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Invited talks

Margit Voigt : List colorings of planar graphs

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ABSTRACT

Let $G = (V, E)$ be a graph, let $f : V(G) \rightarrow \mathbb{N}$, and let $k \geq 0$ be an integer. A *list-assignment* L of G is a function that assigns to each vertex v of G a set (list) $L(v)$ of colors: usually each color is a positive integer. We say that L is an *f-assignment* if $|L(v)| = f(v)$ for all $v \in V$, and a *k-assignment* if $|L(v)| = k$ for all $v \in V$.

A *coloring* of G is a function φ that assigns a color to each vertex of G so that $\varphi(v) \neq \varphi(w)$ whenever $vw \in E$. An *L-coloring* of G is a coloring φ of G such that $\varphi(v) \in L(v)$ for all $v \in V$. If G admits an *L-coloring*, then G is *L-colorable*. The graph G is said to be *f-list-colorable* and f is called a *choice function* if G is *L-colorable* for every *f-assignment* L of G . When $f(v) = k$ for all $v \in V$, the corresponding term becomes *k-list-colorable* or *k-choosable*. The *list-chromatic number* or *choice number* $\chi_\ell(G)$ is the least number k such that G is *k-list-colorable*.

In 1979 Erdős, Rubin and Taylor [1] conjectured that all planar graphs are 5-list-colorable, but that not all planar graphs are 4-list-colorable. Both conjectures were proved in 1993.

Theorem 1 ([4],[5]). *Every planar graph is 5-list colorable but there are planar graphs which are not 4-list colorable.*

Hence the situation seems to be clear for planar graphs. However we may ask for conditions which guarantee the 4-list colorability or even the 3-list colorability of planar graphs.

Many authors investigated the list colorability of planar graphs with forbidden cycle lengths. In this talk, we will discuss some aspects of this topic and give an overview on the recent state of research.

Another interesting direction of research deals with variations of the list lengths. We will consider some problems and results related to this topic. Finally, this investigations leads us to the concept of sum-list colorings. Instead of a fixed list length for all vertices we try to minimize the sum over all list lengths.

For a choice function f define $size(f) = \sum_{v \in V} f(v)$. The *sum choice number* $\chi_{sc}(G)$ is the minimum of $size(f)$ over all choice functions of G . The concept was introduced by Garth Isaak in 2002 [2]. An overview on the recent state of research is given in the PhD Thesis of Michelle Lastrina [3]

It is well-known that $\chi_{sc}(G) \leq |V| + |E|$ for every graph G . To see this consider an arbitrary order v_1, \dots, v_n of the vertices of G and define $f(v_i) := 1 +$ the number of vertices v_j adjacent to v_i with $j < i$. Obviously, $\sum_{v \in V} f(v) = |V| + |E|$ and the graph is *L-colorable* by a greedy algorithm for every list assignment L with $|L(v)| = f(v)$. Therefore, a graph G is called *sc-greedy* if $\chi_{sc}(G) = |V| + |E|$.

Corollary 2. *If G is a planar graph with n vertices then $\chi_{sc}(G) \leq 4n - 6$.*

Corollary 3. *For every planar graph G there is a choice function f_G such that G is *L-list colorable* for every list assignment L with $|L(v)| = f_G(v)$ and the average list length of such an assignment is*

$$\frac{\sum_{v \in V} |L(v)|}{|V|} \leq 4 - \frac{6}{n}.$$

In this talk we will discuss some problems and results concerning sum list colorings of planar graphs.

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Maria Chudnovsky : Induced cycles and coloring

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ABSTRACT

The Strong Perfect Graph Theorem states that a graph and all its induced subgraphs have equal clique number and the chromatic number if and only if the graph contains no induced odd cycle of length at least five, and no complement of one. But what happens if only some induced cycles (and no complements) are excluded? Gyarfás made a number of conjectures on this topic, asserting that in many cases the chromatic number is bounded by a function of the clique number. In this talk we discuss recent progress on some of these conjectures. This is joint work with Alex Scott and Paul Seymour.

Mariusz Woźniak : Local irregularity

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ABSTRACT

A *locally irregular graph* is a graph whose adjacent vertices have distinct degrees. We say that a graph G can be decomposed into k locally irregular subgraphs if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph in G . We characterize all connected graphs which cannot be decomposed into locally irregular subgraphs. These are all of odd size and include paths, cycles and a special class of graphs of maximum degree 3.

Moreover we conjecture that apart from these exceptions all other connected graphs can be decomposed into three locally irregular subgraphs.

We also investigate a total version of this problem, where in some sense also the vertices are being prescribed to particular subgraphs of a decomposition.

The both concepts are closely related to the known 1-2-3 Conjecture and 1-2 Conjecture, as well as to other similar problems concerning edge and arc colorings of graphs and digraphs, respectively.

Xuding Zhu: Online list colouring of graphs

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ABSTRACT

The on-line colouring game on a family \mathcal{F} of graphs is played by two players: Presenter and Algorithm. In each round, Presenter add one new vertex v to the existing graph, including all edges connecting v to earlier vertices. The resulting graph must be in the family \mathcal{F} . Algorithm colours v with a colour not used by any of its existing neighbours. The $(n$ -round) *on-line chromatic number* of \mathcal{F} is the least number of colours sufficient for Algorithm to play the game for n -rounds. Equivalently, it is the maximum number of colours Presenter can force the Algorithm to use in n -rounds. It was proved by Lovasz, Saks and Trotter that the $(n$ -round) on-line chromatic number of bipartite graphs is at most $2\log n$. A recent result of Bianchi shows that it is at least $1.13746\log n - 0.49887$. We improve this lower bound to $2\log n - 10$, which differs from the upper bound by an additive constant. It was proved by Kierstead that graphs of odd girth 7 has $(n$ -round) on-line chromatic number $O(n^{1/2})$, and we give a lower bound and prove that this family of graphs has $(n$ -round) on-line chromatic number $\Omega(\frac{n}{\log n})^{1/3}$.

This is a joint work with Grzegorz Gutowski, Jakub Kozik and Piotr Micek.

Gary MacGillivray : The Firefighter Problem: A survey of directions, results, and problems

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ABSTRACT

Imagine that a fire breaks out at one or more vertices of a graph, and at each time interval spreads to all neighbouring vertices that have not been protected so far. This is the general setup for the Firefighter Problem, which has attracted considerable attention since being introduced by Hartnell in 1995. We shall survey various directions that have been pursued, and results that have been obtained. Examples include whether the fire can be contained on infinite grids, algorithms, and complexity theorems. Open problems and possible directions for future research will also be discussed.

Oriol Serra : Isoperimetric orders

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ABSTRACT

The isoperimetric problem for graphs asks for the cardinality of the minimum (edge or vertex) boundary among sets of vertices with given cardinality. An ordering of the vertex set of a graph is isoperimetric if its initial segments are optimal sets for the isoperimetric problem. The classical theorems of Kruskal-Katona and Harper show that the n -cube admits an isoperimetric order. In the talk we will discuss a local-global principle which extends analogous results for certain classes of cartesian products of graphs and provides simple proofs for several known results. Algorithmic aspects concerning the existence of a set of nested solutions for graphs with bounded treewidth will be also addressed.

The abstracts of the accepted contributions are listed in the order following the schedule: For each half-day block from Wednesday morning through Saturday morning, the abstracts of all the talks scheduled in session A are listed before the abstracts of all the talks in session B (and session C, eventually).

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A sufficient condition leading to the domination number of a bipartite graph

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EXTENDED ABSTRACT

In this paper, a graph $G = (V, E)$ is undirected with no loop. A set of vertices X such that the closed neighborhood satisfies $N_G[X] = V$ is called a dominating set. The minimum and maximum cardinality taken over all minimal dominating sets of G are the domination number $\gamma(G)$ and upper domination number $\Gamma(G)$ respectively. Also the minimum cardinality taken over all maximal independent sets of G is the independent domination number $i(G)$. A sufficient condition for $\gamma(G) = i(G)$ was represented for a general graph G , that is free from an induced subgraph isomorphic to claw $K_{1,3}$ [?]. A bipartite graph G has an induced subgraph isomorphic to $K_{1,3}$ if the maximum degree $\Delta(G) \geq 3$ so that $\gamma(G)$ is not necessarily equal to $i(G)$.

For a graph G , a subgraph I is defined as two adjacent vertices v and w in $V(G)$ and its neighbors such that the degrees on I satisfy $d_I(v) \geq 3$ and $d_I(w) \geq 3$. In this paper, v and w are called core vertices. We observe that I is a forbidden subgraph for $\gamma(G) = i(G)$ with a simplest proof. A property of I is remarkable for dominating sets of a graph. It characterizes 3-connected graphs, all of which are arranged by contracting edges between core vertices [2], where the independent domination number and the domination number are significantly different [3] by existence of I .

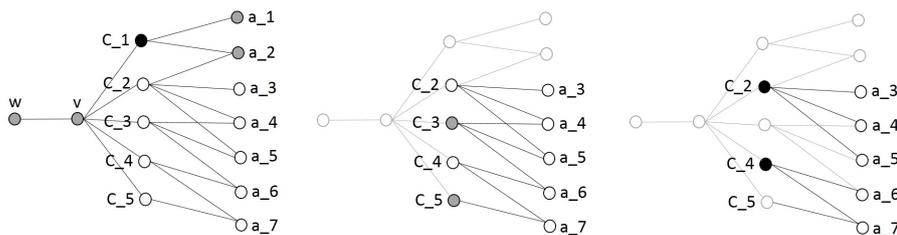
We have a different approach to the domination number formulation. A bipartite graph G is decomposed by a vertex set I , which is $G = I_1 \cup \dots \cup I_k \cup F$ pairwise disjoint. Obviously, $\gamma(G_1 \cup G_2) \leq \gamma(G_1) + \gamma(G_2)$ for arbitrary graphs G_1 and G_2 . On the basis of it, we present a sufficient condition led to an equation $\gamma(G) = \gamma(I_1) + \dots + \gamma(I_k) + \gamma(F) = 2k + i(F)$ as a main theorem. The k -dominating graph $D_k(G)$ explains the proof of the theorem. It is defined by a vertex set in which a vertex is corresponding to a dominating set of G with the cardinality at most k and an edge set of vertex pairs which differ by either adding or deleting a single vertex of corresponding dominating sets. For any non-trivial bipartite graph G , $D_{\Gamma(G)+1}(G)$ is connected [6], so a sequence of minimal dominating sets is a factor.

The independent domination number of a bipartite graph is led in a certain case. The equivalency of $i(G) = |V|/2$ for a bipartite graph G was shown in [4]. From the proof, $i(G)$ is generally formulated with variants represented on a partition by complete bipartite graphs.

The domination number of a bipartite graph is related to a compactness of a topological space. The graphical representation of an instance of the minimum cover problem is a bipartite graph which solves the domination number with the use of statements in this paper.

illustration :

the instance of the minimum cover problem with a minimum cover $\{C_1, C_2, C_4\}$



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Parametrized complexity of length-bounded cuts and multi-cuts

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EXTENDED ABSTRACT

The study of network flows and cuts begun in 1960's by the work of Ford and Fulkerson [4]. It has many generalizations and applications now. We are interested in a generalization of cuts related to the flows using only short paths.

Length bounded cuts. Let $s, t \in V$ be two distinct vertices of a graph $G = (V, E)$ – we call them source and sink, respectively. We call a subset of edges $F \subseteq E$ of G an L -bounded cut (or L -cut for short), if the length of the shortest path between s and t in the graph $(V, E \setminus F)$ is at least $L + 1$. We measure the length of the path by the number of its edges. In particular, we do not require s and t to be in distinct connected components as in the standard cut, instead we do not allow s and t to be close to each other. We call the set F a *minimal L -cut* if it has the minimum size among all L -bounded cuts of the graph G .

We state the cut problem formally:

PROBLEM: MINIMUM LENGTH BOUNDED CUT (MLBC)
Instance: graph $G = (V, E)$, vertices s, t and integer $L \in \mathbb{N}$
Goal: find a minimal L -bounded s, t cut $F \subseteq E$

Length bounded flows were first considered by Adámek and Koubek [1]. They showed that the max-flow min-cut duality cannot hold and also that integral capacities do not imply integral flow. Finding a minimal length bounded cut is NP-hard on general graphs for $L \geq 4$ as was shown by Itai et al. [5] and Baier et al. [2]. Itai et al. [5] also found algorithms for finding L -bounded cut with $L = 1, 2, 3$ in polynomial time by reducing it to the usual network cut in an altered graph. The algorithm of [5] uses the fact that paths of length 1, 2 and 3 are edge disjoint from longer paths, while this does not hold for length at least 4.

Tree-decomposition Let $G = (V, E)$ be a graph. We say that $T = (\mathcal{W}, F)$ is a *tree decomposition* of the graph G if $\mathcal{W} \subseteq 2^V$, T is a tree and the following holds:

1. for each $v \in V$ there exists a $X \in \mathcal{W}$ such that $v \in X$,
2. for each $e \in E$ there exists a $X \in \mathcal{W}$ such that $e \subseteq X$,
3. for each $v \in V$ the graph $T[X_v]$ is connected, where $X_v = \{X \in \mathcal{W} : v \in X\}$.

We call the elements of the set \mathcal{W} the *nodes*, and the elements of the set F the *decomposition edges*.

We define a width of a tree decomposition (\mathcal{W}, F) as $\max_{X \in \mathcal{W}} |X| - 1$ and the *tree-width* $\text{tw}(G)$ of a graph G as the minimal width of a tree decomposition of the graph G . Moreover, if the decomposition is a path we speak about *path-width* of G , which we denote as $\text{pw}(G)$.

Our Results. Our main contribution is an FPT-algorithm for the MLBC problem, its consequences and an FPT-algorithm for a more general multi-terminal version problem. We also prove W[1]-hardness of the MLBC problem. We denote by n the number of vertices of the input graph.

Theorem 1. *Let G be a graph of tree-width k . Let s and t be two distinct vertices of G . Then for any $L \in \mathbb{N}$ an minimal L -CUT between s and t can be found in time $O((L^{k^3})^2 \cdot 2^{k^2} \cdot n)$.*

For the algorithm we use dynamic programming techniques based on a nice tree-decomposition of the input graph, see Kloks [6] for more information about this type of decomposition.

Proposition 2. [6] *Given a tree decomposition of a graph G with n vertices that has width k and $O(n)$ nodes, we can find a nice tree decomposition of G that also has width k and $O(n)$ nodes in time $O(n)$.*

As a corollary when parametrized by the length of paths L only, this algorithm runs in time $n^{f(L)}$ for some computable function f and so the problem is in XP class. We were interested whether there can be a FPT-algorithm for this problem when parametrized only by tree-width or even path-width. However, this seems unlikely due to following theorem.

Another corollary of the Theorem 1 is the following theorem.

Corollary 3. *Let G be a graph that has vertex cover of size k . Let s and t be two distinct vertices of G . Then for any $L \in \mathbb{N}$ an minimal L -CUT between s and t can be found in time $O(f(k) \cdot n)$ for some recursive function f .*

Theorem 4. MINIMAL LENGTH BOUNDED CUT parametrized by path-width is W[1]-hard.

Our ideas for a hardness result were inspired by work of Dom et al. [3]. They proved W[1]-hardness of CAPACITATED VERTEX COVER and CAPACITATED DOMINATING SET parametrized by the tree-width of the input graph by a reduction from MULTICOLOR CLIQUE. We remark that their reduction also proves W[1]-hardness of these problems when parametrized by the path-width of the input graph.

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On parameterized complexity to determine b-chromatic and partial Grundy numbers

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EXTENDED ABSTRACT

The Grundy number of a graph G , denoted by $\Gamma(G)$, is the largest integer k such that there exists a partition of $V(G)$ into k independent sets V_1, \dots, V_k and every vertex of V_i is adjacent to at least one vertex in V_j , for every $j < i$. A vertex v of color i is a *Grundy vertex* if v is adjacent to at least one vertex colored j , for every $j < i$. A *Grundy k -coloring* is a proper k -coloring such that every vertex is a Grundy vertex. A *partial Grundy k -coloring* is a proper k -coloring such that every color class contains at least one Grundy vertex. The *partial Grundy number* of a graph G , denoted by $\partial\Gamma(G)$, is the largest integer k such that there exists a partial Grundy k -coloring of G .

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*, denoted by $\varphi(G)$. A vertex v of color i is a *b-vertex* if v is adjacent to at least one vertex colored j , for every $j \neq i$. A *b- k -coloring* is a proper k -coloring such that every color class contains a b-vertex. The b-chromatic number of a graph G is the largest integer k such that there exists a b- k -coloring of G . The concept of b-coloring has been introduced by Irving and Manlove [2]. Since the introduction of the b-coloring, a large number of papers has appeared [3, 4].

We introduce the following decision problems with parameter t :

b-coloring

Instance : A graph G .

Parameter : An integer t .

Question: Does $\varphi(G) \geq t$?

Partial Grundy coloring

Instance : A graph G .

Parameter : An integer t .

Question: Does $\partial\Gamma(G) \geq t$?

The concept of t -atom was introduced by Zaker [1]. The family of t -atoms is a finite family which only contains finite graphs, for a fixed t . Thus, the presence of a t -atom can be determined in polynomial time for a fixed t .

Theorem 1 (Zaker '06 [1]). *For a given graph G , $\Gamma(G) \geq t$ if and only if G contains an induced t -atom.*

Corollary 2. *For a given graph G of order n , determining if $\Gamma(G) \geq t$ can be done by an algorithm in $O(n^{2^{t-1}})$.*

We introduce the equivalent of the t -atoms for b-coloring and partial Grundy coloring. These concepts are used to prove that determining some properties is polynomial time solvable if the parameter t is fixed.

