{Z}_3 \times \mathbb{Z}_3$

- colours ... elements of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$
- colours meeting at a vertex ... triples that sum to 0

$\Rightarrow$ ϕ is a local Tait colouring by elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$



Geometric viewpoint

$\phi : E(G) \to P$ ... local Tait colouring with colour set P

- view the colours as geometric points
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Every local Tait colouring determines a configuration of points and lines.

Combinatorial viewpoint

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- B ... triples of colours occurring at vertices

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- P ... set of colours
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By Condition (L):

any two elements of P belong to at most one triple listed in B

$\Rightarrow$ the pair (P, B) forms a partial Steiner triple system

Steiner triple systems & configurations

A *Steiner triple system* $\mathcal{S} = (P, B)$ of order n consists of

- set P of n *points*
- collection B of 3-element subsets of P (called *blocks*) s.t. any two points belong to exactly one block

If any two points belong to at most one one block

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$\Rightarrow$ *partial Steiner triple system*

Well-known:

- Any partial STS embeds into some full STS ... Treash (1976) (may need some extra vertices)

$\Rightarrow$ Any partial STS can be viewed as a *configuration* of points and blocks of a full STS

Colourings by configurations

Summing up:

Any local Tait colouring can be viewed as one where

- colours ... points of a (partial) Steiner triple system $\mathcal{S}$
- colours that meet at a vertex form a block of $\mathcal{S}$

$\Rightarrow$ *$\mathcal{S}$-colouring*

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Tait colouring $=$ $\mathcal{I}$-colouring, $\mathcal{I}$ being the trivial STS of order 3

Projective and affine STS's

The *projective* Steiner triple system $PG(n, 2)$, $n \geq 2$, has

- points ... $P = \mathbb{Z}_2^{n+1} - \{0\}$
- blocks ... triples $\{x, y, z\}$ with $x + y + z = 0$

smallest projective STS ... $PG(2, 2)$ ... Fano plane of order 7

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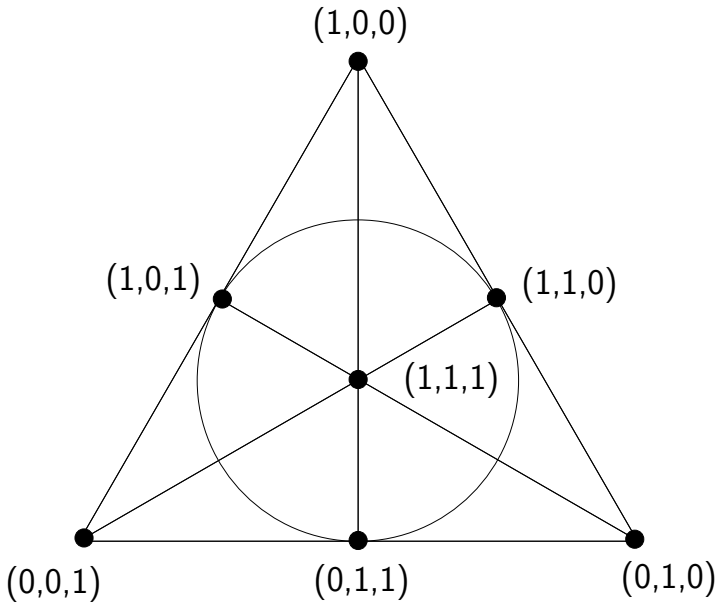
The *affine* Steiner triple system $AG(n, 3)$, $n \geq 2$, has

- points ... $P = \mathbb{Z}_3^n$
- blocks ... triples $\{x, y, z\}$ with $x + y + z = 0$

smallest affine STS ... $AG(2, 3)$ ... affine plane of order 9

Fano plane $PG(2,2)$, the smallest non-trivial STS



Existence of $\mathcal{S}$-colourings: bridgeless graphs

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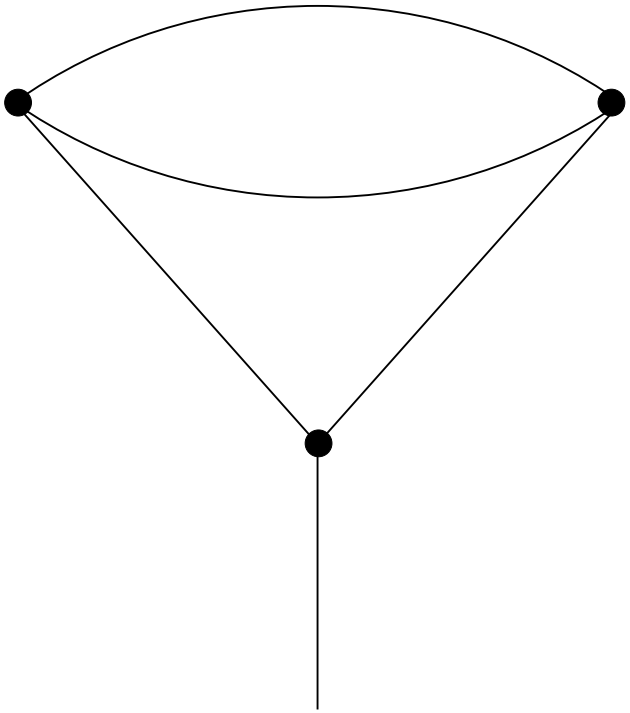
Theorem (Holroyd & S., 2004)

Every bridgeless cubic graph has an $\mathcal{S}$-colouring for every non-trivial Steiner triple system $\mathcal{S}$.