Definition 3. *Let t be an integer with $t \geq 2$. We begin by taking an independent set of t vertices, denoted by $C(t)$, called the center of the b- t -atom and associating integers of $\{1, \dots, t\}$ to each vertex of H . The vertices should have pairwise different associated integers. Let c_k be the vertex of $C(t)$ with associated integer k , $1 \leq k \leq t$. Let k be an integer with $1 \leq k \leq t$. A k -dominating operation on a graph G containing $C(t)$, consists in adding a set D of ℓ vertices, with $0 \leq \ell \leq t - 1$ and associated*

integer k and adding edges between $G - \{c_k\}$ and $D \cup \{c_k\}$ such that every vertex of $C(t) - \{c_k\}$ is adjacent to a vertex of $D \cup \{c_k\}$ and $D \cup \{c_k\}$ is independent. Note that there can be edges between vertices of $D \cup \{c_k\}$ and vertices of $G - C(t)$. The family of graphs $\mathcal{A}^b(t)$ is obtained as follows: let $\mathcal{G}_1 = \{C(t)\}$. The family of graphs \mathcal{G}_i , for $i \geq 2$, is obtained from the graphs in \mathcal{G}_{i-1} by doing an i -dominating operation on these graphs. The graphs in \mathcal{G}_t are the graphs in $\mathcal{A}^b(t)$. A graph in $\mathcal{A}^b(t)$ is called a b - t -atom.

Note that the associated integers correspond to a b -coloring of the b - t -atom. The associated integers of the b - t -atom are not considered anymore when we use the notion of induced subgraph.

Property 4. For every graph G in $\mathcal{A}^b(t)$, we have $|V(G)| \leq t^2$.

Definition 5. Let G be graph. A b - t -atom G is minimal if G is not contained as induced subgraph in another b - t -atom G' , for $G \neq G'$.

Note that the only minimal b -2-atom is P_2 . The minimal b -3-atoms are C_3 , P_3 , C_5 , $P_3 \cup P_4$ and $P_3 \cup P_3 \cup P_3$.

For an induced subgraph A of G , let $N(A) = \{v \in V(G) \mid uv \in E(G), u \in V(A)\}$. A b - t -atom A is *feasible* in G if there exists a b - t -coloring of $V(A)$ that can be extended to a coloring of $N(A)$ without using new colors.

Theorem 6. We have $\varphi(G) = t$ for a given graph G if and only if G contains an induced feasible minimal b - t -atom and no induced feasible minimal b - t' -atom, for $t' > t$.

For a given graph G , let $\varphi_r(G)$ be the maximum value of the b -chromatic number of an induced subgraph of G .

Theorem 7. For a graph G , we have $\varphi_r(G) \geq t$ if and only if G contains an induced b - t -atom.

Corollary 8. Let G be a graph of order n . There exists an algorithm in $O(n^{t^2})$ to determine if $\varphi_r(G) \geq t$.

Moreover, we can construct the equivalent of the b - t -atoms for partial Grundy coloring. We call these graphs the partial-Grundy- t -atoms. We have the following results for the partial Grundy number:

Theorem 9. For a graph G , we have $\partial\Gamma(G) \geq t$ if and only if G contains an induced partial-Grundy- t -atom.

Corollary 10. Let G be a graph of order n . There exists an algorithm in $O(n^{t^2})$ to determine if $\partial\Gamma(G) \geq t$.

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Towards bounds of relational complexity

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EXTENDED ABSTRACT

A *relational structure* (or simply *structure*) \mathbf{A} is a pair $(A, (R_{\mathbf{A}}^i : i \in I))$, where $R_{\mathbf{A}}^i \subseteq A^{\delta_i}$ (i.e., $R_{\mathbf{A}}^i$ is a δ_i -ary relation on A). The family $(\delta_i : i \in I)$ is called the *type* Δ and it is usually assumed to be fixed and well understood from the context. Let moreover $\text{Rel}(\Delta)$ denotes the class of all (countable) relational structures of type Δ . The underlying set A is called the domain of \mathbf{A} , whose cardinality is the corresponding cardinality of the whole relational structure. Here we consider only countable or finite structures. Moreover we see relational structures as a generalization of digraphs and adopt standard graph theoretic terms (such as isomorphism, homomorphism or connected structures). For more details, see [6].

A relational structure \mathbf{A} is called *ultrahomogeneous* if every isomorphism between two induced finite substructures of \mathbf{A} can be extended to an automorphism of \mathbf{A} . Ultrahomogeneity describes a high degree of symmetry for relational structures far beyond vertex or edge transitivity. For countable structures ultrahomogeneity is usually studied for class \mathcal{K} of structures and corresponding universal structure – such structure contains all structures from class \mathcal{K} as induced substructures. For a class \mathcal{K} of relational structures, we denote by $\text{Age}(\mathcal{K})$ the class of all finite structures isomorphic to an (induced) substructure of some $\mathbf{A} \in \mathcal{K}$ and call it the *age* of \mathcal{K} . For a structure \mathbf{A} , the age of \mathbf{A} , $\text{Age}(\mathbf{A})$, is $\text{Age}(\{\mathbf{A}\})$. The classification program for ultrahomogeneous structures is one of the celebrated lines of research in the model theory, see [1, 8]. Namely classical result of Fraïssé [4] provides conditions for age to generate ultrahomogeneous universal structure. The class of structures studied in this work is the class of all structures \mathbf{A} for which there is no homomorphism $\mathbf{F} \rightarrow \mathbf{A}$, $\mathbf{F} \in \mathcal{F}$ denoted as $\text{Forb}_{\text{h}}(\mathcal{F})$. The existence of a universal structure for this class is guaranteed for every finite \mathcal{F} , see [3].

Let $\Delta' = (\delta'_i : i \in I')$ be a type containing type Δ . (That is $I \subseteq I'$ and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{X} \in \text{Rel}(\Delta')$ may be viewed as a structure $\mathbf{A} = (A, (R_{\mathbf{A}}^i : i \in I)) \in \text{Rel}(\Delta)$ together with some additional relations for $i \in I' \setminus I$. We will thus also write $\mathbf{X} = (A, (R_{\mathbf{A}}^i : i \in I), (R_{\mathbf{X}}^i : i \in I' \setminus I))$. We call \mathbf{X} a *lift* of \mathbf{A} and \mathbf{A} is called the *shadow* of \mathbf{X} . Note that a lift is also in the model-theoretic setting called an *expansion*.

Let \mathbf{A} be a relational structure and let $\text{Aut}(\mathbf{A})$ be the automorphism group of \mathbf{A} . A k -ary relation $\rho \subseteq A^k$ is an *invariant* of $\text{Aut}(\mathbf{A})$ if $(\alpha(x_1), \dots, \alpha(x_k)) \in \rho$ for all $\alpha \in \text{Aut}(\mathbf{A})$ and all $(x_1, \dots, x_k) \in \rho$. Let $\text{Inv}_k(\mathbf{A})$ denote the set of all k -ary invariants of $\text{Aut}(\mathbf{A})$ and let $\text{Inv}(\mathbf{A}) = \bigcup_{k \geq 1} \text{Inv}_k(\mathbf{A})$, $\text{Inv}_{\leq k} = \bigcup_{1 \leq k' \leq k} \text{Inv}_{k'}(\mathbf{A})$. It can be easily shown that for every structure $\mathbf{A} = (A, (R_{\mathbf{A}}^i : i \in I))$ the lift $(A, (R_{\mathbf{A}}^i : i \in I), \text{Inv}(\mathbf{A}))$ (considering possibly an infinite type) is an ultrahomogeneous structure. For a structure \mathbf{A} the *relational complexity*, $\text{rc}(\mathbf{A})$, of a \mathbf{A} is the least k such that $(A, (R_{\mathbf{A}}^i : i \in I), \text{Inv}_{\leq k}(\mathbf{A}))$ is ultrahomogeneous, if such a k exist. If no such k exist, we say that the relational complexity of \mathbf{A} is not finite and write $\text{rc}(\mathbf{A}) = \infty$. Note that if $\text{rc}(\mathbf{A})$ is less than the arity of some relation in Δ , then $\text{rc}(\mathbf{A})$ may be lower than the relational complexity of $\text{Aut}(\mathbf{A})$ as defined in [2].

We can restate the result of Fraïssé as definition of the critical property of age \mathcal{K} such that there exists structure \mathbf{U} , $\text{Age}(\mathbf{U}) = \mathcal{K}$, satisfying $\text{rc}(\mathbf{A}) = 0$. We seek, for given n , the structural properties of age \mathcal{K} such that there exists structure \mathbf{U} , $\text{Age}(\mathbf{U}) = \mathcal{K}$, $\text{rc}(\mathbf{A}) = n$.

Since for many ages there exists structures with trivial automorphism group we restrict ourselves to ω -categorical structures. Recall that structure is ω -categorical if and only if it has only finitely many orbits on n -tuples, for every n , and thus also there are only finitely many invariant relations of arity n , see [6].

Since there is no 1-1 correspondence in between ω -categorical structures and their ages, we can further restrict ourselves even more. Structure \mathbf{A} with $\text{Age}(\mathbf{A}) = \mathcal{K}$ is *existentially complete* if for

every structure \mathbf{B} , such that $\text{Age}(\mathbf{B}) = \mathcal{K}$ and the identity mapping (of A) is an embedding $\mathbf{A} \rightarrow \mathbf{B}$, every existential statement ψ which is defined in \mathbf{A} and true in \mathbf{B} is also true in \mathbf{A} . Moreover by [3] for every age \mathcal{K} defined by forbidden monomorphisms with ω -categorical universal structure there is also up to isomorphism unique ω -categorical, existentially complete, and ω -saturated universal structure. This in fact holds more generally. In such cases, for a given age \mathcal{K} , the *canonical universal structure* of age \mathcal{K} is the unique ω -categorical, existentially complete, and ω -saturated structure \mathbf{U} such that $\text{Age}(\mathbf{U}) = \mathcal{K}$. Given an age \mathcal{K} we can thus ask:

- I. What is the minimal relational complexity of an ω -categorical structure \mathbf{U} such that $\text{Age}(\mathbf{U}) = \mathcal{K}$?
- II. What is the relational complexity of the canonical universal structure of age \mathcal{K} ?

We consider universal structures for class $\text{Forb}_h(\mathcal{F})$ where \mathcal{F} is a family of connected structures. For a structure $\mathbf{A} = (A, (R_{\mathbf{A}}^i, i \in I))$, the *Gaiřman graph* is the graph $G_{\mathbf{A}}$ with vertices A and all those edges which are a subset of a tuple of a relation of \mathbf{A} , i.e., $G = (A, E)$, where the neighborhood of $\{x, y\} \in E$ if and only if $x \neq y$ and there exists a tuple $\vec{v} \in R_{\mathbf{A}}^i$, $i \in I$, such that $x, y \in \vec{v}$. For a structure \mathbf{A} and a subset of its vertices $B \subseteq A$, the *neighborhood* of set B is the set of all vertices of $A \setminus B$ connected in $G_{\mathbf{A}}$ by an edge to a vertex of B . We denote by $G_{\mathbf{A}} \setminus B$ the graph created from $G_{\mathbf{A}}$ by removing the vertices in B .

A *g-cut* in \mathbf{A} is a subset C of A that is a vertex cut of $G_{\mathbf{A}}$. A *g-cut* C is *minimal g-separating* in \mathbf{A} if there exists structures $\mathbf{A}_1 \neq \mathbf{A}_2$ induced by \mathbf{A} on two connected components of $G_{\mathbf{A}} \setminus C$ such that C is the intersection of the neighborhood of A_1 and the neighborhood of A_2 in \mathbf{A} . A family of structures is called *minimal* if and only if all structures in \mathcal{F} are cores and there is no homomorphism between two structures in \mathcal{F} .

Theorem 1 (Hartman, Hubička, Nešetřil to appear [5]). *Let \mathcal{F} be a finite minimal family of finite connected relational structures and \mathbf{U} an ω -categorical universal structure for $\text{Forb}_h(\mathcal{F})$. Denote by n the size of the largest minimal *g-separating g-cut* in \mathcal{F} . Then (a) $\text{rc}(\mathbf{U}) \geq n$; (b) if \mathbf{U} is the canonical universal structure for $\text{Forb}_h(\mathcal{F})$, then $\text{rc}(\mathbf{U}) = n$.*

For further related things see [7]. This work has been supported by by grant ERC-CZ LL-1201 of the Czech Ministry of Education and CE-ITI P202/16/6061 of GAČR.

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On oriented and 2-edge colored graphs

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EXTENDED ABSTRACT

We consider simple graphs. A graph parameter p is *monotonous* if for every subgraph G' of G , we have $p(G') \leq p(G)$. Two monotonous parameters p_1 and p_2 are *equivalent* if for every graph class \mathcal{G} , we have $\max\{p_1(G) \mid G \in \mathcal{G}\} = O(1)$ if and only if $\max\{p_2(G) \mid G \in \mathcal{G}\} = O(1)$. We note $p_1 \sim p_2$ if p_1 and p_2 are equivalent. Notice that $p_1 \sim p_2$ if and only if for every graph G , we can bound $p_1(G)$ in terms of $p_2(G)$ and vice-versa.

In this paper, we investigate the family \mathcal{F}_a of graph parameters equivalent to the acyclic chromatic number χ_a .

An (n, m) -mixed graph M is a graph in which some pairs of vertices are linked by arcs and some are linked by edges such that the arcs are colored with n colors, the edges are colored with m colors, and the underlying graph of M is simple. For $(n, m) \neq (0, 0)$, the mixed chromatic number $\chi_{(n,m)}(M)$ is the minimum of vertices of an (n, m) -mixed graph T such that there exists a homomorphism from M to T that is compatible with the colors of the arcs and the edges and with the orientation of the arcs. We define $\chi_{(n,m)}(G)$ as the maximum of $\chi_{(n,m)}(M)$ over the (n, m) -mixed graphs M having G as underlying simple graph of M . We have that

1. $\chi = \chi_{(0,1)}$ corresponds to chromatic number,
2. $\chi_o = \chi_{(1,0)}$ corresponds to the oriented chromatic number,
3. $\chi_2 = \chi_{(0,2)}$ corresponds to the 2-edge-colored chromatic number,
4. for $n' \leq n$ and $m' \leq m$, $\chi_{(n',m')} \leq \chi_{(n,m)}$,
5. $\chi_{(n,m)} \leq \chi_a \cdot (2n + m)^{\chi_a - 1}$ [5],
6. $\chi_a \leq (\chi_o)^2 + (\chi_o)^{3 + \lceil \log_2 \log_2 \chi_o \rceil}$ [3],
7. $\chi_{P_4} \leq \chi_a \leq \chi_a \cdot (2\chi_a - 1)$ [1], where χ_{P_4} is the star chromatic number,
8. $\chi_p \leq \chi_o \leq 2\chi_p$ [2], where χ_p is the pushable chromatic number,
9. $\chi_s \leq \chi_2 \leq 2\chi_s$ [4], where χ_s is the signed chromatic number.

By (5) and (6), we have $\chi_o \sim \chi_a$, so that $\chi_o \in \mathcal{F}_a$. We have $\chi_{P_4} \in \mathcal{F}_a$ since $\chi_{P_4} \sim \chi_o$ by (7) and $\chi_p \in \mathcal{F}_a$ since $\chi_p \sim \chi_o$ by (8). Obviously, $\chi \notin \mathcal{F}_a$ since $\chi(K_{n,n}) = 2$ whereas $\chi_a(K_{n,n}) = n + 1$.

Theorem 1. For every graph G , $\chi_o(G) \leq \chi(G) \cdot \chi_2(G)$.

Proof. Consider a proper coloring $c : V(G) \mapsto \{1, \dots, \chi(G)\}$ of the simple graph G . Consider an oriented graph J_o whose underlying simple graph is G . Consider a 2-edge-colored graph J_2 whose underlying simple graph is G . We say that J_2 corresponds to J_o if for every edge uv of G such that $c(u) < c(v)$, the edge uv is colored + in J_2 if there is the arc uv in J_o and uv is colored - if there is the arc vu in J_o . Let G_o be an orientation of G such that $\chi_o(G_o) = \chi_o(G)$. Let G_2 be the 2-edge-colored graph corresponding to G_o . We consider a 2-edge-colored homomorphism h_2 from G_2 to a 2-edge-colored T_2 such that $V(T_2) = \{1, \dots, \chi_2(G_2)\}$. We define the mapping h_o as the cartesian product of c and h_2 , that is, for every vertex $v \in V(G)$, $h_o(v) = (c(v), h_2(v))$. It is not hard to check that h_o defines an oriented homomorphism from G_o to the target graph T_o defined as follows:

- $V(T_o) = [1, \dots, \chi(G)] \times [1, \dots, \chi_2(G_2)]$,
- For every i, j, m, n such that $1 \leq i \leq j \leq \chi(G)$, $1 \leq m \leq \chi_2(G_2)$, $1 \leq n \leq \chi_2(G_2)$, $m \neq n$, we add in T_o the arc $(i, m)(j, n)$ if mn is an edge colored $+$ in T_2 and we add in T_o the arc $(j, n)(i, m)$ if mn is an edge colored $-$ in T_2 .

We thus have $\chi_o(G) = \chi_o(G_o) \leq \chi(G) \cdot \chi_2(G_2) \leq \chi(G) \cdot \chi_2(G)$. □

By Theorem 1, we have $\chi_o \leq (\chi_2)^2$. Since $\chi_2 \leq \chi_a \cdot 2^{\chi_a - 1}$ by (5), we obtain $\chi_2 \in \mathcal{F}_a$. For every $(n, m) \neq (0, 1)$, we have $\chi_o \leq \chi_{(n, m)}$ or $\chi_2 \leq \chi_{(n, m)}$ by (4). Since $\chi_o \in \mathcal{F}_a$, $\chi_2 \in \mathcal{F}_a$, and $\chi_{(n, m)} \leq \chi_a \cdot (2n + m)^{\chi_a - 1}$ by (5), this implies that $\chi_{(n, m)} \in \mathcal{F}_a$ for every $(n, m) \neq (0, 1)$. Finally, since $\chi_s \sim \chi_2$ by (9), we have that $\chi_s \in \mathcal{F}_a$.

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On the incidence chromatic number of sparse graphs

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EXTENDED ABSTRACT

Let G be a simple, non-empty graph. We let $V(G)$ and $E(G)$ denote its vertex and edge set, respectively.

An *incidence* in G is a pair (u, e) with $u \in V(G)$ and $e \in E(G)$, where u is an endpoint of e . In other words, incidences of G are in natural bijection with edges in the graph G_s obtained from G by subdividing every edge once. The set of all incidences in G is denoted by $I(G)$, where $I(G) = \{(v, e) \in V(G) \times E(G) : \text{edge } e \text{ is incident to } v\}$.

Two incidences (u, e) and (v, f) are adjacent if one of the following holds :

- i*) $u = v$, *ii*) $e = f$ and *iii*) the edge $uv = e$ or $uv = f$.

A *k-incidence coloring* of a graph G is defined as a function ϕ on $I(G)$ into a set of colors $C = \{1, 2, \dots, k\}$, such that adjacent incidences are assigned distinct colors. We denote by $\chi_i(G)$ the *incidence chromatic number* of G which is the smallest integer k such that G admits a k -incidence coloring. The notion of incidence coloring was introduced by Brualdi and Massey [1] in 1993. They proved the following theorem :

Theorem 1 (Brualdi and Massey [1]). *For every graph G , $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.*

And they posed the Incidence Coloring Conjecture, which states that:

Conjecture 2 (Brualdi and Massey [1]). *For every graph G , $\chi_i(G) \leq \Delta(G) + 2$.*

However, in 1997, Guiduli [4] disproved the Incidence Coloring Conjecture showing that Paley graphs have an incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. He also improved the upper bound proposed by Brualdy and Massey in Theorem 1.

Theorem 3 (Guiduli [4]). *For every graph G , $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$.*

The maximum average degree is defined as $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$. This a conventional measure of sparseness of arbitrarily graphs (not necessary planar). An upper bound for the incidence chromatic number in terms of the maximum average degree was given by Dolama and Sopena [3] in 2005. They started looking for such relationships in the case of graphs with low maximum average degree (*i.e.* not only bounded, but bounded by a small constant). However, earlier theorems have implications on graphs with bounded maximum average degree (*i.e.* bounded by any constant).

Theorem 4. [2] *Let $k \in \mathbb{N}$, and G be a k -degenerate graph. Then $\chi_i(G) \leq \Delta(G) + 2k - 1$.*

We know that, for every integer k , a graph G with $\text{mad}(G) < k$ is $(k-1)$ -degenerate, the following corollary, can be easily derived from Theorem 4

Corollary 5. *Let $k \in \mathbb{N}$, if G is a graph with $\text{mad}(G) < k$, then $\chi_i(G) \leq \Delta(G) + 2k - 3$.*

By allowing a lower bound on the maximum degree $\Delta(G)$, we seek to reduce the number of colors necessary for an incidence coloring.

Theorem 6. Let $k \in \mathbb{N}$, and G be a graph with maximum degree $\Delta(G)$ and maximum average degree $\text{mad}(G) < k$.

1. If $\Delta(G) \geq \frac{k^2}{2} + \frac{3k}{2}$, then $\chi_i(G) \leq \Delta(G) + k - 1$.
2. For all $\alpha > 0$, if $\Delta(G) \geq \frac{3\alpha+1}{2\alpha}k$, then $\chi_i(G) \leq \Delta(G) + \lceil (1 + \alpha)k \rceil - 1$.

However, Theorem 6 can be proved with a relatively simple discharging argument, and gives hope that further development of more complicated reducible configurations and discharging rules might unlock generic results with consequences beyond purely extremal.

When considering small values of k , Theorem 6.1 cannot compete with specific results, as we can for example obtain the following results through discharging arguments that are purely local.

Theorem 7.

1. Let G be a graph with $\text{mad}(G) < 4$, then $\chi_i(G) \leq \Delta(G) + 3$ for every $\Delta(G) \geq 7$.
2. Let G be a graph with $\text{mad}(G) < \frac{9}{2}$, then $\chi_i(G) \leq \Delta(G) + 4$ for every $\Delta(G) \geq 9$ or $\Delta(G) \leq 4$.
3. Let G be a graph with $\text{mad}(G) < 5$, then $\chi_i(G) \leq \Delta(G) + 5$ for every $\Delta(G) \leq 5$ or $9 \leq \Delta(G) \leq 19$. If $6 \leq \Delta(G) \leq 8$ then $\chi_i(G) \leq \Delta(G) + 6$. If $\Delta(G) \geq 20$ then $\chi_i(G) \leq \Delta(G) + 4$.
4. Let G be a graph with $\text{mad}(G) < 6$, then $\chi_i(G) \leq \Delta(G) + 6$ for every $\Delta(G) \leq 6$ or $12 \leq \Delta(G) \leq 26$. If $7 \leq \Delta(G) \leq 11$ then $\chi_i(G) \leq \Delta(G) + 7$. If $\Delta(G) \geq 27$ then $\chi_i(G) \leq \Delta(G) + 5$.

As every planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$, the following corollary, can be easily derived from Theorem 7

Corollary 8.

1. Let G be a triangle free planar graph with $\Delta(G) \geq 7$. Then, $\chi_i(G) \leq \Delta(G) + 3$.
2. Let G be a planar graph with girth $g \geq 5$ and $\Delta(G) \geq 9$. Then, $\chi_i(G) \leq \Delta(G) + 4$.

This Corollary improves, for a restricted family of planar graphs, a result of Yang in [5], where it is proved by using the link between the incidence chromatic number, the star arboricity and the chromatic index of a graph, that every planar graph has an incidence coloring with at most $\Delta + 5$ colors for every $\Delta \neq 6$.

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Oriented Incidence Colouring of Digraphs

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EXTENDED ABSTRACT

Let G be an directed graph without loops. We define incidences of two types:

1. An ordered pair (u, uv) where u is a vertex of G and uv is an arc of G
2. An ordered pair (xy, y) where xy is an arc of G and y is a vertex of G

Let \mathcal{I}_G denote the set of incidences of a digraph, G . A pair of incidences (a_1, b_1) and (a_2, b_2) are adjacent if one of the following conditions hold:

- (a_1, b_1) is of type (1) with $(a_1, b_1) = (u, uv)$ and (a_2, b_2) is of type (2) with $(a_2, b_2) = (uv, v)$
- (a_1, b_1) is of type (1) with $(a_1, b_1) = (u, uv)$ and (a_2, b_2) is of type (2) with $(a_2, b_2) = (wu, u)$
- (a_1, b_1) is of type (1) with $(a_1, b_1) = (u, uv)$ and (a_2, b_2) is of type (1) with $(a_2, b_2) = (v, vx)$
- (a_1, b_1) is of type (2) with $(a_1, b_1) = (uv, v)$ and (a_2, b_2) is of type (2) with $(a_2, b_2) = (vx, x)$

An oriented incidence colouring of G assigns to each of the incidences of G a colour such that adjacent incidences receive different colours. That is, if \mathcal{I}_G is the set of incidences of G then an oriented incidence colouring of G with k colours is a function

$c: \mathcal{I}_G \rightarrow \{1, 2, \dots, k-1, k\}$ such that if α and β are adjacent incidences, then $c(\alpha) \neq c(\beta)$. For a digraph G we define the *oriented incidence chromatic number* to be the least k such that G has an oriented incidence colouring using k colours. We denote this value as $\overline{\chi}_i(G)$.

By way of example consider the oriented incidence chromatic number of the family of orientations of stars. Let S_k be an oriented star on $k+1$ vertices. Let u be the centre vertex of S_k , A be the set of out-neighbours of u and B be the set of in-neighbours of u . Consider a function, $c: \mathcal{I}_{S_k} \rightarrow \{1, 2, 3\}$ defined as follows:

For all $a \in A$ and all $b \in B$ let

- $c((u, ua)) = 3$,
- $c((ua, a)) = 1$,
- $c((bu, u)) = 2$, and
- $c((b, bu)) = 1$.

It is easy to observe that this is an oriented incidence colouring of S_k .

The notion of colouring incidences arose in 1993 when Brualdi and Quinn Massey first defined the incidence chromatic number (then called the incidence colouring number), denoted $\chi_i(G)$ [1]. In this first paper they gave upper and lower bounds for $\chi_i(G)$ based on the maximum degree of the graph. The authors used their results as a method to improve a bound for the the strong chromatic index of bipartite graphs. Since then, bounds have been given for a variety of families of graphs including planar graphs, k -trees, k -regular graphs and k -degenerate graphs ([2], [3], [4]). Our notion of oriented incidence colouring considers only incidences that arise when the tail of an arc is incident with the tip of another arc. By restricting the set of incidences in this way we are able to bound the oriented incidence chromatic number as a function of the chromatic number of the underlying graph.

Returning to our example on orientations of stars, consider a homomorphism that maps S_k to P_2 , the directed path on 3 vertices. We notice that we can obtain the oriented incidence colouring of S_k exhibited above by lifting back an oriented incidence colouring of P_2 to the incidences of S_k . This idea leads us to the following result.

Theorem 1. *If G and H are digraphs such that $G \rightarrow H$, then $\vec{\chi}_i(G) \leq \vec{\chi}_i(H)$*

Using this result we are able to find upper bounds on the oriented incidence chromatic number for a variety of families of digraphs by finding the oriented incidence chromatic number for symmetric complete graphs and tournaments. For example, the symmetric graph on 4 vertices has an oriented incidence chromatic colouring using 5 colours. Since every orientation of a planar graph admits a homomorphism to \vec{K}_4 , the symmetric complete graph on 4 vertices, every orientation of a planar graph admits a oriented incidence colouring using at most 5 colours.

In addition to studying the oriented incidence chromatic number of a families of graphs, we consider the oriented incidence chromatic number of graph products and decompositions. For example, we show that 6 colours suffice to colour any digraph who arcs can be partitioned into a pair of oriented trees.

Finally, we consider the problem of identifying oriented graphs H_k such that an oriented graph G has oriented incidence chromatic number at most k if and only if $G \rightarrow H_k$. For $k = 2$ and $k = 3$ we construct oriented graphs H_2 and H_3 . Further, for any input oriented graph, G , we are able to check in polynomial time if G admits a homomorphism to H_2 or H_3 . For $k > 3$ we show that no such target, H_k , exists.

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Total colorings of cartesian products of graphs

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EXTENDED ABSTRACT

A *total coloring* of a finite and simple graph G is an assignment of colors to the elements (vertices and edges) of G such that neighbored elements (two adjacent vertices or two adjacent edges or a vertex and an incident edge) are colored differently. The *total chromatic number* $\chi''(G)$ of G is defined to be the minimum numbers of colors in a total coloring of G . Obviously, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G , and Behzad and Vizing independently conjectured that $\chi''(G) \leq \Delta(G) + 2$ for every graph G . The truth of this so-called *total coloring conjecture* would imply that $\chi''(G)$ attains one of two possible values for every graph G : G is called a *type-1 graph* if $\chi''(G) = \Delta(G) + 1$ and a *type-2 graph* if $\chi''(G) = \Delta(G) + 2$, respectively.

The total coloring conjecture is proved so far for some specific classes of graphs, e. g., for complete graphs, for bipartite graphs, for complete multipartite graphs, for graphs G with $\Delta(G) \geq \frac{3}{4}|V(G)|$ or $\Delta(G) \leq 5$, and for planar graphs G with $\Delta(G) \neq 6$.

The *cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i = u_k$ and v_j is adjacent to v_l in H or if $v_j = v_l$ and u_i is adjacent to u_k in G .

If the total coloring conjecture is true for graphs G and H then it is also true for their cartesian product $G \square H$ (see [3]).

In this talk we give some general upper bounds for $\chi''(G \square H)$ improving results of [1]-[3]. Moreover, we determine the total chromatic number of cartesian products $K_n \square K_m$ of complete graphs.

Theorem 1. *If n is odd and $n \geq m$ then $K_n \square K_m$ is of type 1.*

Theorem 2. *If n and m are even and $n \geq m \geq 4$, $n \equiv 0 \pmod{4}$ or $n \equiv 6 \pmod{8}$ or $n > m \geq 4$, $n \equiv 2 \pmod{8}$ then $K_n \square K_m$ is of type 1.*

Theorem 3. *If n is even and m odd and $n > (m - 1)^2$ then $K_n \square K_m$ is of type 2.*

Furthermore, we determine the total chromatic number of cartesian products $G \square H$ where H is a bipartite graph.

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Forbidden pairs of disconnected graphs implying hamiltonicity

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EXTENDED ABSTRACT

Let H be a given graph. A graph G is said to be H -free if G contains no induced copies of H . If G is H -free, then we call H a *forbidden subgraph* of G . Note that if H_1 is an induced subgraph of H_2 , then an H_1 -free graph is also H_2 -free. For a class \mathcal{H} of graphs, the graph G is \mathcal{H} -free if G is H -free for every $H \in \mathcal{H}$.

As usual, K_n and P_n denote respectively the *complete graph* and the *path* on n vertices. We write Z_n ($n \geq 1$) for the graph obtained by identifying a vertex of a K_3 with an end-vertex of a P_{n+1} , and L_n ($n \geq 2$) for the graph obtained from K_n by adding a pendant edge. The graph $K_{1,n}$ ($n \geq 2$) is called a *star*. We denote by W the graph obtained by identifying two vertices of a K_3 with an end-vertex of a P_2 and an end-vertices of a P_3 , respectively, and by N the graph obtained from a K_3 by adding three disjoint pendant edges (these two graphs are sometimes called *wounded* and *net*, respectively).

If a graph is K_2 -free, then it is an *empty graph* (contains no edges). To avoid the discussion of this trivial case, in the following, we throughout assume that our forbidden subgraphs have at least 3 vertices. If a connected graph G is P_3 -free, then G is complete, and then is hamiltonian if $n(G) \geq 3$. It is in fact not difficult to show that the only connected graph S of order at least 3 such that every 2-connected S -free graph is hamiltonian, is P_3 . The following results involving forbidden pairs for hamiltonicity of graphs were proved by Duffus et al., Broersma and Veldman, Bedrossian and Faudree et al., respectively.

Theorem 1. *Let G be a 2-connected graph.*

- (1) ([3]) *If G is $\{K_{1,3}, N\}$ -free, then G is hamiltonian.*
- (2) ([2]) *If G is $\{K_{1,3}, P_6\}$ -free, then G is hamiltonian.*
- (3) ([1]) *If G is $\{K_{1,3}, W\}$ -free, then G is hamiltonian.*
- (4) ([5]) *If G is $\{K_{1,3}, Z_3\}$ -free, then G is hamiltonian or $G = H_1$ or H_2 .*

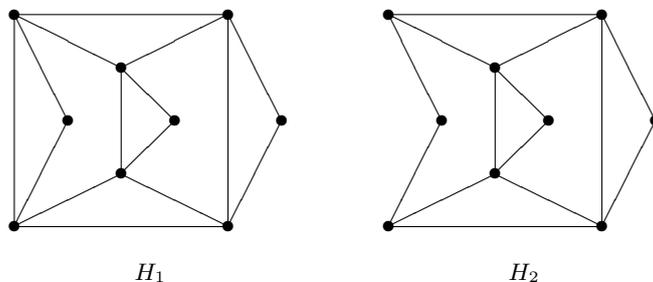


Figure 1. Graphs H_1 and H_2 .

Bedrossian [1] characterized all pairs of forbidden connected subgraphs for hamiltonicity. Faudree and Gould [4] extended Bedrossian's result by proving the necessity part of the theorem based on infinite families of non-hamiltonian graphs.

Theorem 2 (Faudree and Gould, 1997, [4]). *Let R, S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph of order $n \geq 10$. Then G is R -free and S -free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of P_6, Z_3, W or N .*

In the above theorems, the forbidden subgraphs are considered to be connected. So what about the disconnected forbidden subgraphs? In this paper, we consider the forbidden subgraph conditions in terms of not necessary connected subgraphs.

Theorem 3. *The only graphs S of order at least 3 such that every 2-connected S -free graph is hamiltonian is P_3 and $3K_1$.*

Theorem 4. *Let R, S be graphs of order at least 3 other than P_3 and $3K_1$. Then there is an integer n_0 such that every 2-connected $\{R, S\}$ -free graph of order at least n_0 is hamiltonian, if and only if, one of the following is true (up to symmetry):*

- (1) $R = K_{1,3}$ and S is an induced subgraph of $P_6, Z_3, W, N, K_1 \cup Z_2, K_2 \cup Z_1$ or $K_3 \cup P_4$;
- (2) $R = K_{1,r}$ with $r \geq 4$ and S is an induced subgraph of $2K_1 \cup K_2$;
- (3) $R = rK_1$ with $r \geq 4$ and S is an induced subgraph of L_s with $s \geq 3$;
- (4) $R = rK_1$ with $r \geq 4$ and S is an induced subgraph of $2K_1 \cup K_s$ with $s \geq 2$.