Existence of $\mathcal{S}$-colourings: graphs with bridges

There exist graphs with no local Tait colouring!



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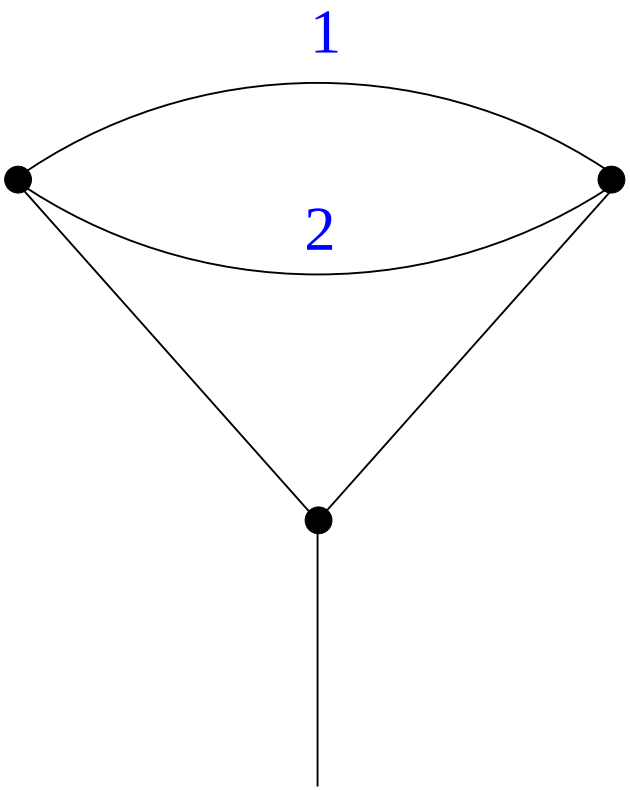

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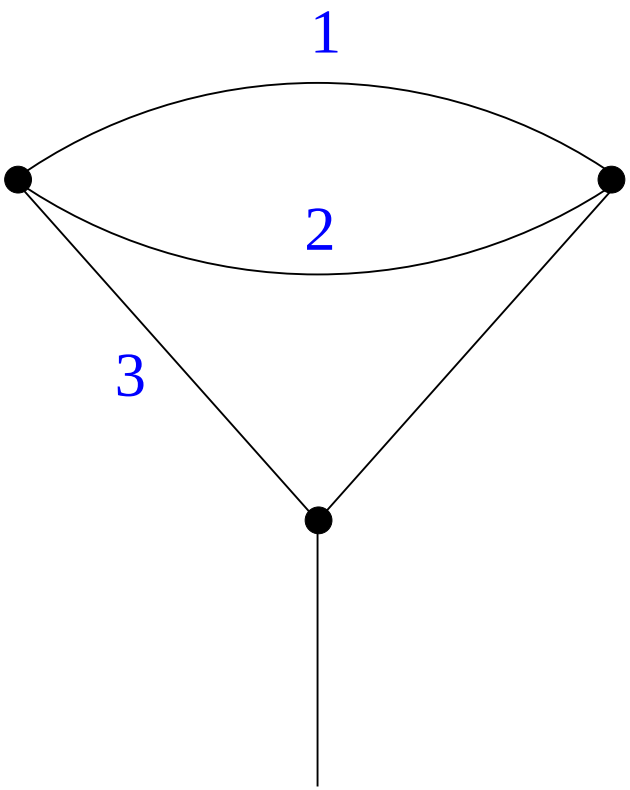

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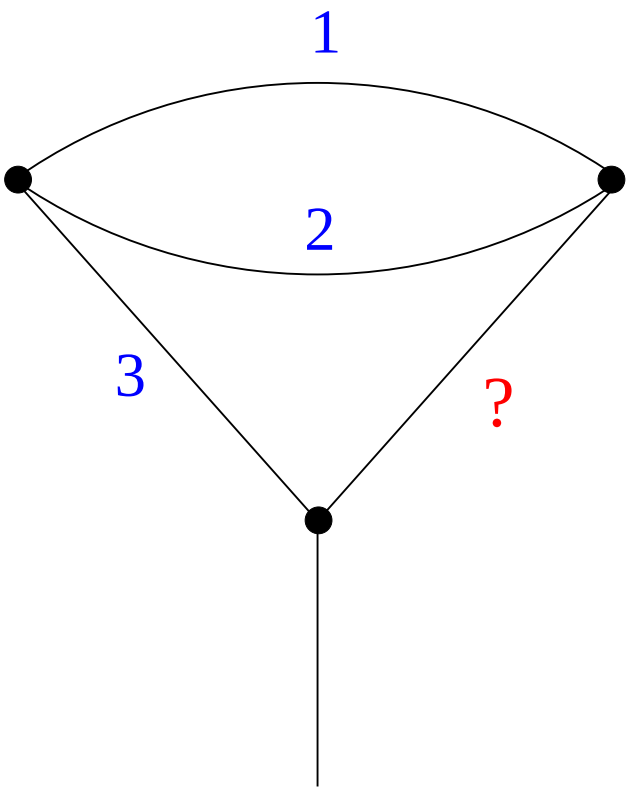