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Partitioning 3-edge-coloured complete bipartite graphs into monochromatic cycles

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EXTENDED ABSTRACT

Given any colouring of the edges of a graph G with r colours, how many monochromatic cycles¹ are needed to partition the vertices of G ? This question has been given a considerable amount of attention during the last years, especially for the case of G being a complete graph. As an easy construction shows, there are r -edge-colourings² of the complete graph K_n , such that at least r paths are needed to cover the vertices of K_n , see [5]. On the other hand, it is far from obvious, that an r -edge-coloured K_n can be vertex-partitioned into a number of monochromatic cycles independent of n . However, this is true as the following theorem states.

Theorem 1 ([5]). *For any colouring of the edges of K_n with r colours, there exists a vertex-partition of K_n into $25r^2 \log r$ or less monochromatic cycles.*³

The approach of Theorem 1 has been refined and complemented with ideas of [13] and [15] to obtain the following best known general upper bound:

Theorem 2 ([9]). *Let $r \in \mathbb{N}$ and n be sufficiently large. Then for any r -edge-colouring of K_n , there is a vertex-partition of K_n into at most $100r \log r$ monochromatic cycles.*

After showing Theorem 1, Erdős et al. asked if the lower bound of r monochromatic cycles needed to vertex-partition an r -edge-coloured K_n , could always be achieved:

Conjecture 3 ([5]). *For any colouring of the edges of K_n with r colours, one can partition the vertices of K_n into r or less monochromatic cycles.*

For $r = 2$, Conjecture 3 was known as Lehel's conjecture, see [2], which, after intermediate results by Gerencsér and Gyárfás [6], Łuczak et al. [16] and Allen [1], has been resolved by Bessy and Thomassé [4] in a stronger manner, that is every graph can be vertex-partitioned into a cycle and an anti-cycle. Additional results are known for the case of $r = 3$:

Theorem 4 ([12]). *For $n \in \mathbb{N}$ sufficiently large, let the edges of K_n be coloured with 3 colours. Then the following hold:*

1. *There is a partition of all but $o(n)$ vertices of K_n into 3 or less monochromatic cycles;*
2. *the vertices of K_n can be partitioned into at most 16 monochromatic cycles.*

These results having been obtained, it was somewhat surprising that [17] found a counterexample to Conjecture 3 by constructing r -edge-colourings of K_n for every $r \geq 3$, such that a partition of the vertices into less than $r + 1$ monochromatic cycles is not possible. Nonetheless, in these colourings one can partition all but one vertices into r monochromatic cycles. In light of this, it has been proposed to tone down Conjecture 3 to the following:

Conjecture 5 ([3]). *For every r there is a constant $c(r)$ such that for any colouring of the edges of K_n with r colours, one can partition all but $c(r)$ vertices of K_n into r or less monochromatic cycles.*

¹To omit some trivial obstacles, here single vertices and edges count as cycles too.

²An r -edge-colouring is an edge-colouring using r colours.

³The constant of 25 has been calculated later by Sárközy [18].

Results: In many proofs on this field one-sided vertex-partitions of edge-coloured unbalanced bipartite graphs into monochromatic cycles play a crucial role, see [5, 13, 9, 10, 12]. But also for balanced bipartite graphs similar to the above questions have been raised. Specifically, given a colouring of the edges of a balanced complete bipartite graph $K_{n,n}$ with r colours, is it possible to partition the vertices of $K_{n,n}$ into a number of monochromatic cycles independent of n ? see [5] Erdős proposed 25\$ to anyone who could answer the related question of [7], if any 3-edge-coloured $K_{n,n}$ could be partitioned into less than 1995 cycles, see [13]. Haxell responded positively by showing:

Theorem 6 ([13]). *For large r and any r -edge-colouring of $K_{n,n}$ it is possible to partition the vertices of $K_{n,n}$ into $O((r \log r)^2)$ monochromatic cycles.*

Furthermore her proof yields that a 3-edge-coloured $K_{n,n}$ can be partitioned into at most 1695 monochromatic cycles. Nevertheless, this is still far from the best known lower bound of 5 cycles, or in general $2r - 1$ cycles, as obtained from another easy construction.⁴ In the case of $r = 2$, [17] showed that a 2-edge-coloured $K_{n,n}$ can be partitioned into 2 monochromatic paths, unless there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a properly edge-coloured $K_{2,2}$, in which case 3 paths suffice though. In this work we significantly improve the bound of [13] for $r = 3$ by obtaining the following analogue to Theorem 4:

Theorem 7 (Main result). *For $n \in \mathbb{N}$ sufficiently large, let a 3-edge-colouring of $K_{n,n}$ be given. Then the following hold:*

- (a) *There is a partition of all but $o(n)$ vertices of $K_{n,n}$ into 5 or less monochromatic cycles;*
- (b) *the vertices of $K_{n,n}$ can be partitioned into at most 18 monochromatic cycles.*

The proof of Theorem 7(a) involves the notion of monochromatic connected matchings and an application of the Regularity Lemma of [19]. A method which has been introduced by [15] and has become a standard approach in this and related fields, see [8]. A monochromatic connected matching is a matching in a connected component of the graph spanned by the edges of a single colour. The following is our key Lemma. Its proof is involved but purely combinatorial.

Lemma 8. *Let the edges of $K_{n,n}$ be coloured with 3 colours. Then there is a partition of the vertices of $K_{n,n}$ into 5 or less monochromatic connected matchings.*

An application of the Regularity Lemma provides a vertex-partition of $K_{n,n}$ which is highly regular with respect to each of the graphs, which are induced by the edges of single colours in $K_{n,n}$. In addition the 3-edge-colouring of $K_{n,n}$ induces a 3-edge-colouring on the respective reduced graph R , which is, due to the high regularity, almost complete bipartite. Having established these structures, we are ready to make use a robust version of Lemma 8, which permits us to partition R into 5 monochromatic connected matchings. In the subsequent step, we apply a specific case of the Blow Up Lemma (see [14, 15, 11]) together with some further ideas to find 5 monochromatic cycles, which together form a partition of almost all vertices of $K_{n,n}$, using the monochromatic connected matchings of R as a roadmap.

The proof of Theorem 7(b) combines ideas of Haxell [13] and Gyárfás et al. [12] on basis of Theorem 7(a). In a first step we fix a large monochromatic uniform subgraph H , more precisely H has the property that it is hamiltonian and remains so even if some of the vertices are deleted from it. Secondly we cover almost all vertices of $K_{n,n} - V(H)$ with Theorem 7(a) using at most five vertex-disjoint monochromatic cycles. We denote by S the remaining not yet covered vertices of $K_{n,n} - V(H)$. Since the size of S is much smaller than the order of H , we can in a third step apply a Lemma of Gyárfás et al. [9] to the graph induced by the edges between S and H , in order to cover the vertices of S with at most 12 vertex-disjoint monochromatic cycles $C = C_1 + \dots + C_{12}$. The uniformity of H guarantees that $H - V(C)$ is hamiltonian. As H is monochromatic we can finish by taking one more monochromatic cycle, which covers $H - V(C)$.

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⁴Indeed, take proper r -edge-coloured $K_{r,r}$, replace each of the vertices in one bipart with r new vertices and replace one vertex in the other bipart with $r(r - 1) + 1$ new vertices. Then add coloured edges to this new graph H until it is complete bipartite and such that each edge of H has the same colour as its projection onto $K_{r,r}$.

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Proof of the middle levels conjecture

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EXTENDED ABSTRACT

The middle levels conjecture. The question whether a graph has a Hamilton cycle is a fundamental graph theoretical problem, and answering this question is one of the prototypical NP-complete problems. Even for families of graphs defined by very simple algebraic constructions this question turns out to be surprisingly difficult. One prominent example is the *middle layer graph* whose vertex set consists of all bitstrings of length $2n + 1$ that have exactly n or $n + 1$ entries equal to 1, with an edge between any two vertices for which the corresponding bitstrings differ in exactly one bit. Note that the middle layer graph is a subgraph of the discrete cube of dimension $2n + 1$, the graph whose vertex set are *all* bitstrings of length $2n + 1$, with an edge between any two vertices that differ in exactly one bit. The middle layer graph is bipartite, connected, the number of vertices is $N := \binom{2n+1}{n} + \binom{2n+1}{n+1} = 2^{\Theta(n)}$, and all vertices have degree $n + 1 = \Theta(\log(N))$ (i.e., the graph is sparse). Moreover, the middle layer graph is vertex-transitive, i.e., any pair of vertices can be mapped onto each other by an automorphism (informally speaking, the graph ‘looks’ the same from the point of view of any vertex). The *middle levels conjecture*, also known as revolving door conjecture, asserts that the middle layer graph has a Hamilton cycle for every $n \geq 1$. This conjecture originated probably with Havel [6] and Buck and Wiedemann [1], but has also been attributed to Dejter, Erdős, Trotter [8] and various others. It also appears as Exercise 56 in Knuth’s book [9, Section 7.2.1.3].

The bigger picture. There are two main motivations for tackling the middle levels conjecture. The first motivation are Gray codes: In its simplest form, a Gray code is a cyclic list of all binary code words (=bitstrings) of a certain length such that any two adjacent code words in the list differ in exactly one bit. Clearly, such a Gray code corresponds to a Hamilton cycle in the entire cube, and a Hamilton cycle in the middle layer graph is a restricted Gray code. The second motivation is a classical conjecture due to Lovász [10], which asserts that every connected vertex-transitive graph (as e.g. the middle layer graph) has a Hamilton path and, apart from five exceptional graphs, even a Hamilton cycle. This vastly more general conjecture is still wide open today: Even for explicit families of vertex-transitive graphs as e.g. Kneser graphs and bipartite Kneser graphs, only the denser ones are known to have a Hamilton cycle (see e.g. [2]). In fact, the middle layer graph is the sparsest bipartite Kneser graph, so in some sense it is the hardest obstacle in proving Hamiltonicity for this family of graphs.

Previous work. The middle levels conjecture has attracted considerable attention over the last 30 years. With the availability of more powerful computers, so far the conjecture has been verified for all $n \leq 19$ [15, 16] (for $n = 19$ the middle layer graph has $N = 137.846.528.820$ vertices). The first notable asymptotic result is [13], where it was shown that the middle layer graph has a cycle of length $N^{0.836}$. Improving on this, it was shown in [5] that there is a cycle that visits $0.25N$ many vertices of the middle layer graph, and in [14] that there is a cycle that visits $0.839N$ many vertices. Another major step towards the conjecture was [7], where the existence of a cycle of length $(1 - c/\sqrt{n})N$ was established, where c is some constant. Unfortunately, attempts to obtain a Hamilton cycle from the union of two perfect matchings in the middle layer graph have not been successful so far [3, 4, 8]. More references on partial results and relaxations of the middle levels conjecture can be found in [11].

Our results. In this work we prove the middle levels conjecture.

Theorem 1. *For any $n \geq 1$, the middle layer graph has a Hamilton cycle.*

In fact, we prove the following more general result:

Theorem 2. For any $n \geq 1$, the middle layer graph has at least $\frac{1}{4}2^{2^{\lfloor (n+1)/4 \rfloor}} = 2^{2^{\Omega(n)}}$ different Hamilton cycles.

Note that $2^{2^{\Omega(n)}}$ Hamilton cycles are substantially more than we get from applying all $2(2n+1)! = 2^{\Theta(n \log n)}$ automorphisms of the middle layer graph to a single Hamilton cycle (these automorphisms are given by bit permutations and possibly bit inversion). In fact, any graph G has at most $|V(G)|!$ different Hamilton cycles, where $V(G)$ denotes the vertex set of G . This establishes an upper bound of $N! = 2^{2^{cO(n)}}$ for the number of Hamilton cycles in the middle layer graph and shows that Theorem 2 is best possible.

Our arguments are constructive and yield an algorithm that outputs each of the Hamilton cycles referred to in Theorem 2 in polynomial time per cycle (polynomial in the size of the middle layer graph, which is exponential in n).

The techniques we use to prove these results are an extension of our earlier work [12] (joint with Franziska Weber). The key idea is to reduce the problem of finding a Hamilton cycle in the middle layer graph to the problem of finding a spanning tree in a suitably defined auxiliary graph (which is considerably easier). The preprint [11] contains all proofs and also a more detailed informal explanation of this reduction.

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2-distance coloring of not-so-sparse graphs

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EXTENDED ABSTRACT

The *square* G^2 of a graph G is the graph obtained from G by adding an edge between every pair of vertices having a common neighbor. A proper coloring of G^2 is also called a *2-distance coloring* of G .

The *maximum average degree* $\text{Mad}(G)$ of a graph G is the maximum among the average degrees of the subgraphs of G , i.e. $\text{Mad}(G) = \max \left\{ \frac{2|E(H)|}{V(H)} \mid H \subseteq G \right\}$. Graphs with bounded maximum average degree are often called *sparse graphs*.

There are several results about the chromatic number of G^2 when the maximum average degree of G is bounded (see for example [1, 2]). However, in all these results, the bound for $\text{Mad}(G)$ is strictly less than 4.

We present upper and lower bounds for larger values of $\text{Mad}(G)$, and conjecture that our lower bound is best possible.

Conjecture 1. *There exists an integer D such that every graph G with $\Delta(G) \geq D$ and $\text{Mad}(G) < 4$ has $\chi(G^2) \leq 2\Delta(G)$.*

Theorem 2. *There is an integer Δ such that every graph G with $\Delta(G) \geq \Delta$ and $\text{Mad}(G) < 4$ has $\chi(G^2) \leq 3\Delta(G) + 3$.*

Conjecture 3. *Let $k \geq 3$ be an integer. There is an integer D_k such that every graph G with $\Delta(G) \geq D_k$ and $\text{Mad}(G) < 2k$ has $\chi(G^2) \leq k\Delta(G) - k$.*

Theorem 4. *Let $k \geq 3$ be an integer. There is an integer D_k such that every graph G with $\Delta(G) \geq D_k$ and $\text{Mad}(G) < 2k$ has $\chi(G^2) \leq 2k\Delta(G)$.*

The construction we use proving our lower bound is inspired by a construction by Alon and Mohar [3], but we use a different (and seemingly unrelated) notion to construct our graph: graeco-latin squares.

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Circular flow numbers of (signed) regular graphs

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EXTENDED ABSTRACT

We consider finite (multi-) graphs G with vertex set $V(G)$ and edge set $E(G)$. The set of edges which are incident to vertex v is denoted by $E(v)$.

Vizing [8] proved that the edge-chromatic number $\chi'(G)$ of a graph G with maximum vertex degree $\Delta(G)$ is an element of $\{\Delta(G), \dots, \Delta(G) + \mu(G)\}$, where $\mu(G)$ is the maximum multiplicity of an edge of G . We say that G is a class 1 graph if $\chi'(G) = \Delta(G)$ and it is a class 2 graph if $\chi'(G) > \Delta(G)$.

An orientation D of G is an assignment of a direction to each edge, and for $v \in V(G)$, $E^-(v)$ is the set of edges of $E(v)$ with head v and $E^+(v)$ is the set of edges with tail v . The oriented graph is denoted by $D(G)$.

A nowhere-zero r -flow $(D(G), \phi)$ on G is an orientation D of G together with a function ϕ from the edge set of G into the real numbers such that $1 \leq |\phi(e)| \leq r - 1$, for all $e \in E(G)$, and $\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e)$, for all $v \in V(G)$. The circular flow number of G is $\inf\{r \mid G \text{ has a nowhere-zero } r\text{-flow}\}$, and it is denoted by $F_c(G)$. It is known [1], that $F_c(G)$ is always a minimum and that it is a rational number.

For $i \in \{3, 4\}$ there are characterizations of cubic graphs with nowhere-zero i -flow. These results are due to Tutte [6][7], see also [2].

Theorem 1 ([6][7]). (A) A cubic graph G is bipartite if and only if $F_c(G) = 3$.
(B) A cubic graph G is a class 1 graph if and only if $F_c(G) \leq 4$.

The following theorem generalizes Theorem 1(A) to $(2t + 1)$ -regular graphs.

Theorem 2 ([5]). Let $t \geq 1$ be an integer. A $(2t + 1)$ -regular graph G is bipartite if and only if $F_c(G) = 2 + \frac{1}{t}$. Furthermore, if G is not bipartite, then $F_c(G) \geq 2 + \frac{2}{2t-1}$.

Flow numbers of graphs have attracted considerable attention over the last decades. Pan and Zhu [3] proved that for every rational number r with $2 \leq r \leq 5$ there is a graph G with $F_c(G) = r$. This result is used in [4] to prove that for every integer $t \geq 1$ and every rational number $r \in \{2 + \frac{1}{t}\} \cup [2 + \frac{2}{2t-1}, 5]$, there exists a $(2t + 1)$ -regular graph G with $F_c(G) = r$.

If G is a cubic graph, then $F_c(G) \leq 4$ if and only if G is class 1. Hence, Theorem 1(B) implies that the flow number 4 separates class 1 and class 2 cubic graphs from each other. We prove the following generalization of Theorem 1(B).

Theorem 3. Let $t \geq 1$ be an integer. A non-bipartite $(2t + 1)$ -regular graph G has a 1-factor F such that $G - F$ is bipartite if and only if $F_c(G) = 2 + \frac{2}{2t-1}$.