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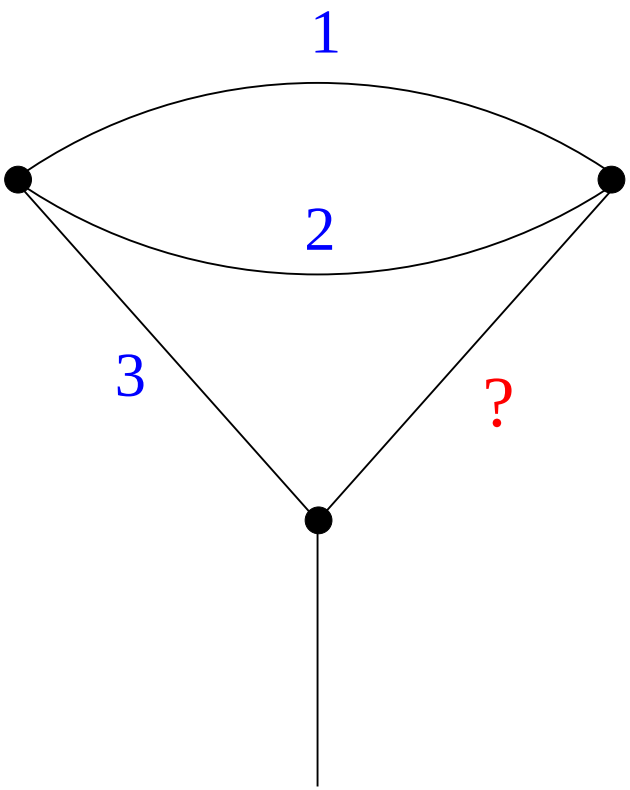

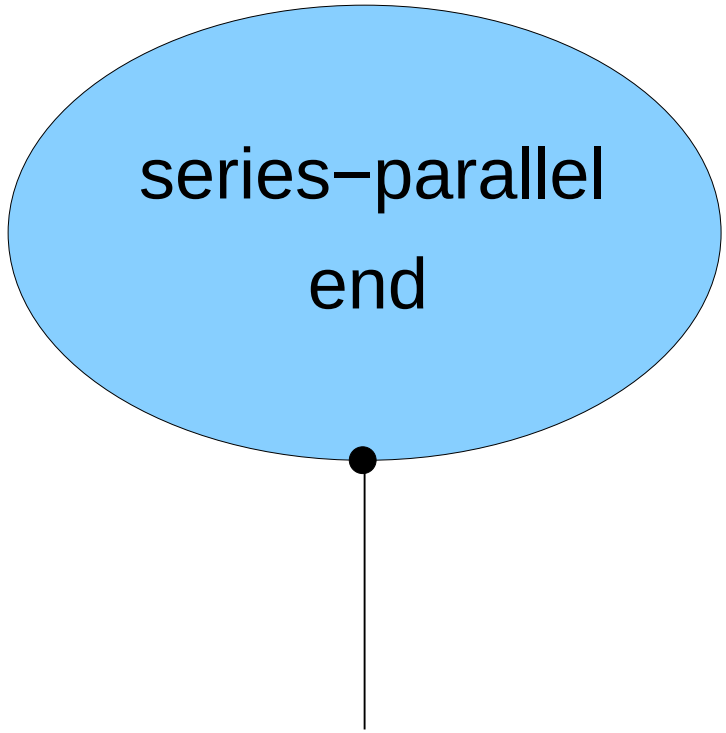

series-parallel = no subdivision of K_4

Existence of $\mathcal{S}$-colourings: graphs with bridges

- Every cubic graph with no series-parallel end admits a local Tait colouring.

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- Projective and affine systems are not universal!

  Projective systems do not colour graphs with bridges
  Affine systems do not colour graphs with bipartite ends

Universal Steiner triple systems

- Universal STS of order 381 (Grannell, Griggs, Knor, S., 2004)

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Question: Point-intransitive systems?

Universal Steiner triple systems

Universality of point-intransitive STS's
(Grannell, Griggs, Máčajová, S., 2010+):

- Infinitely many point-intransitive inclusion-minimal universal systems (Wilson-Schreiber systems = abelian group + two extra points)
- Infinitely many point-intransitive systems that are not universal (projective systems $PG(2, n)$, $n \geq 3$, modified by one Pasch switch)

PART II

Local Tait colourings of bridgeless cubic graphs

Colourings by configurations in STS's

Theorem (Holroyd & S., 2004)

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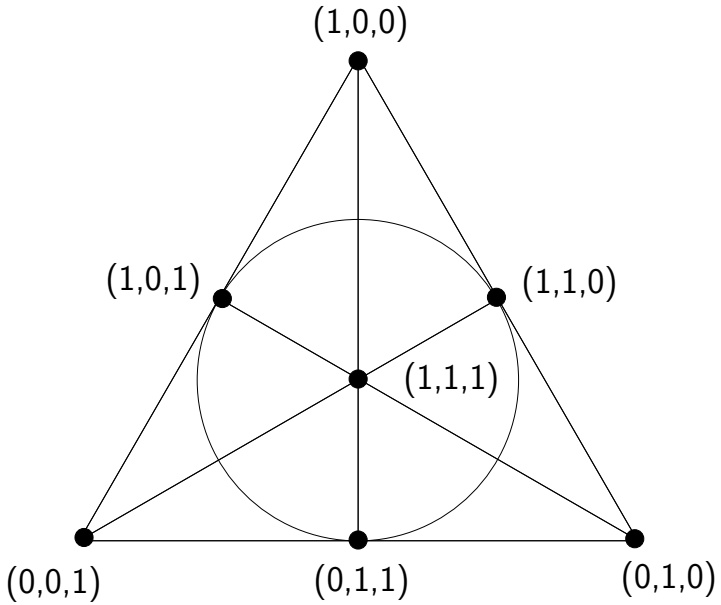
Question: What happens if a Steiner triple system $\mathcal{S}$ is replaced by a suitable configuration?

1. Fano colourings



Fano colouring – proper edge-colouring of a cubic graph

- colours – points of the Fano plane
- the colours around each vertex form a line

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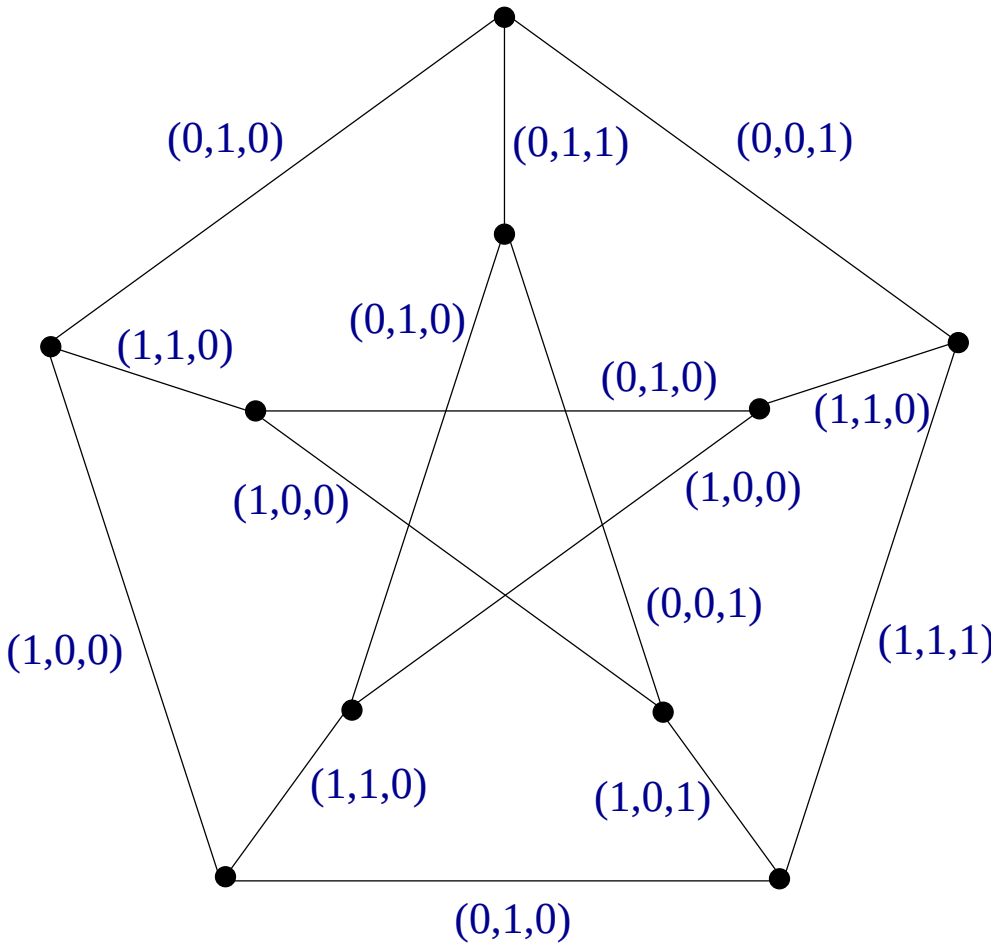
F_i-colouring – colouring using at most i lines of the Fano plane

F_5-colouring of the Petersen graph



F_i-colourings

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Theorem (Máčajová & S., 2005) F_6T

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We believe that four lines are always enough:

4-Line Conjecture (Máčajová & S., 2005) F_4C

Every bridgeless cubic graph admits an F_4-colouring.

Conjecture of Fan and Raspaud

Theorem (Máčajová & S., 2005)

The 4-Line Conjecture is equivalent to the following conjecture of Fan and Raspaud:

Conjecture (Fan & Raspaud, 1994) F&RC

Every bridgeless cubic graphs contains three perfect matchings with no edge in common.

Conjecture of Fan and Raspaud – background

Theorem (Petersen, 1891)

Every bridgeless cubic graphs contains a perfect matching.

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Theorem (Petersen, 1891)

Every bridgeless cubic graphs contains a perfect matching.

- A cubic graphs has two disjoint perfect matchings $\Leftrightarrow$ 3-edge-colourable.
- Every two perfect matchings in a non-3-edge-colourable graph have an edge in common.

Conjecture of Fan and Raspaud – background

Conjecture (Fan & Raspaud, 1994) F&RC

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Conjecture (Fan & Raspaud, 1994) F&RC

Every bridgeless cubic graphs contains three perfect matchings with no edge in common.

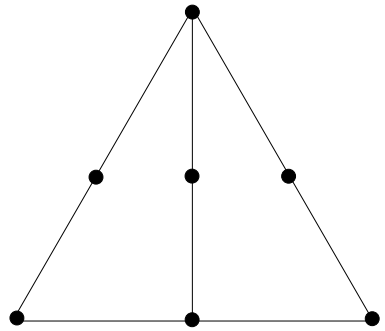
Conjecture (Berge, Fulkerson, 1971) FC

Every bridgeless cubic graphs contains a family of six perfect matchings which together cover each edge exactly twice.