We further show that the case of cubic graphs is exceptional in the sense that for every $t > 1$ there is no flow number that separates $(2t + 1)$ -regular class 1 graphs and class 2 graphs. However, our results imply that a $(2t + 1)$ -regular graph G with $F_c(G) \leq 2 + \frac{2}{2t-1}$ is a class 1 graph.

We study the question whether there are $(2t + 1)$ -regular graphs H_1 and H_2 such that H_1 is class 1, H_2 is class 2, and $F_c(H_1) = F_c(H_2) = r > 2 + \frac{2}{2t-1}$. We give an affirmative answer to this question for some specific values of r . Figure 1 sketches some results.

In [5] it is shown that $F_c(K_{2t+2}) = 2 + \frac{2}{t}$ for the complete graph K_{2t+2} on $2t + 2$ vertices. K_{2k+2} is a class 1 graph and we conjecture that the circular flow number of any $(2t + 1)$ -regular class 1 graph is at most $2 + \frac{2}{t}$. We relate this conjecture to other conjectures on flows on graphs.

In the second part of the talk, we study structural properties of signed graphs that have the aforementioned numbers in their flow spectrum. For instance, we prove the following theorem:

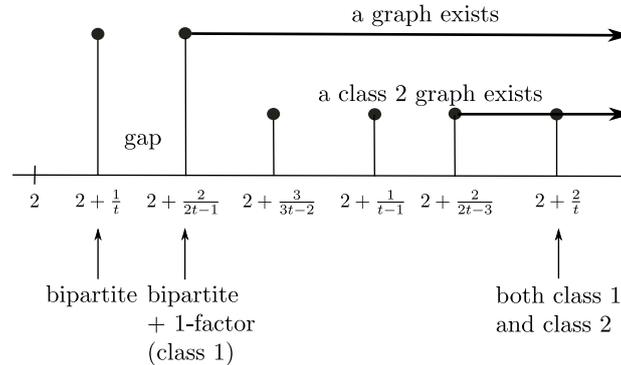


Figure 1: Sketch of flow numbers of $(2t+1)$ -regular graphs ($t > 3$)

Theorem 4. Let $t \geq 1$ be an integer.

1) A $(2t+1)$ -regular graph G has a t -factor if and only if there is a signature σ such (G, σ) has a nowhere-zero $(2 + \frac{1}{t})$ -flow.

2) If G is $(2t+1)$ -regular and has a 1-factor, then there is a signature σ such (G, σ) has a nowhere-zero 3-flow.

Furthermore, for each $t > 1$ there is a $(2t+1)$ -regular graph G_t that has a signature σ such that (G_t, σ) has a nowhere-zero 3-flow and G_t has no 1-factor.

Furthermore, we generalize Theorem 2 to signed graphs.

Theorem 5. Let $t \geq 1$ be an integer and (G, σ) be a signed $(2t+1)$ -regular graph. If $F_c((G, \sigma)) = r$, then $r = 2 + \frac{1}{t}$ or $r \geq 2 + \frac{2}{2k-1}$.

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The chromatic number of a signed graph

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EXTENDED ABSTRACT

We study vertex-colourings of signed graphs as they were introduced by Zaslavsky in [4]. Let G be a signed graph, that is, a graph in which each edge is labelled with $+1$ or -1 . A *proper vertex-colouring* of G , or simply a *colouring*, is a mapping $\phi: V(G) \rightarrow \mathbb{Z}$ such that for each edge $e = uv$ of G the colour $\phi(u)$ is distinct from the colour $\sigma(e)\phi(v)$, where $\sigma(e)$ is the sign of e . In other words, the colours of vertices joined by a positive edge must not coincide while those joined by a negative edge must not be opposite to each other.

This definition is natural for several reasons extensively discussed in [4, 5, 6]. In particular, colourings are well behaved under switching. Recall that *switching* of G at a vertex v reverses the sign of each non-loop edge incident with v . The switching operation does not essentially change the signed graph, because it preserves the sign product on each circuit. If we switch a signed graph endowed with a colouring, the colouring must be switched together with the signature, that is to say, we have to replace the colour $\phi(v)$ with $-\phi(v)$. It is easy to see that the result of switching is a proper colouring on the resulting signed graph. Furthermore, a proper vertex-colouring of a balanced signed graph (one where the sign product on each circuit is positive) is just a proper vertex-colouring of the underlying unsigned graph.

The next step is to define the chromatic number of a signed graph. We wish to obtain an extension of the usual chromatic number of an unsigned graph that agrees with it on every balanced signed graph. We therefore divert from the approach taken by Zaslavsky in [4, 5, 6], which, roughly speaking, only counts the absolute values of colours. For this purpose we define, for each $n \geq 1$, a subset $M_n \subseteq \mathbb{Z}$ by setting $M_n = \{\pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k$, and $M_n = \{0, \pm 1, \pm 2, \dots, \pm k\}$ if $n = 2k + 1$. A proper colouring of G that uses colours from M_n will be called an *n -colouring*. Note that if G admits an n -colouring, then it also admits an m -colouring for each $m \geq n$. The smallest n such that G admits an n -colouring will be called the *signed chromatic number of G* and will be denoted by $\chi_{\pm}(G)$.

For a signed graph G let $\chi(G)$ denote the usual chromatic number of its underlying graph. Our first result relates $\chi_{\pm}(G)$ to $\chi(G)$.

Theorem 1. *For every signed graph G one has $\chi_{\pm}(G) \leq 2\chi(G) - 1$. Moreover, the bound is sharp.*

Proof. Take a proper colouring ϕ of the underlying graph of G with colours $0, 1, \dots, k - 1$, where $k = \chi(G)$. Clearly, ϕ is a proper colouring of G , as well. Since all colours are contained in $\{0, \pm 1, \dots, \pm(k - 1)\}$, it is a $(2k - 1)$ -colouring. The inequality follows.

The bound is reached by the family of signed graphs $\{G_n\}_{n \geq 2}$ which can be constructed as follows. Take one all-positive copy of K_n , denoted by H_1^n , and $n - 1$ all-negative copies of K_n , denoted by $H_2^n, H_3^n, \dots, H_n^n$. Let $v_{i,1}, v_{i,2}, \dots, v_{i,n}$ denote the vertices of H_i^n , and call two vertices $v_{i,j}$ and $v_{k,l}$ *corresponding* whenever $j = l$. To construct G_n , take $\bigcup_{i=1}^n H_i^n$ and add one positive edge between each pair of non-corresponding vertices from different copies of K_n . It can be shown that $\chi_{\pm}(G_n) = 2\chi(G_n) - 1$. The details will appear in [3]. \square

Our main result is a theorem that bounds the chromatic number of a signed graph in a similar manner as does the celebrated theorem of Brooks [1] for unsigned graphs. By $\Delta(G)$ we denote the maximum degree of a signed graph G .

Theorem 2. *Let G be a simple connected signed graph different from a balanced complete graph, a balanced circuit of odd length, and an unbalanced circuit of even length. Then*

$$\chi_{\pm}(G) \leq \Delta(G).$$

Sketch of proof. The proof is straightforward if G has maximum degree at most 2. If G is an unbalanced complete graph, the bound can be established by induction. Henceforth we can therefore assume that G is a signed graph with maximum degree at least 3 different from a signed complete graph.

If G is 2-connected, we follow the ideas of the proof of Lovász [2]. We find an ordering x_1, x_2, \dots, x_n of the vertex set of G such that x_1 and x_2 are not adjacent, x_n is adjacent to both of them, and for every x_i with $i < n$ there exists $j > i$ such that x_j is adjacent to x_i . If the edges x_1x_n and x_2x_n have the same sign, we colour x_1 and x_2 with the same colour, say $+1$; if they have different signs, we colour x_1 and x_2 with opposite colours, say $+1$ and -1 , respectively. Then we colour the vertices x_3, x_4, \dots, x_{n-1} one by one in the described order. Set $\Delta = \Delta(G)$, and assume that the vertices x_1, x_2, \dots, x_{i-1} have already been coloured. At most $\Delta - 1$ neighbours of x_i have already received a colour, and each such neighbour forbids one colour. Therefore at most $\Delta - 1$ values from M_Δ are forbidden for x_i , leaving at least one colour in M_Δ available for x_i . At last, we colour x_n . Since the vertices x_1 and x_2 are distinct and coloured as mentioned, there is one colour available for x_n , and the colouring of G can be completed.

Next assume that G has a cut-vertex, say v . Let V_1, V_2, \dots, V_s denote the vertex sets of the components of $G - v$, and let G_i be the subgraph induced by $V_i \cup \{v\}$. If $\deg_{G_i}(v) \leq \Delta - 2$ for each i , then all the graphs G_i are $(\Delta - 1)$ -degenerate, and thus admit a Δ -colouring. Since $\deg_{G_i}(v) \leq \Delta - 2$, we can choose a non-zero colour for v within each G_i and then permute the colours in such a way that v receives the same colour from each G_i . By combining these colourings we get a Δ -colouring of G .

We are left with the case where G has a cut-vertex v and there exists an index j such that $\deg_{G_j}(v) > \Delta - 2$. It follows that $\deg_{G_j}(v) = \Delta - 1$, $s = 2$, and the vertex v is incident with a bridge joining v to a vertex u of G . Clearly, we can switch the signature to make uv positive. In this situation we carefully analyse the structure of G and show that in all subcases that may occur at least one of the components of $G - uv$ has a Δ -colouring which colours the bridge end by a non-zero colour. After applying a permutation of colours, if necessary, the two colourings can be combined into a Δ -colouring of G . We defer the details, which are quite involved, to [3]. \square

From the the four colour theorem we know that the underlying graph of every signed planar graph G can be properly coloured by colours from the set $\{+1, -1, +2, -2\}$. It is natural to ask whether it is possible to find a 4-colouring of G that respects the constraints of the signature. Towards this end, we would like to propose the following conjecture:

Conjecture 3. *Every simple signed planar graph G has $\chi_\pm(G) \leq 4$.*

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Signed graphs with two negative edges

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EXTENDED ABSTRACT

A *nowhere-zero k -flow on a graph G* is an orientation of edges of G and a valuation of them by non-zero integers whose absolute value is smaller than k in such a way that for every vertex of G the sum of the incoming values is equal to the sum of the outgoing ones. The smallest k , if it exists, for which a graph G admits a nowhere-zero k -flow is called the *flow number of G* . If G admits a nowhere-zero k -flow for some integer k , then G is called *flow-admissible*.

In the presented work, we study a generalization of nowhere-zero flows on graphs – nowhere-zero flows on signed graphs. A *signed graph (G, σ)* is a graph G with a signature $\sigma : E(G) \rightarrow \{-1, 1\}$. Two signed graphs (G, σ_1) and (G, σ_2) are considered to be *equivalent* if the symmetric difference of $\sigma_1^{-1}(-1)$ and $\sigma_2^{-1}(-1)$ is an edge-cut of G . Equivalent signed graphs have the same flow number. A nowhere-zero k -flow on a signed graph differs from one on a graph only in the definition of an orientation of edges. An *orientation* of edges of a signed graph is obtained as follows. Consider each edge as two half-edges and orient each half-edge independently in such a way that for a positive edge the orientations are alike while for a negative edge the orientations are opposite.

Bouchet [1] conjectured the following:

Conjecture 1 (Bouchet 1983 [1]). *Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

The best current approach to Conjecture 1 is a 12-flow theorem by Matt DeVos [2].

If n_σ denotes the number of negative edges of (G, σ) , then, according to Seymour [4], the conjecture is true for signed graphs with $n_\sigma = 0$, because they correspond to an unsigned case. Since (G, σ) with $n_\sigma = 1$ does not admit a flow, two is the smallest value of n_σ for which the Bouchet's conjecture is open.

We study flow-admissible signed graphs that are equivalent to (G, σ) with $n_\sigma = 2$. We restrict our study to cubic graphs, because for each non-cubic signed graph (G, σ) there is a set $\mathcal{G}(G, \sigma)$ of cubic graphs such that $F(G, \sigma) \leq \min\{F(H, \sigma_H) : (H, \sigma_H) \in \mathcal{G}(G, \sigma)\}$. We prove the following:

Theorem 2. *If (G, σ) is a flow-admissible signed cubic graph with $n_\sigma = 2$, then (G, σ) has a nowhere-zero 8-flow such that the negative edges receive flow value 1.*

The bound given by a previous theorem can be improved to 6 if G has a sufficient cyclic edge-connectivity. Furthermore we prove better bounds if there is an element (H, σ_H) of $\mathcal{G}(G, \sigma)$ that satisfies an additional condition. Namely:

Theorem 3. *Let (H, σ) be a flow-admissible signed cubic graph with $n_\sigma = 2$. If H is bipartite, then $F(H, \sigma) \leq 4$.*

The bound given in the previous theorem is tight. It is reached, for example, on the signed graph of Figure .2.

In our work we further distinguish cubic graphs according to their chromatic index. An *edge-coloring* of a graph G is to set a color to every edge of G in such a way that two adjacent edges receive different colors. We say that G is *c -edge-colorable* if there exists an edge-coloring of G that uses at most c colors. The smallest number of colors needed to edge-color G is *chromatic index of G* . By Vizing's theorem the chromatic index of a cubic graph is either 3 or 4. Bridgeless cubic graphs with chromatic index 4 are also called *snarks*. We say that a snark G is *critical* if $G - e$ admits a nowhere-zero 4-flow for every edge e .

Theorem 4. *Let (G, σ) be a flow-admissible signed cubic graph with $n_\sigma = 2$. If G is 3-edge-colorable or a critical snark, then (G, σ) has a nowhere-zero 6-flow such that the negative edges receive flow value 1.*

We would like to note that a restriction of a flow value on a particular edge of a signed graph is crucial. The signed graph on Figure .2 is an example of a signed graph that does not admit a nowhere-zero 4-flow that assigns 1 to negative edges even though its flow number is 4 according to Theorem 3.

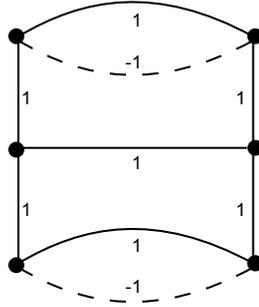


Figure .2: A signed graph for which a choice of flow value on negative edges is important

This extended abstract is based on a manuscript [3].

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Balance in Random Signed Graphs

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EXTENDED ABSTRACT

We study balance in random graphs. The theory of balance in graphs was first motivated by Heider [9] who defined balance as absence of tension in a social system. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists [3, 10]. From a mathematical point of view, the most appropriate model for studying structural balance is that of signed graphs. Formally, a signed graph (G, σ) is a graph $G = (V, E)$ together with a function $\sigma : E \rightarrow \{+, -\}$, which associates each edge with the sign $+$ or $-$. In such a signed graph, a subset H of $E(G)$ is said to be *positive* if it contains an even number of negative edges; otherwise it is said to be negative. A signed graph G is *balanced* if each cycle of G is positive. Otherwise it is unbalanced. In 1956, Cartwright and Harary [1] obtained the following important result.

Theorem 1 (Cartwright, Harary '56 [1]). *A signed graph is balanced if and only if its vertex set can be partitioned into two classes so that every edge joining vertices within a class is positive and every edge joining vertices between classes is negative.*

A signed graph is called *weakly balanced* if it contains no cycle with exactly one negative edge. We have the following nice characterization of weakly balanced graphs.

Theorem 2 (Davis '67 [2]). *A signed graph is weakly balanced if and only if its vertex set can be partitioned into different classes so that every edge connecting two vertices that belong to the same class is positive and every edge connecting two vertices that belong to different classes is negative.*

Balance in random social systems was first considered in [6]. In [5] the authors defined the following random signed graph model $G(n, p, q)$. Let $p, q > 0$, $0 < p + q < 1$. Given a set of n vertices, between each pair of distinct vertices there is either a positive edge with probability p or a negative edge with probability q or no edge with probability $1 - (p + q)$. The edges between different pairs of vertices are chosen independently.

It is well-known that the threshold for the existence of cycles in the random graph model $G(n, p)$ is $1/n$. Thus, if $p + q = o(1/n)$ then $G(n, p, q)$ is almost surely balanced. The following theorems show that essentially this is the best that one could hope for.

Theorem 3 (El Maftouhi, Harutyunyan, Manoussakis '14 [4]). *Let $\epsilon, \delta > 0$ be fixed. If $p + q \geq (1 + \epsilon)/n$ and $q \geq \delta/n$ then $G(n, p, q)$ is almost surely unbalanced.*

Theorem 4 (El Maftouhi, Harutyunyan, Manoussakis '14 [4]). *Let $\epsilon, \delta > 0$ be fixed. If $p \geq (1 + \epsilon)/n$ and $q \geq \delta/n$ then $G(n, p, q)$ is almost surely weakly unbalanced.*

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A New Game Invariant of Graphs : the Game Distinguishing Number

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EXTENDED ABSTRACT

The distinguishing number of a graph G is a symmetry related graph invariant. Its study starts a decade ago in a work by Albertson and Collins [1]. Given a graph G the distinguishing number $D(G)$ is the least integer d such that G has a d -distinguishing coloring. A d -distinguishing coloring is a coloring $c : V(G) \rightarrow \{1, \dots, d\}$ invariant only under the trivial automorphism. More generally we say that an automorphism σ of a graph G preserves the coloring c or is a colors preserving automorphism, if for all $u \in V(G)$, $c(u) = c(\sigma(u))$.

In the last couple of years the study of this invariant was really flourishing. See in particular [7] and [6] for the work of Imrich, Jerebic and Klazar on the distinguishing number of Cartesian products or [3] for an analog of Brook's theorem. Some variant of this distinguishing number were already introduced. In [2], Chan extends the distinguishing notion to any action of a group H on a set X and in [3] Collins and Trenk study distinguishing coloring that must be proper coloring. Our goal will be to introduce a game variant of this invariant in the spirit of the game chromatic number χ_G (see [4]) and try to show this approach raises up a lot of promising and interesting questions.

The distinguishing game is a game with two players, the Gentle and the Rascal, with antagonist goals. This game is played on a graph G with a set of $d \in \mathbb{N}^*$ colors. Alternatively, the two players choose a vertex of G and color it with one of the d colors. The game ends when all the vertices have been colored. Then the Gentle wins if the coloration is d -distinguishing and the Rascal wins otherwise.

This game leads to a definition of two new invariants for a graph G . Those invariants are the minimum numbers of colors needed to ensure that the Gentle has a winning strategy. This means that the Gentle has a way to win the game whatever his opponent will play. Note that this number could eventually be infinite and that it really depends on who starts the game.

Definition 1. *The game distinguishing numbers of a graph G are defined as follows :*

1. $GD_G(G)$ is the minimum of colors needed to ensure that the Gentle has a winning strategy assuming he is playing first. If the Rascal is sure to win whatever the number of colors we allow, then $GD_G(G) = \infty$.
2. $GD_{\mathcal{R}}(G)$ is the minimum of colors needed to ensure that the Gentle has a winning strategy assuming the Rascal is playing first. If the Rascal is sure to win whatever the number of colors we allow, then $GD_{\mathcal{R}}(G) = \infty$.

With the convention that ∞ is greater than any natural number, the following property arises directly from the definition.

Proposition 2. *For any graph G , we have $D(G) \leq GD_G(G)$ and $D(G) \leq GD_{\mathcal{R}}(G)$.*

We first try to figure out what happens when at least one game distinguishing number is infinite. We give sufficient conditions, but a complete characterization of graphs with this property is still open.

After that, we introduce a new class of graphs, called involutive graphs. For this class, which contains cycles, hypercubes and more generally diametrical graphs, we are able to bound quadratically the game distinguishing number using the classical one. We will prove the following theorem.

Theorem 3 (Gravier, Meslem, S., Slimani 2014). *Let G be an involutive graph whose classical distinguishing number is d .*

1. *If bar is an automorphism, then $GD_{\mathcal{G}}(G) = \infty$.*
2. *We have $GD_{\mathcal{R}}(G) \leq d^2 + d - 2$.*

A slight improvement of the ideas used to prove this theorem leads to sharper results in the particular cases of even cycles and hypercubes.

Theorem 4 (Gravier, Meslem, S., Slimani 2014). *Let C_n be an even cycle of order n .*

1. $GD_{\mathcal{G}}(C_n) = \infty$,
2. *If $n \geq 8$ then $GD_{\mathcal{R}}(C_n) = 2$,*
3. *If $n = 4, 6$ then $GD_{\mathcal{R}}(C_n) = 3$.*

Theorem 5 (Gravier, Meslem, S., Slimani 2014). *Let Q_n be the hypercube of dimension n .*

1. *We have $GD_{\mathcal{G}}(Q_n) = \infty$.*
2. *If $n \geq 5$, then $GD_{\mathcal{R}}(Q_n) = 2$. Moreover $GD_{\mathcal{R}}(Q_2) = GD_{\mathcal{R}}(Q_3) = 3$.*
3. *We have $2 \leq GD_{\mathcal{R}}(Q_4) \leq 4$.*

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On the game colouring number of powers of graphs

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EXTENDED ABSTRACT

We generalize the methods of Esperet and Zhu [2] providing an upper bound for the game colouring number of squares of graphs to obtain upper bounds for the game colouring number of m -th powers of graphs, $m \geq 3$, which rely on the maximum degree and the game colouring number of the underlying graph. Furthermore, we improve these bounds in case the underlying graph is a forest.

The game colouring number of a graph, introduced by Zhu [4], is defined by the following maker-breaker marking game. We are given a finite graph $G = (V, E)$. At the beginning every vertex $v \in V$ is unmarked. Two players, Alice and Bob, alternately mark vertices, one vertex in a turn, with Alice beginning. The game ends if every vertex is marked. The players thereby create a linear ordering L of the vertices. The *back degree* $bd_L(v)$ of a vertex v (with respect to L) is the number of neighbours of v that precede v in L , i.e. the number of previously marked neighbours of v in the game. The *score* $sc(G, L)$ (with respect to L) is the maximum back degree over all vertices. Alice tries to minimize the score, Bob tries to maximize the score. The smallest score Alice can achieve with some strategy against every strategy of Bob is called *game score* $sc_g(G)$ of the graph G , i.e. the game score is the score obtained if both players use optimal strategies.

The *game colouring number* $\text{col}_g(G)$ of G is defined as

$$\text{col}_g(G) = 1 + sc_g(G).$$

The game colouring number is a useful tool (cf. [3, 4]) to bound the game chromatic number which was introduced by Bodlaender [1].

The m -th *power* of a graph $G = (V, E)$, denoted by G^m , is a graph (V, E_m) with $vw \in E_m$ if and only if $1 \leq \text{dist}_G(v, w) \leq m$. The 2nd power is also called *square* of a graph.

Esperet and Zhu [2] obtained the following upper bound for the game colouring number of the square of a graph G , which uses the game colouring number of G and the maximum degree of G .

Theorem 1 (Esperet and Zhu '09 [2]). *Let G be a graph with maximum degree Δ . Then*

$$\text{col}_g(G^2) \leq (\text{col}_g(G) - 1)(2\Delta - \text{col}_g(G) + 1) + 1.$$

We extend the methods of Esperet and Zhu and obtain the following generalization of Theorem 1 to arbitrary powers of a graph.

Theorem 2. *Let G be a graph with maximum degree $\Delta \geq 3$ and $m \in \mathbb{N} \setminus \{0\}$.*

(a) *If $\text{col}_g(G) \in \{\Delta, \Delta + 1\}$, then*

$$\text{col}_g(G^m) \leq \text{col}_g(G) + \Delta(\text{col}_g(G) - 1) \frac{(\Delta - 1)^{m-1} - 1}{\Delta - 2}.$$

(b) *If $\text{col}_g(G) = \Delta - 1$, then*

$$\text{col}_g(G^m) \leq 1 - 2m + \Delta \frac{(\Delta - 1)^m - 1}{\Delta - 2}.$$

(c) If $\text{col}_g(G) \leq \Delta - 2$, then

$$\text{col}_g(G^m) \leq 1 + \Delta \frac{(\Delta - 1)^m - 1}{\Delta - 2} - (\Delta - \text{col}_g(G) + 1) \frac{(\Delta - \text{col}_g(G))^m - 1}{\Delta - \text{col}_g(G) - 1}.$$

The main idea of the proof is to estimate the number of unmarked and marked vertices at distance $1, 2, 3, \dots, m$ from an unmarked vertex during the game in a recursive way, and simplifying a system of two recursions to a single recursion.

In case the graph of the game is a power of a forest, Theorem 2 can be improved in the following way.

Theorem 3. *Let F be a forest with maximum degree $\Delta \geq 3$. Let $m \in \mathbb{N}$. Then we have*

$$\text{col}_g(F^m) \leq \frac{2(\Delta - 1)^m - 2}{\Delta - 2} + 2.$$

To obtain the upper bound in Theorem 3 we use a variation of the well-known activation strategy for trees [3].

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On the unilateral colouring game

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EXTENDED ABSTRACT

Let be given a graph G , an additive hereditary property \mathcal{P} and a set of colours $\{1, \dots, r\}$. The two players are Alice and Bob and they play alternately with Alice having the first move. The player may colour a vertex v with colour $i \in \{1, \dots, r\}$ if after the move a subgraph induced by vertices with colour i has the property \mathcal{P} . If after $|V(G)|$ moves the graph G is coloured, then Alice wins, otherwise Bob wins. The above game is called the \mathcal{P} -colouring game. The \mathcal{P} -game chromatic number $\chi_g^{\mathcal{P}}(G)$ of G is the least number r for which Alice has a winning strategy for the \mathcal{P} -colouring game with r colours.

Let us denote $\mathcal{O} = \{G : E(G) = \emptyset\}$ and $\mathcal{S}_d = \{G : \Delta(G) \leq d\}$.

If $\mathcal{P} = \mathcal{O}$, then we obtain the most known variant of the \mathcal{P} -game – the r -colouring game. The variant of the \mathcal{P} -game such that $\mathcal{P} = \mathcal{S}_d$ is called the (r, d) -relaxed colouring game. The parameter associated with these games $\chi_g(G) = \chi_g^{\mathcal{O}}(G)$ and $\chi_g^{(d)}(G) = \chi_g^{\mathcal{S}_d}(G)$ are called the *game chromatic number* and the *d -relaxed game chromatic number*, respectively.

We introduce an *unilateral game* in which Bob is more powerful than Alice, he has more freedom in colour selection and can miss his move. We say that colour i is (\mathcal{P}, r) -admissible for an uncoloured vertex v if $i \in \{1, \dots, r\}$ and after colouring v with i no monochromatic minimal forbidden subgraph for the property \mathcal{P} containing v appears. Again two players Alice and Bob colour vertices of G . Alice starts the game. The players play alternately, but Bob can miss his move. Bob colours a vertex with an arbitrary colour from $\{1, \dots, r\}$, while Alice must colour a vertex with a colour that is (\mathcal{P}, r) -admissible for this vertex. If all vertices of the graph G are coloured, then Alice wins the game, otherwise Bob wins. Thus, Bob wins the game if after a move of some player there is an uncoloured vertex v which has no (\mathcal{P}, r) -admissible colour. The above game is called the *unilateral \mathcal{P} -game*. The *\mathcal{P} -unilateral game chromatic number*, denoted by $\chi_{ug}^{\mathcal{P}}(G)$, is the least number r such that Alice has a winning strategy, when playing the unilateral \mathcal{P} -game with r colours on G .

Since in the unilateral \mathcal{P} -game Bob is more powerful than Alice, the \mathcal{P} -unilateral game chromatic number is the upper bound on the the \mathcal{P} -game chromatic number.

Theorem 1 (E.S.'14 [2]). *For any graph G , it holds $\chi_g^{\mathcal{P}}(G) \leq \chi_{ug}^{\mathcal{P}}(G)$.*

Moreover, we prove other useful properties of the unilateral game chromatic number.

Theorem 2 (E.S.'14 [2]). *For any graph G ,*

1. $\chi_g(G) \leq \chi_{ug}^{\mathcal{O}}(G) \leq \text{col}_g(G)$,
2. $\chi_g^{(d)}(G) \leq \chi_{ug}^{\mathcal{S}_d}(G)$.

Theorem 3 (E.S.'14 [2]). *If G' is a subgraph of G , then $\chi_{ug}^{\mathcal{P}}(G') \leq \chi_{ug}^{\mathcal{P}}(G)$.*

Theorem 4 (E.S.'14 [2]). *If Alice has a winning strategy for the unilateral \mathcal{P} -game on G with r colours, then she also has a winning strategy for the unilateral \mathcal{P} -game on G with t colours for any $t \geq r$.*

Let $\mathcal{H}_k = \{G : \text{every block of } G \text{ has at most } k \text{ vertices}\}$.

We consider the unilateral \mathcal{S}_d -game on the class \mathcal{H}_k that contains many well-known classes of graphs. For example \mathcal{H}_2 is the class of forests. If $G \in \mathcal{H}_k$ and every block is a complete graph, then G is a Husimi tree. If $G \in \mathcal{H}_k$ and every block is a cycle or K_2 , then G is a cactus graph. The class \mathcal{H}_k is also interesting because it contains line graphs of forests. If every block of $G \in \mathcal{H}_k$ is a complete graph and every vertex of G is in at most two blocks, then G is the line graph of a forest. We prove the following theorems:

Theorem 5. *Let $k \geq 2$, $0 \leq d \leq k - 1$ and $G \in \mathcal{H}_k$. Then $\chi_{ug}^{S_d}(G) \leq k + 2 - d$.*

Theorem 6. *Let $k \geq 5$, $1 \leq d \leq k - 2$ and $G \in \mathcal{H}_k$. Then $\chi_{ug}^{S_d}(G) \leq k + 1 - d$.*

From Theorem 5 immediately follows result for the d -relaxed game chromatic number:

Corollary 7. *Let $k \geq 2$, $0 \leq d \leq k - 1$ and $G \in \mathcal{H}_k$. Then $\chi_g^{(d)}(G) \leq k + 2 - d$.*

In [1] was proved:

Theorem 8 (E.S.'10 [1]). *Let $k \geq 6$ and $G \in \mathcal{H}_k$. Then $\chi_g(G) \leq k + 1$.*

Thus, from Theorem 6 and Theorem 8 we obtain:

Corollary 9. *Let $k \geq 6$, $0 \leq d \leq k - 2$ and $G \in \mathcal{H}_k$. Then $\chi_g^{(d)}(G) \leq k + 1 - d$.*

An (r, d) -relaxed edge-colouring game is the version of the (r, d) -relaxed colouring game which is played on the edge set of a graph G . The d -relaxed game chromatic index of G is denoted by ${}^{(d)}\chi'_g(G)$. Since the line graph of the forest with maximum degree k belongs to \mathcal{H}_k , from results for \mathcal{H}_k we obtain some new results for line graphs of forests, i.e., for the relaxed game chromatic index of forests.

Corollary 10. *Let T be a forest and $0 \leq d \leq \Delta(T) - 1$. Then ${}^{(d)}\chi'_g(T) \leq \Delta(T) + 2 - d$.*

Corollary 11. *Let T be a forest with $\Delta(T) \geq 5$ and $0 \leq d \leq \Delta(T) - 2$. Then ${}^{(d)}\chi'_g(T) \leq \Delta(T) + 1 - d$.*

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On achromatic and harmonious numbers of powers of paths and cycles

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EXTENDED ABSTRACT

The achromatic (resp. harmonious) number of a graph G , denoted by $\psi(G)$ (resp. $h(G)$) is the largest (resp. the smallest) number of colors in a good vertex coloring of G such that for any pair of colors there is at least one (resp. at most one) edge of G whose endpoints are colored with this pair of colors. By P_n^k (resp. C_n^k) we denote the k th power of the n -vertex path (resp. n -vertex cycle), i.e. the graph obtained from the path P_n (resp. the cycle C_n) by joining with edges all pairs of vertices at distance not exceeding k . The problem of finding the exact values of achromatic and harmonious numbers is not fully solved even for the graphs P_n^2 and C_n^2 . We show that for every fixed k and $\epsilon > 0$, $\psi(P_n^k) = \sqrt{2kn} - O(n^\epsilon)$ and $h(P_n^k) = \sqrt{2kn} + O(n^{0.263})$. Moreover, for $k = n^\alpha$, we prove that $\psi(P_n^k) = \sqrt{2kn}(1 - o(1))$, if α is any constant satisfying $0 < \alpha < 1$ and $h(P_n^k) = \sqrt{2kn}(1 + o(1))$, if $0 < \alpha < 1/3$. We get similar results for the graphs C_n^k . The research is motivated by some problems occurring in large data transfer.

Rank-width for countable graphs and compactness

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EXTENDED ABSTRACT

We define several notions of rank-width for countable graphs. We compare, for each of them, the width of a countable graph with the least upper-bound of the widths of its finite induced subgraphs. A width has the *compactness property* if these two values are equal.

Tree-width has the compactness property [7]. Clique-width has the compactness property *for some gap function* [1], which means that the clique-width of a countable graph is bounded by a function of the least upper-bound of the widths of its finite induced subgraphs.

Rank-width is a width measure on finite graphs investigated first by Oum and Seymour [8, 9]. Its variant called *linear rank-width* (similar to path-width) has been investigated in [3, 5]. Our main notion of *rank-width* for countable graphs uses subcubic *quasi-trees*: "trees" where "paths" between two nodes may have the order type of rational numbers. It has the compactness property. So has *linear rank-width*, based on arbitrary linear orders.

A more natural notion of rank-width based on countable subcubic trees that we call *discrete rank-width*, has compactness with gap : the discrete rank-width is at most twice the least upper bound of the rank-widths of the finite induced subgraphs. The notion of *discrete linear rank-width*, based on discrete linear orders (e.g. the integers) has no compactness property. Our results are in this table.

<i>measure</i>	<i>compactness</i>
rank-width	yes
discrete rank-width	yes for gap $\lambda n.2n$
linear rank-width	yes
discrete linear rank-width	no

To prove that discrete linear rank-width has no compactness property, we consider the disjoint union of countably many copy of an infinite path: it has infinite discrete linear rank-width but its finite induced subgraphs have linear rank-width 1.

This research is based on ideas used for studying the clique-width of countable graphs and for defining the modular decomposition of such graphs [2].

We now define quasi-trees. We associate with each tree T its ternary *betweenness relation*, $B_T \subseteq N_T \times N_T \times N_T$ (N_T is the set of nodes of T) defined as follows:

$B_T(x, y, z)$ holds if and only if x, y, z are pairwise different and y is on the unique path in T between x and z .

We let $Q(T) := (N_T, B_T)$.