Conjecture of Fan and Raspaud – background

Definition. A configuration $\mathcal{C}$ is *Class 1* if $\mathcal{C} \to \mathcal{I}$, otherwise it is *Class 2*.

Example: The smallest Class 2 configuration is $C_{15} \cong F_4$, the *sail*.



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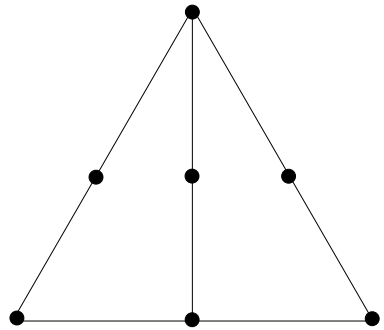

4-Line-Conjecture (rephrased)

The smallest Class 2 configuration colours every bridgeless cubic graph.

Conjecture of Fan and Raspaud

$$\mathbf{FC} \Rightarrow \mathbf{F\&RC} \Leftrightarrow F_4\mathrm{C} \Rightarrow F_5\mathrm{C} \Rightarrow F_6\mathrm{T} \equiv \mathbf{TRUE}$$

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Theorem (Kaiser & Raspaud, 2007, 2010)

Every bridgeless cubic graph of oddness ≤ 2 admits an F_5-colouring.

oddness – minimum number of odd circuits in a 2-factor

oddness 0 $\Leftrightarrow$ Class 1

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Theorem (Máčajová & S., 2009+)

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Equivalently:

Every bridgeless cubic graph of oddness ≤ 2 has three perfect matchings with no edge in common.

2. Abelian colourings

- Given an abelian group A, an *A-colouring* of a cubic graph is a proper colouring by elements of $A - 0$ such that around each vertex the colours sum to zero.

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  Abelian colourings are local Tait colourings:

- Let A be an abelian group. Define $\mathcal{C}(A)$ to be the partial STS where
  points ... $A - 0$
  blocks ... triples $\{x, y, z\}$ with $x + y + z = 0$

  Clearly, A-colouring $=$ $\mathcal{C}(A)$-colouring

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Remark.
A-colouring $\leftrightarrow$ n.-z. A-flow on a cubic graph with *antibalanced bidirection*

2. Abelian colourings

- If A is one of $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$, then $\mathcal{C}(A) = \emptyset \Rightarrow$ no A-colouring
- If A is one of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, and $\mathbb{Z}_9$, then
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Theorem (Máčajová, Raspaud & S., 2005)

If $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|A| \geq 12 \Rightarrow$ every bridgeless cubic graph is A-colourable.

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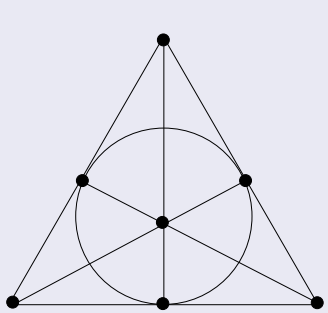
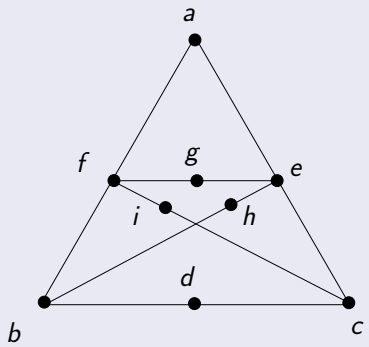
Proof.

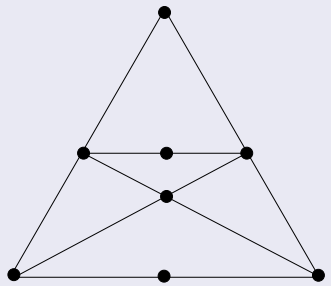
Each of F_7, D_9, or D_8 colours all bridgeless cubic graphs (Holroyd & S.):



F_7



D_9



D_8

The four remaining groups

Remaining groups:

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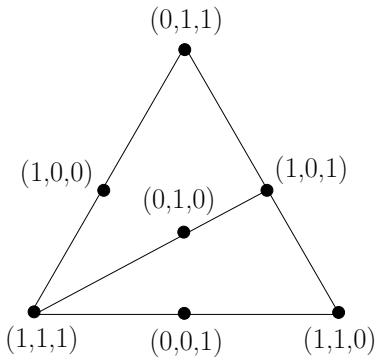
Theorem (Král', Máčajová, Pangrác, Raspaud, Sereni & S., 2008)

The 5-Line Conjecture implies the existence of an A-colouring of every bridgeless cubic graph for each $A \in \{\mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}\}$

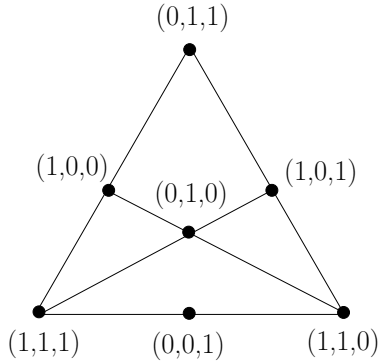
Equivalence of colourings by Fano plane and by affine plane

Theorem (Král', Máčajová, Pangrác, Raspaud, Sereni & S., 2008)

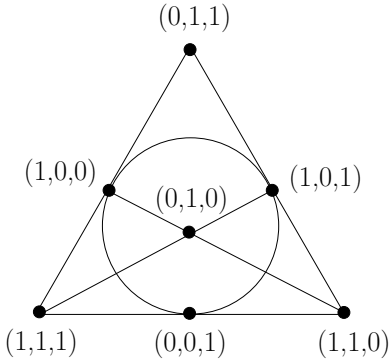
There exist i-line configurations $A_i \subseteq AG(2,3)$ and $F_i \subseteq PG(2,2)$, $4 \leq i \leq 6$, s. t. a cubic graph has an F_i-colouring $\Leftrightarrow$ has an A_i-colouring.