Proposition : The betweenness relation $B = B_T$ of a finite or countable tree T satisfies the following properties for all u, x, y, z in N_T :

- A1 : $B(x, y, z) \Rightarrow x \neq y \neq z \neq x$.
- A2 : $B(x, y, z) \Rightarrow B(z, y, x)$.
- A3 : $B(x, y, z) \Rightarrow \neg B(x, z, y)$.
- A4 : $B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u)$.
- A5 : $B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z)$.

$$\begin{aligned}
A6 : B(x, y, z) \wedge B(x, u, z) \Rightarrow \\
y = u \vee (B(x, u, y) \wedge B(u, y, z)) \vee (B(x, y, u) \wedge B(y, u, z)). \\
A7 : x \neq y \neq z \neq x \Rightarrow \\
B(x, y, z) \vee B(x, z, y) \vee B(y, x, z) \vee (\exists w. B(x, w, y) \wedge B(y, w, z) \wedge B(x, w, z)).
\end{aligned}$$

Definitions : A *quasi-tree* is a structure $S = (N_S, B_S)$ such that N_S , called the set of *nodes*, has at least 3 elements, and B_S is a ternary relation that satisfies conditions A1-A7. A *leaf* of T is a node z such that $B_S(x, z, y)$ holds for no x, y .

Let x a node of S . We say that $y, z \in N_S - \{x\}$ are in the *same direction relative to x* if, either $y = z$ or $B_S(y, z, x)$ or $B_S(z, y, x)$ or $B_S(y, u, x) \wedge B_S(z, u, x)$ holds for some node u . This relation is an equivalence, denoted by $y \sim_x z$, and its classes are the *directions relative to x* . The *degree* of x is the number of classes of \sim_x . A node has degree 1 if and only if it is a leaf. We say that S is *cubic* (resp. *subcubic*) if its nodes have degree 1 or 3 (resp. degree at most 3).

A *cut* of a quasi-tree S is a partition $\{X, X^c\}$ of N_S into two convex sets. X is *convex* if: $B_S(x, y, z) \wedge x \in X \wedge z \in X \implies y \in X$.

Definition : *Rank-width.* A *layout* of a graph G is a subcubic quasi-tree S whose set of leaves L_S contains V_G . Its *width* $rdw(G, S)$ is the least upperbound of the ranks of the adjacency matrices $M_G[X \cap V_G, X^c \cap V_G]$ over all cuts $\{X, X^c\}$ of S . The rank-width of G , denoted by $rdw(G)$, is the minimal width of all its layouts.

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Minimal Obstructions for Partial Representations of Interval Graphs

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EXTENDED ABSTRACT

An *interval representation* \mathcal{R} is a collection of intervals $\{\langle x \rangle : x \in V(G)\}$ where $\langle x \rangle \cap \langle y \rangle \neq \emptyset$ if and only if $xy \in E(G)$. A graph is an *interval graph* (INT) if it has an interval representation.

Partial Representations. The partial representation extension problem was introduced by Klavík et al. [9]. For interval graphs, a *partial representation* \mathcal{R}' is an interval representation $\{\langle x' \rangle : x \in V(G')\}$ of an induced subgraph G' of G . The vertices of G' are called *pre-drawn*. A representation \mathcal{R} of G *extends* \mathcal{R}' if and only if it assigns the same intervals to the vertices of G' , i.e., $\langle x \rangle = \langle x' \rangle$ for every $x \in V(G')$. The *partial representation extension problem* asks whether there exists an interval representation of G extending \mathcal{R}' .

For interval graphs, the partial representation extension problem was solved in (n^2) time in [9], and currently there are two linear-time algorithms [1, 8] for this problem. A linear-time algorithm for proper interval graphs and an almost quadratic-time algorithm for unit interval graphs are given in [6]. Polynomial-time algorithms are further known for circle graphs [4], and permutation and function graphs [5]. The partial representation extension problems for chordal graphs [7] and contact representations of planar graphs [3] are NP-hard.

Our Results. Lekkerkerker and Boland [10] describe INT by five types of minimal forbidden induced subgraphs which we call *Lekkerkerker-Boland obstructions* (LB). We generalize [10] to extendible partial representations. We add ten new classes of minimal obstructions (see [arXiv:1406.6228](https://arxiv.org/abs/1406.6228) for details):

- *SE obstructions:* These are two simple *shared endpoint obstructions* which deal with shared endpoints in \mathcal{R}' . They are depicted in Fig. .3a.
- *k-FAT obstructions:* The class of *forced asteroidal triple obstructions* is defined inductively. The 1-FAT obstruction contains pre-drawn non-adjacent vertices x_1, y_1 and z_1 such that y_1 is between x_1 and z_1 . Further, x_1 and z_1 are connected by an induced path P_1 and y_1 is non-adjacent to the inner vertices of P_1 . The k -FAT obstruction is constructed from the $(k-1)$ -FAT obstruction as shown in Fig. .3b.
- *Derived Classes:* The remaining eight classes, depicted in Fig. .4, are derived from k -FAT obstructions.

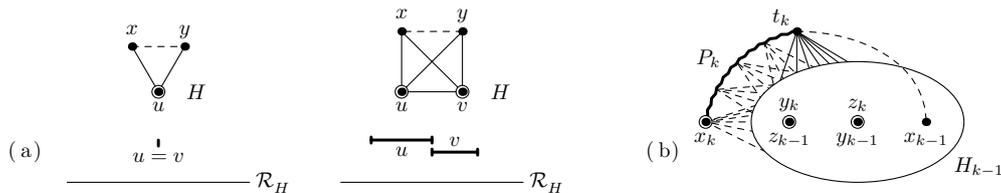
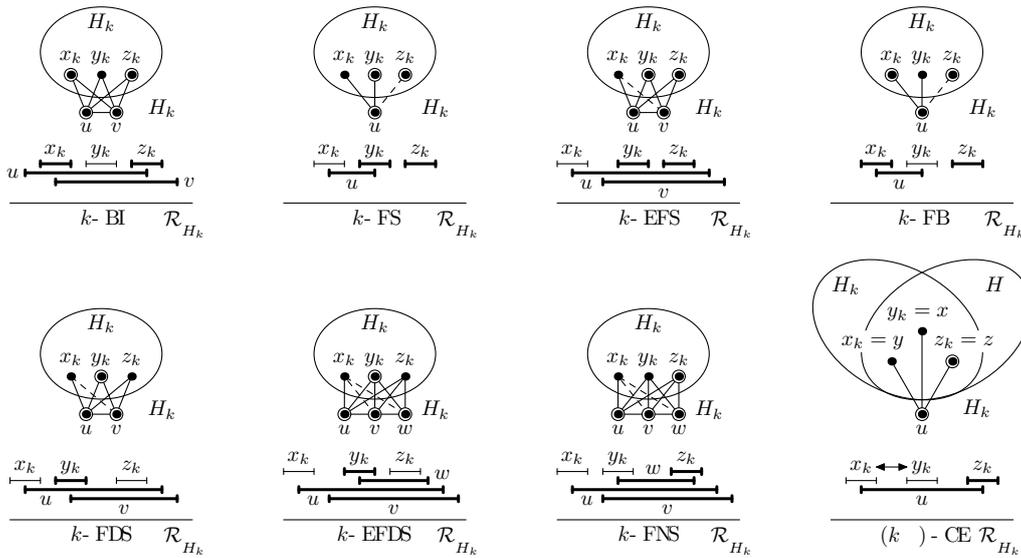


Figure .3: (a) Two SE obstructions. (b) A k -FAT obstruction is created from a $(k-1)$ -FAT obstruction by adding vertices x_k and t_k connected by an induced path P_k . Curly lines denote induced paths, dashed edges are non-edges and pre-drawn intervals are bold.

Figure 4: The classes of obstructions derived from k -FAT.

Theorem 1. A partial representation \mathcal{R}' of G is extendible if and only if G and \mathcal{R}' contain no LB, SE, k -FAT, k -BI, k -FS, k -EFS, k -FB, k -FDS, k -EFDS, k -FNS and (k, ℓ) -CE obstructions.

Corollary 2. A partial representation is extendible if and only if every four pre-drawn intervals are extendible by themselves.

Corollary 3. There exists an (nm) certifying algorithm for partial representation extension, where n is the number of vertices and m is the number of edges of the input graph.

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Multigraphs without large bonds are wqo by contraction

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EXTENDED ABSTRACT

Well-quasi-order theory has been the subject of an increasing interest in the last sixty years, from its introduction in the 50's to recent results on graph containment relations. Formally, a *well-quasi-order* (*wqo* for short) is a partial order which contains no infinite decreasing sequence, nor infinite set of pairwise incomparable elements. The most significant result on wqos is certainly the celebrated Graph Minor Theorem, proved at the culminating point of the Graph Minors series of Robertson and Seymour and which can be stated as follows: graphs are well-quasi-ordered by the minor relation [10]. Robertson and Seymour later proved in [9] that this result also holds for the immersion relation. Prior work includes results of Higman on sequences from a wqo [4] and Kruskal's Tree Theorem [6], as well as techniques that are now standard tools in the study of wqos, e.g. the *minimal bad sequence* argument of Nash-Williams [7]. Unlike minors and immersions, most of the common containment relations do not well-quasi-order the class of all graphs, and therefore attention has been naturally brought to subclasses that are well-quasi-ordered. In this direction, Damaschke proved that cographs are well-quasi-ordered by induced subgraphs [1] and Ding characterized ideals that are well-quasi-ordered by the subgraph relation [3]. Last, Liu and Thomas recently announced that graphs not containing the elements of a set of graphs called "Robertson chain" are well-quasi-ordered by the topological minor relation [5].

In this paper, we consider graphs where parallel edges are allowed, but not loops. The usual contraction relation is extended to this setting in the obvious way: an *edge contraction* is the operation that identifies two adjacent vertices and deletes the possibly created loops (but keeps multiple edges), and a graph H is said to be a *contraction* of a graph G , denoted $H \trianglelefteq G$, if it can be obtained from G by a sequence of edge contractions. A *bond* is a minimal non-empty collection of edges disconnecting the graph. The contraction relation defines a partial order on the class of graphs. However, this order is not a wqo, and an illustration of this fact is the infinite sequence of incomparable graphs $\{\theta_i\}_{i \in \mathbb{N}}$, where θ_k is the graph with two vertices and k edges, for every integer k . Our main result is the following.

Theorem 1. *Let \mathcal{G} be a class of connected graphs and let $k \geq 1$ be an integer. If no graph of \mathcal{G} has a bond of order more than k , then \mathcal{G} is well-quasi-ordered by \trianglelefteq .*

Remark that a graph does not have a bond of order more than k iff it contains no graph of $\{\theta_i\}_{i > k}$ as contraction. The *torso* of a bag X_t of a tree-decomposition $(T, (X_t)_{t \in V(T)})$ of a graph G is the graph obtained from $G[X_t]$ by adding all edges $\{u, v\} \in X_t \cap X_{t'}$ for some neighbor t' of t . Our proof of Theorem 1 uses the two following propositions.

Proposition 2 (Tutte, [11], see also [2, Exercise 20 of Chapter 12]). *Every 2-connected graph has a tree-decomposition $(T, (X_t)_{t \in V(T)})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and all torsos are either 3-connected, or a cycle.*

In the sequel, such a decomposition is called a *Tutte decomposition*.

Proposition 3 (Oporowski et al., [8]). *For every $k \in \mathbb{N}^*$ there is a $w_k \in \mathbb{N}$ such that every 3-connected graph of order at least w_k has a bond of order k .*

Sketch of the proof of Theorem 1. A sequence of graphs $\{G_i\}_{i \in \mathbb{N}}$ is said to be *bad* if there is no pair $(i, j) \in \mathbb{N}$ with $i < j$ and $G_i \trianglelefteq G_j$. If additionally for every $i \in \mathbb{N}$, G_i is a smallest graph such a bad sequence starts with G_1, \dots, G_i , then this bad sequence is *minimal*. Let \mathcal{G}_k be the class of graphs not containing a bond of order more than k , for some $k \in \mathbb{N}$. In order to be able to use Proposition 2, we first prove a structural lemma that reduces the study of (unlabeled) graphs of \mathcal{G}_k to the case of 2-connected graphs of \mathcal{G} that are labeled by the elements of a wqo.

In the sequel, we deal with labeled graphs. Remark that we can encode every 2-connected G graph of \mathcal{G}_k as the sequence $\langle J, G_1, \dots, G_l \rangle$, where J is the torso of an arbitrarily chosen bag X_t in a Tutte decomposition $(T, (X_t)_{t \in V(T)})$ of G , and G_1, \dots, G_l are the subgraphs induced in G by bags corresponding to the connected components of $T \setminus t$. Roughly speaking, constructing a graph from such an encoding can be done by gluing each G_i to the edge of J it corresponds to. Proposition 3 ensures that J will be either a cycle, or one of the finitely many possible 3-connected graphs of \mathcal{G}_k .

By contradiction, let us consider a minimum bad sequence $\{A_i\}_{i \in \mathbb{N}}$ of \mathcal{G}_k and for each $i \in \mathbb{N}$ a sequence Q_i encoding A_i . We are able to prove that the set of graphs used in these encodings of graphs is well-quasi-ordered by \trianglelefteq , by an argument similar to the one of Nash-Williams in [7] with some more technicalities due to the nature of the contraction relation. Let us say that two encodings of $\{Q_i\}_{i \in \mathbb{N}}$ are *related* if their first elements are isomorphic, or if both are a cycle. Remark that this is an equivalence relation, and that there is a finite number of equivalence classes, as a consequence of Proposition 3. Therefore, $\{Q_i\}_{i \in \mathbb{N}}$ has a subsequence $\{Q_{\varphi(i)}\}_{i \in \mathbb{N}}$ (where $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is an increasing function) whose elements are all related, otherwise it would be finite. The following lemma shows that the subsequence $\{A_{\varphi(i)}\}_{i \in \mathbb{N}}$ is well-quasi-ordered by \trianglelefteq , a contradiction.

Lemma 4. *Let $(\mathcal{H}, \trianglelefteq)$ be a wqo of 2-rooted graphs. The class of graphs obtained by attaching graphs of \mathcal{H} to the edges of cycles (resp. to of a fixed graph J) is well-quasi-ordered by \trianglelefteq .*

Therefore, there is no bad sequence in \mathcal{G}_k , and this concludes the proof.

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Structure of complement bipartite graphs

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EXTENDED ABSTRACT

From the starting point of Hall's stable marriage theorem, a new notion of complement graph is introduced. This concept is defined only in the Category of Bipartite Graphs, which means that all graphs considered, even the new complement graph should be bipartite, except if the general case is stated explicitly.

If $G = (V, W; E)$ is a bipartite and connected, we define the bipartite complement graph of G as the bipartite graph G_b^c with the same bipartition $V \cup W$ and where $u \in V$ and $w \in W$ are adjacent in G_b^c if and only if they are not adjacent in G . The basic properties of this new concept are established. The bipartite complement is directly used to prove the Complete Unstable Marriage Theorem. Many examples illustrate this notion.

If G is not connected there is, in general, not a unique bipartite complement associated to G . We clarify this case and give several results on the structure of the bipartite complement graph in case where G is not connected. The main results are the case where special connected components are considered and when we establish the conditions under which the bipartite complement is connected while the original graph is not. This is an analogous of the classical result saying that the usual complement of a non connected graph is connected. This new concept in bipartite graphs theory raises many new problems, some of which are given in the conclusion, especially the notion of b -self complement graph. The notion of bipartite matrix is helpful in an algorithmic approach of bipartite complementation.

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The Rectangle Covering Number of Random Boolean Matrices

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EXTENDED ABSTRACT

The *rectangle covering number* of an $n \times n$ Boolean matrix M is the smallest number of 1-rectangles which are needed to cover all the 1-entries of M . The concept is an important lower bound in Communication Complexity and Combinatorial Optimization [11]. It is also known as the *Boolean rank* [1] of M : the smallest k such that M is the product of an $n \times k$ and a $k \times n$ matrix under Boolean arithmetic ($1 + 1 = 1$). Under the name of *biclique covering number* of a bipartite graph, the concept is important in graph theory and algorithmics (e.g., [8, 1]).

By a construction of Lovász and Saks [7], the rectangle covering number is the chromatic number of a graph constructed from M : The Lovász-Saks matrix-graph $G_{\mathbb{B}}(M)$ has as its vertex set the set of pairs (k, ℓ) with $M_{k, \ell} = 1$, and two vertices $(k, \ell), (k', \ell')$ are adjacent iff $M_{k, \ell'} M_{k', \ell} = 0$. Every graph is an induced subgraph of a Lovász-Saks matrix-graph of some matrix.

Generally speaking, the importance of the Lovász-Saks construction lies in making available the tools and approaches of graph coloring to the application areas of rectangle covering (cf., e.g., [5]). (Specifically for Communication Complexity, however, its overwhelming importance comes from the fact that it connects the famous log-rank conjecture in Communication Complexity to the equivalent conjecture in graph theory: the chromatic number of a graph is at most 2 to a polylog in the rank of the adjacency matrix.)

In our paper, we study the rectangle covering number of random $n \times n$ matrices, where each entry is 1 with probability p , and the entries are independent.

As usual, we are interested in $n \rightarrow \infty$ and p is allowed to vary with n . To bound the rectangle covering number, i.e., the chromatic number of the Lovász-Saks graph, we also study its clique number (cliques are called “fooling set” in Communication Complexity vernacular [6]; and “sets of independent entries” in Matrix Theory [3]) and its independence number.

In some aspects, the random Lovász-Saks graph behaves like similar to an Erdős-Renyi random graph with density $1 - p^2$, but the differences are more profound.

The clique number of the random Lovász-Saks graph with $p = 1/2$ has been discussed by Dietzfelbinger et al. [4] (see also [10]). The independence number is known for constant p from work of Park & Szpankowski [9]; and the chromatic number for constant $p = 1/2$ appears in Sherstov’s lecture notes [10].