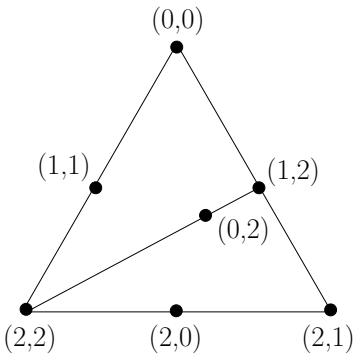
F_4



F_5



F_6



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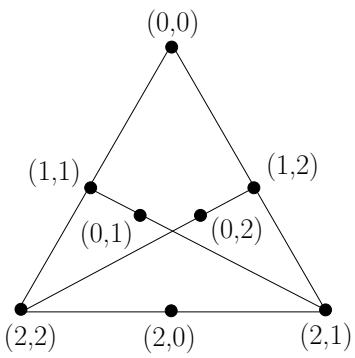
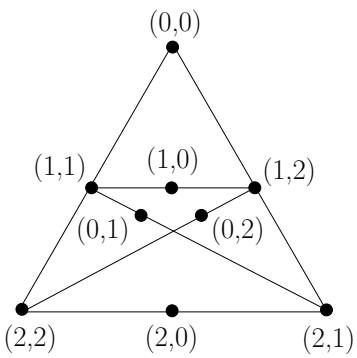


A_5



A_6

3. Integer colourings

- An *integer k-colouring* of a cubic graph is a proper $\mathbb{Z}$-colouring σ satisfying the condition $0 < |\sigma(e)| < k$ for each edge e.

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  Integer k-colourings are local Tait colourings:

- Define $\mathcal{I}_k$ to be the partial STS where

  points ... all integers n with $|n| < k$
  blocks ... triples $\{x, y, z\}$ with $x + y + z = 0$

  Clearly, integer k-colouring $= \mathcal{I}_k$-colouring

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Theorem (Král', Máčajová, Pangrác, Raspaud, Sereni & S., 2008)

Let G be a bridgeless cubic graph.
(1) *If G admits an F_i-colouring, $i \in \{4, 5, 6\}$, then it also admits an integer $(i + 2)$-colouring.*

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4. Colourings by symmetric configurations

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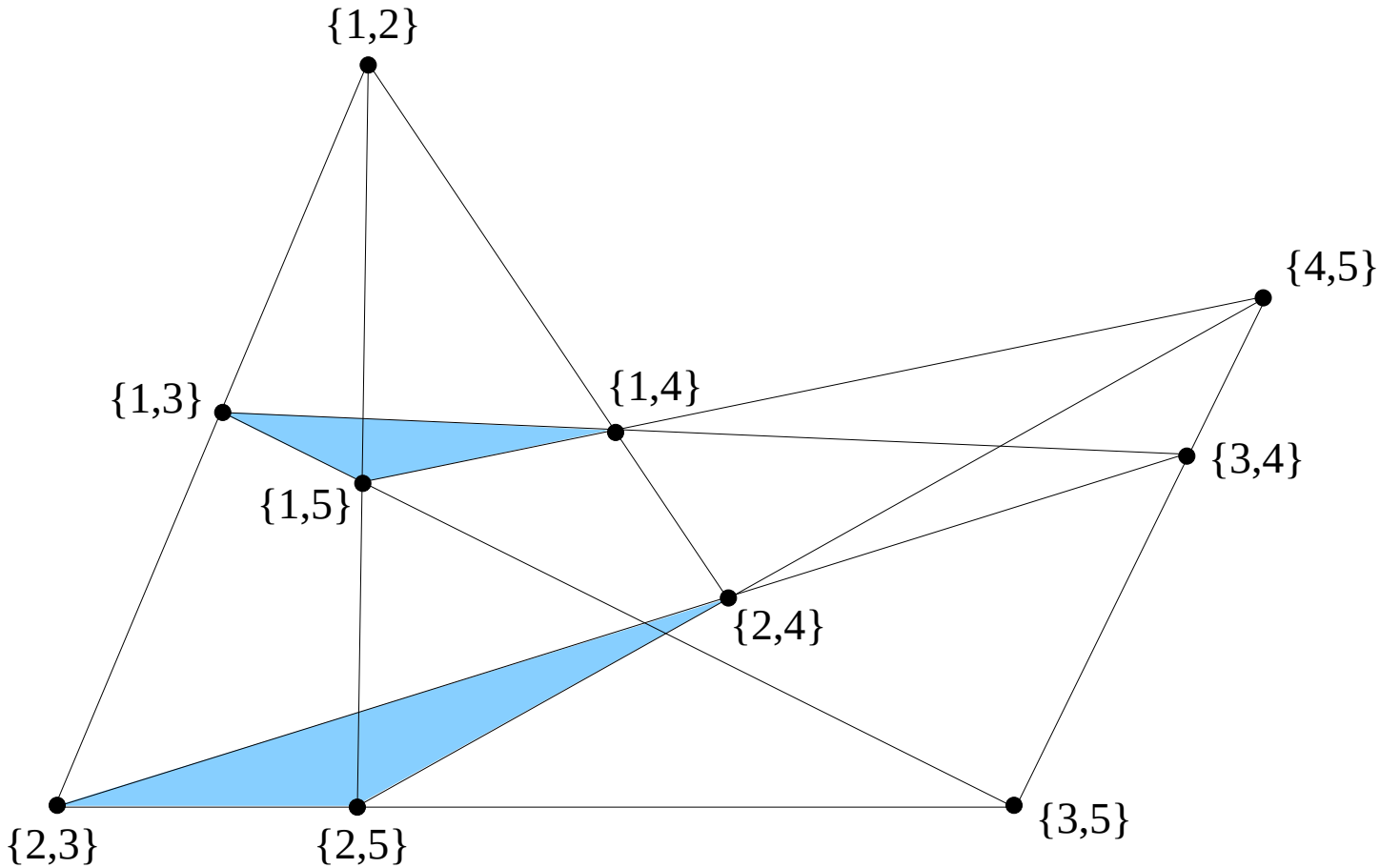
Examples:

7_3 = Fano plane (Class 2)
8_3 = Moebius-Kantor configuration = $AG(2,3)-$pt (Class 2)
9_3 = Pappus configuration (Class 1)
10_3 = Desargues configuration (Class 2)
15_3 = Cremona-Richmond configuration (Class 2)

Desargues configuration 10_3



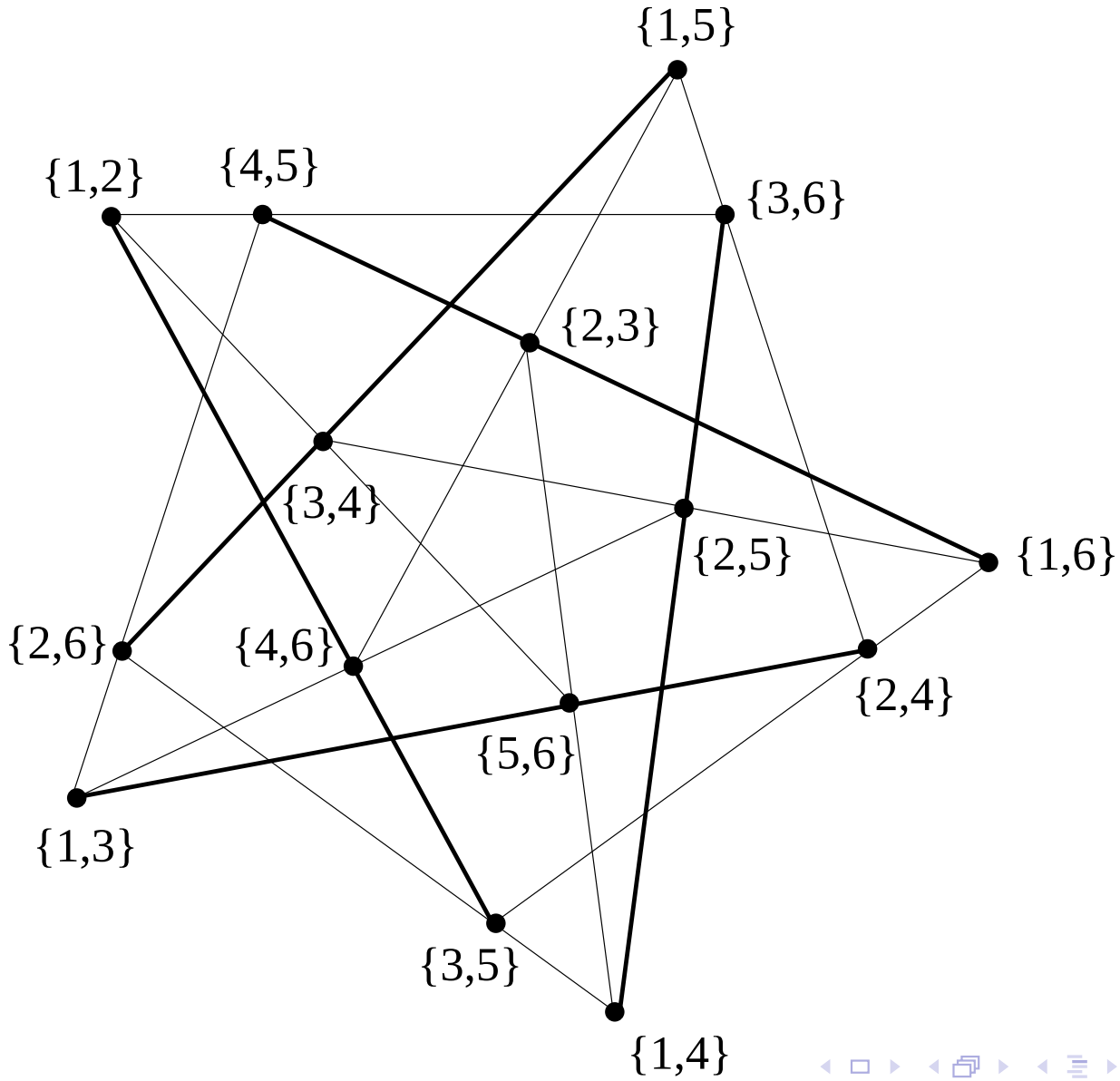
Desargues colouring and 5-CDC

5-Cycle-Double-Cover Conjecture. *Every bridgeless graph contains a collection of ≤ 5 cycles (even subgraphs) s.t. each edge belongs to exactly two of them.*

Theorem

A cubic graph has a Desargues colouring $\Leftrightarrow$ it has a 5-CDC.

Cremona-Richmond configuration 15_3



Cremona-Richmond colouring and Fulkerson's conjecture

Fulkerson's Conjecture. *Every bridgeless cubic graph contains a collection of six perfect matchings s.t. each edge is contained in exactly two of them.*

Theorem

A cubic graph has a Cremona-Richmond colouring $\Leftrightarrow$ it has a double covering of its edges by six perfect matchings.

Cremona-Richmond configuration and Petersen colourings

Petersen colouring conjecture

The edges of every bridgeless cubic graph G can be mapped into the edges of the Petersen graph in such a way that any three mutually incident edges of G are mapped to three mutually incident edges of the Petersen graph.

Definition. Such a mapping is called a *Petersen colouring* of G.

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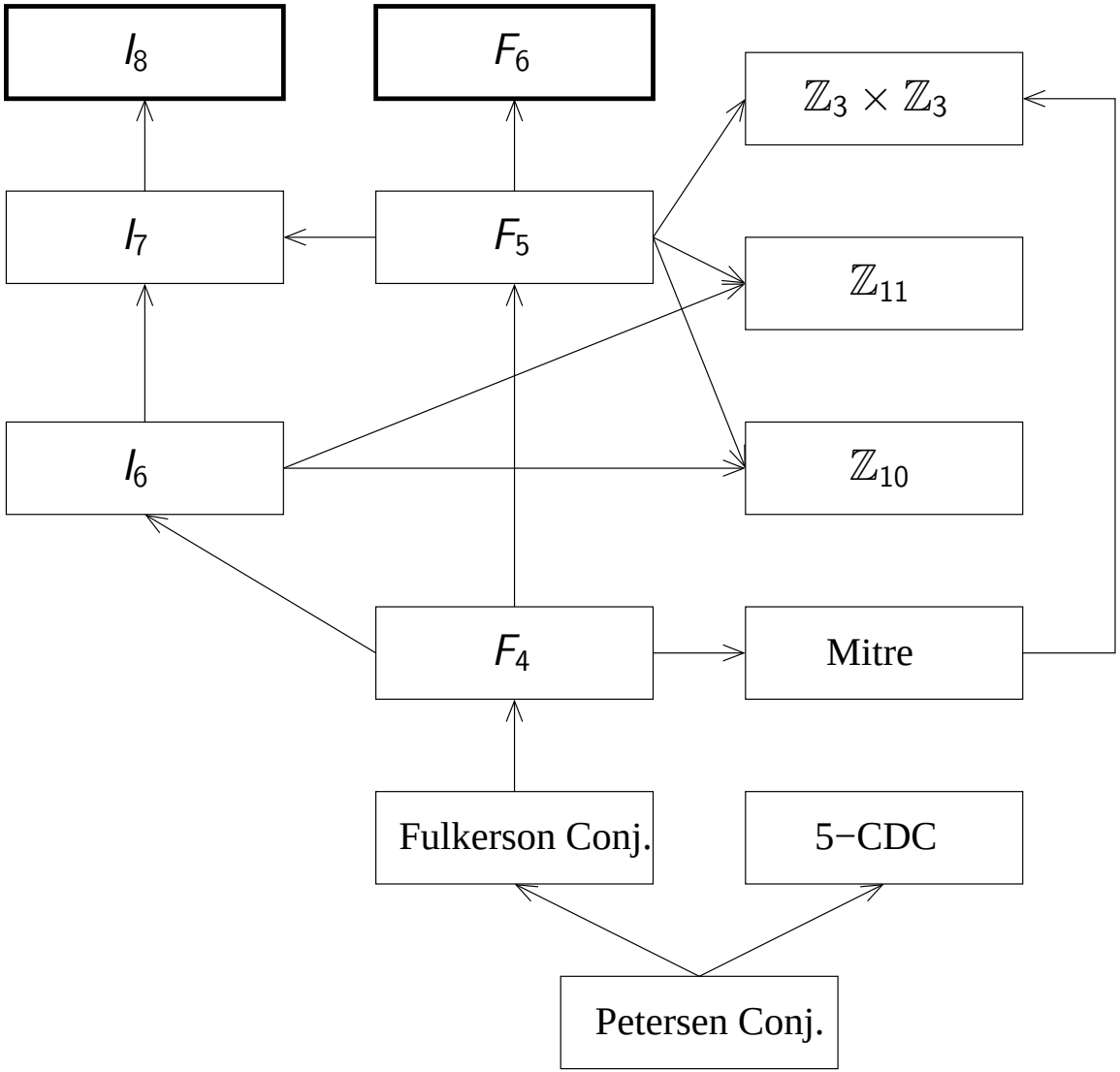
$\mathcal{P}$ = *depleted Cremona-Richmond configuration*

$= 15_3 - \{\text{parallel class of blocks}\}$.

PART III

Conclusion

III. Conclusion: hierarchy of conjectures



THANK YOU!