As in the Erdős-Renyi model, things get more difficult with $p \rightarrow 0$ and, especially, $p \rightarrow 1$. For example, the clique number, for $p \rightarrow 0$, is dominated by the largest matching in the bipartite graph defined by matrix, which is well studied. For $p = \Omega(1)$, we prove lower bounds using the first moment method, and upper bounds by analyzing a greedy algorithm. For $p \rightarrow 1$ fast enough, the clique number becomes a constant. Our upper and lower bound differ by a factor of 4. Whereas for the independence number, we rely entirely on concentration bounds to prove that the size of the largest independent set is near $-1/(e \ln p)$, and it is of a particular form.

We conclude with a number of open questions.

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Colorings of iterated subdivided-line graphs

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EXTENDED ABSTRACT

The notion of the Sierpiński graph $S(n, k)$ was introduced by Klavžar and Milutinović [5]. The extended Sierpiński graph $S^{++}(n, k)$ has also been introduced by Klavžar and Mohar [6]. Because of their interesting self-similar structures, various properties on the Sierpiński-like graphs have been investigated: hamiltonicity, hub numbers, colorings, covering codes, average eccentricity, and crossing numbers (e.g., see [2, 8]). Motivated by self-similar structures of Sierpiński graphs, the subdivided-line graph operation Γ has been introduced in [2]. The *subdivided-line graph* $\Gamma(G)$ of an undirected graph $G = (V, E)$ is defined to be the line graph of the barycentric subdivision of G , and Γ is called the *subdivided-line graph operation*. For $n \geq 1$, the *n-iterated subdivided-line graph* $\Gamma^n(G)$ of G is the graph obtained from G by iteratively applying Γ n times. Then, it holds that $S^\circ(n, k) \cong \Gamma^n(K_1^k)$, where K_1^k is the graph with one vertex and k self-loops and $S^\circ(n, k)$ is the graph obtained from $S(n, k)$ by adding self-loops to vertices called extreme vertices. We also have $S^{++}(n, k) \cong \Gamma^{n-1}(K_{k+1})$, where K_{k+1} is the complete graph with $k+1$ vertices (see Figure 1). Thus, the class of iterated subdivided-line graphs generalizes the class of Sierpiński-like graphs. In this paper, we study various colorings (labelings) of iterated subdivided-line graphs.

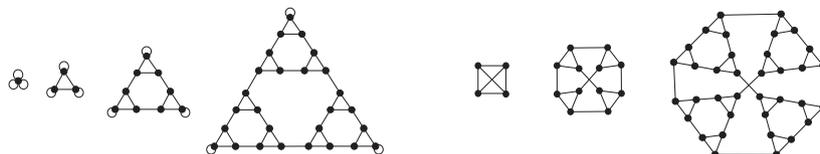


Figure .5: K_1^3 , $\Gamma(K_1^3)$, $\Gamma^2(K_1^3)$, $\Gamma^3(K_1^3)$, K_4 , $\Gamma(K_4)$, and $\Gamma^2(K_4)$.

The disjoint union of m copies of G is denoted by mG . We denote by $\Delta(G)$ the maximum number of edges incident to a vertex in G . A vertex incident to $\Delta(G)$ edges is called a major vertex of G . Let G_{maj} be the graph obtained from G by deleting every edge not incident to a major vertex. For vertex-colorings, we have the following.

Theorem 1. $\chi(\Gamma^n(G)) = \Delta(G)$ for $n \geq 1$.

From Theorem 1, the facts that $\chi(S(n, k)) = k$ and $\chi(S^{++}(n, k)) = k$ on Sierpiński-like graphs shown in [7] and [4], respectively, are obtained as corollaries. A *linear-coloring* of G is a vertex-coloring such that for any pair of colors, the subgraph of G induced by the set of vertices with one of the two colors is a disjoint union of paths. The minimum number of colors in a linear-coloring of G is denoted by $\chi_\ell(G)$. Xue et al. [8] showed the next result.

Theorem 2. [8] $\chi_\ell(S(n, k)) = k$ for $n \geq 1$ and $\chi_\ell(S^{++}(n, k)) = k$ for $n \geq 2$.

In $\Gamma(G)$, each vertex v in G is replaced with the complete graph $K(v)$, and an edge in some $K(v)$ is called a generated edge, while an edge not in any $K(v)$ is called an original edge. For a $\Delta(G)$ -vertex-coloring of $\Gamma^n(G)$ and $S \subseteq \{1, 2, \dots, \Delta(G)\}$, let $\langle S \rangle_n^\circ$ be the graph obtained from the subgraph induced by the vertices with a color in S by adding a self-loop to each vertex not incident to an original edge. Let $K_k^\circ = \Gamma(K_1^k)$. Then, we can show the following result. Considering the case that $|S| = 2$ in Theorem 3, Theorem 2 for $n \geq 3$ is obtained.

Theorem 3. *Let $n \geq 2$. Given a $\Delta(G)$ -vertex-coloring of $\Gamma(G)$, there exists a $\Delta(G)$ -vertex-coloring of $\Gamma^n(G)$ such that for any $S \subseteq \{1, 2, \dots, \Delta(G)\}$,*

$$\langle S \rangle_n^\circ \cong \Gamma^{n-1}(\langle S \rangle_1^\circ) \cup (\cup_{v \in V(G), 0 \leq i \leq n-2} |K(v) - \langle S \rangle_1^\circ|^i \cdot |K(v)|^i \cdot \Gamma^{n-2-i}(K_{|K(v) \cap \langle S \rangle_1^\circ}^\circ)).$$

For edge-colorings, we have the next result which generalizes the results that $\chi'(S(n, k)) = k$ and $\chi'(S^{++}(n, k)) = k$ on edge-colorings of Sierpiński-like graphs proved in [4].

Theorem 4. *Let $n \geq 1$. If $\Delta(G)$ is even, then $\chi'(\Gamma^n(G)) = \Delta(G)$. If $\Delta(G)$ is odd, then $\Delta(G) \leq \chi'(\Gamma^n(G)) \leq \chi'(G_{\text{maj}})$.*

A k - $L(p, q)$ -labeling of G is a function $f : V(G) \mapsto \{0, 1, \dots, k\}$ such that $|f(u) - f(v)| \geq p$ for any pair of adjacent vertices u and v , and $|f(u) - f(v)| \geq q$ for any pair of vertices u and v at distance two. The minimum k for a k - $L(p, q)$ -labeling of G is denoted by $\lambda_{p,q}(G)$. For $L(1, 1)$ -labelings, we have the next result from which the results that $\lambda_{1,1}(S(n, k)) = k$ and $\lambda_{1,1}(S^{++}(n, k)) = k$ follow.

Theorem 5.

$$\Delta(G) \leq \lambda_{1,1}(\Gamma^n(G)) \leq \begin{cases} \lambda_{1,1}(G) & \text{if } G \text{ has a major vertex without a self-loop,} \\ \lambda_{1,1}(G) + 1 & \text{if every major vertex of } G \text{ has a self-loop.} \end{cases}$$

Multiplying the upper bounds in Theorem 5 by two, we have upper bounds on $\lambda_{2,1}(\Gamma^n(G))$. Gravier, Klavžar, and Mollard [1] showed that $\lambda_{2,1}(S(n, k)) = 2k$ for $n \geq 2$ and $k \geq 3$. From their result, a general lower bound on $\lambda_{2,1}(\Gamma^n(G))$ is obtained, since $S(2, \Delta(G)) \subseteq \Gamma^n(G)$ for $n \geq 2$. Therefore, we have the following result.

Theorem 6. *Let $n \geq 2$ and $\Delta(G) \geq 3$.*

$$2\Delta(G) \leq \lambda_{2,1}(\Gamma^n(G)) \leq \begin{cases} 2\lambda_{1,1}(G) & \text{if } G \text{ has a major vertex without a self-loop,} \\ 2\lambda_{1,1}(G) + 2 & \text{if every major vertex of } G \text{ has a self-loop.} \end{cases}$$

From Theorem 6, the new result that $\lambda_{2,1}(S^{++}(n, k)) = 2k$ for $n \geq 2$ and $k \geq 3$ is obtained. Besides, using Theorem 5, we can show the next result on total-colorings.

Theorem 7. *Let $n \geq 1$. If $\Delta(G)$ is odd, then $\chi''(\Gamma^n(G)) = \Delta(G) + 1$. If $\Delta(G)$ is even, then*

$$\Delta(G) + 1 \leq \chi''(\Gamma^n(G)) \leq \begin{cases} \lambda_{1,1}(G) + 1 & \text{if } G \text{ has a major vertex without a self-loop,} \\ \lambda_{1,1}(G) + 2 & \text{if every major vertex of } G \text{ has a self-loop.} \end{cases}$$

Applying Theorem 7 to K_k° and K_{k+1} , the result that $\chi''(S(n, k)) = k + 1$ shown in [3] and the new result that $\chi''(S^{++}(n, k)) = k + 1$ are obtained, respectively.

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Chromatic number of P_5 -free graphs: Reed's conjecture

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EXTENDED ABSTRACT

In this paper we study the chromatic number of P_5 -free graphs. Our work was motivated by the following conjecture of Gyárfas.

Conjecture 1. (*Gyárfas' conjecture [1]*)

Let T be any tree (or forest). Then there is a function f_T such that every T -free graph G satisfies $\chi(G) \leq f_T(\omega(G))$.

Gyárfas [1] proved this conjecture when T is a path P_k for all $k \geq 3$ by showing $\chi(G) \leq (k-1)\omega(G)^{k-1}$. The currently best known upper bound for P_5 -free graphs is due to Esperet, Lemoine, Maffray, and Morel [2]

Theorem 2. *Let G be a P_5 -free graph with clique number $\omega(G)$. Then $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$.*

One may wonder whether this exponential bound can be improved. In particular:

Question 3. *Is there a polynomial (χ -bounding) function f_k such that every P_k -free graph G satisfies $\chi(G) \leq f_k(\omega(G))$?*

Our first main result is the following:

Theorem 4. *There is no linear χ -bounding function for the class of P_5 -free graphs.*

In 1998, Reed proposed the following Conjecture which gives, for any graph G , an upper bound for the chromatic number $\chi(G)$ in terms of the clique number $\omega(G)$ and the maximum degree $\Delta(G)$.

Conjecture 5. (*Reed's conjecture [3]*)

Every graph G satisfies $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$.

Reed's conjecture is still open in general. It has shown to be true for some classes of graphs, especially for several subclasses of P_5 -free graphs like (P_5, C_4) -free graphs, $(P_5, \text{kite}, \text{bull})$ -free graphs (Fouquet and Vanherpe [4]), and $(P_5, \text{house}, \text{dart})$ -free graphs (Fouquet and Vanherpe [4]).

Question 6. *Does Reed's conjecture hold for the class of P_5 -free graphs?*

Our second main result is that the answer is affirmative in the asymptotic sense.

Theorem 7. *For every fixed $\omega \geq 3$ there exists $n(\omega)$ such that if G is a connected P_5 -free graph of order $n \geq n(\omega)$ and clique number ω , then $\chi(G) \leq \lceil \frac{\omega(G) + \Delta(G) + 1}{2} \rceil$.*

We now relax question 6.

Question 8. *Does Reed's conjecture hold for the class of (P_5, F) -free graphs, where $F \in \{\text{House}, \text{bull}, \text{dart}, \text{kite}\}$?*

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Steinberg’s Conjecture, the Bordeaux Coloring Conjecture and Near-Coloring

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EXTENDED ABSTRACT

It was shown by Garey, Johnson, and Stockmeyer [6] that the 3-coloring decision problem for planar graphs is NP-complete. However, there are some interesting and well-studied characterizations of planar graphs that are 3-colorable. One famous condition is a result of Grötzsch [7] who proved that every planar triangle-free graph is 3-colorable. In 1976, Steinberg [9] conjectured the following related result:

Conjecture 1. *Steinberg’s conjecture.* *Every planar graph without 4- and 5- cycles is 3-colorable.*

Steinberg’s conjecture is still open. Much interest has been given to this and related 3-coloring problems in recent years. Most partial results involve excluding additional cycle lengths, such as excluding cycles of length 4 through 7 [3] or adding other structural restrictions to the graphs as in [1], [2], or [4]. There even exist results for graphs on arbitrary surfaces. For instance, Zhao [10] showed that for every surface Σ , there exists some k so that if G is a graph embedded on Σ and there are no cycles of length 4 up to length k , then G is 3-colorable.

In 2003, Raspaud and Borodin [4] conjectured another related result that every planar graph with minimum distance between triangles at least one and without 5-cycles is 3-colorable. This conjecture, called the Bordeaux coloring conjecture, is stated as follows:

Conjecture 2. *Bordeaux coloring conjecture.* *Every planar graph without intersecting triangles and 5-cycles is 3-colorable.*

By intersecting triangles we mean triangles sharing edges or vertices. Another stronger version of the Bordeaux conjecture is described as follows:

Conjecture 3. *Strong Bordeaux coloring conjecture.* *Every planar graph with no adjacent triangles and no 5-cycles is 3-colorable.*

Adjacent triangles refer to several triangles sharing edges. Both Conjectures 1 and 2 follow if the strong Bordeaux conjecture holds. In this presentation we will focus on the Bordeaux conjecture. Rather than prove a theorem about 3-colorability, we will prove a related statement about the family of planar graphs satisfying the hypotheses of the Bordeaux conjecture.

We now define properties associated with the concept of near-coloring. A graph G is called *improperly* (d_1, d_2, \dots, d_k) -colorable, or just (d_1, d_2, \dots, d_k) -colorable, if the vertex set of G can be partitioned into k sets V_1, V_2, \dots, V_k , such that for every i , $1 \leq i \leq k$, the subgraph $G[V_i]$ has maximum degree at most d_i . Such an improper coloring is defined as a *near-coloring*. As an example, the Four Color Theorem can be restated by saying that every planar graph is $(0, 0, 0, 0)$ -colorable. Similarly, Steinberg’s conjecture states that every planar graph without cycles of length 4 and 5 is $(0, 0, 0)$ -colorable.

A recent paper of Chang, Havet, Montassier, and Raspaud [5] relates Steinberg’s conjecture and near-coloring. The authors prove that if \mathcal{F} is the family of planar graphs without cycles of 4 and 5, every graph of \mathcal{F} is $(2, 1, 0)$ -colorable and $(4, 0, 0)$ -colorable. Furthermore, O. Hill and G. Yu [8] recently improved this result to show that all graphs of \mathcal{F} are, in fact, both $(1, 1, 0)$ -colorable and $(3, 0, 0)$ -colorable, a step closer to Steinberg’s conjecture.

In light of the aforementioned results, our aim is to prove analogous results related to the Bordeaux conjecture. Define \mathcal{H} as the family of planar graphs without cycles of length 5 and without intersecting triangles. In this presentation, we will describe aspects of the proof of the following theorem:

Theorem 4. *Every graph of \mathcal{H} is $(4,0,0)$ -colorable and $(2,1,0)$ -colorable.*

Notice that the set of graphs in \mathcal{H} is significantly different than the graphs in \mathcal{F} because \mathcal{H} includes cycles of length four. Based on this difference, the discharging rules and structural lemmas are more complicated than in the near-coloring based papers associated primarily with Steinberg's conjecture. We will discuss some modifications of the discharging rules to handle cycles of length four and give a description of structural results associated with our proof. If time permits, we will also discuss open problems including some involving coloring by an independent set and a set that induces a forest.

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On the Number of Optimal Identifying Codes in Twin-Free Graphs

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EXTENDED ABSTRACT

1 Introduction, notation, and basic definitions

Let $G = (V, E)$ be a simple, undirected, connected graph with vertex set V and edge set E . The *distance* between any two vertices x and y , denoted by $d_G(x, y)$, is the length of any shortest path between x and y .

For any vertex $v \in V$ and integer $r \geq 1$, the *ball of radius r and centre v* , denoted by $B_{G,r}(v)$, is the set of vertices within distance r from v :

$$B_{G,r}(v) = \{x \in V : d_G(v, x) \leq r\}.$$

Two vertices x and y with $B_{G,r}(x) = B_{G,r}(y)$ are called (G, r) -*twins*. If G has no (G, r) -twins, then we say that G is *r -twin-free* or *r -identifiable*.

Whenever two vertices x and y are within distance r from each other in G , i.e., $x \in B_{G,r}(y)$ and $y \in B_{G,r}(x)$, we say that x and y *r -cover* each other. When three vertices x, y, z are such that $x \in B_{G,r}(z)$ and $y \notin B_{G,r}(z)$, we say that z *r -separates* x and y in G (note that $z = x$ is possible). A set is said to *r -separate* x and y in G if it contains at least one vertex which does.

A *code* C is simply a subset of V , and its elements are called *codewords*. For each vertex $v \in V$, the *r -identifying set* of v , with respect to C , is the set of codewords r -covering v , and is denoted by $I_{G,C,r}(v)$:

$$I_{G,C,r}(v) = B_{G,r}(v) \cap C.$$

We say that C is an *r -dominating* code in G if all the sets $I_{G,C,r}(v)$, $v \in V$, are nonempty.

We say that C is an *r -identifying* code ([5]; see also the bibliography at [6] for references on identifying codes and related topics) if all the sets $I_{G,C,r}(v)$, $v \in V$, are nonempty and distinct: in other words, every vertex is r -covered by at least one codeword, and every pair of vertices is r -separated by at least one codeword. Or: given the (nonempty) identifying set $I_{G,C,r}(v)$ of an unknown vertex $v \in V$, we can uniquely recover v (we also say that we *r -identify* v).

It is quite easy to observe that a graph G admits an r -identifying code if and only if G is r -twin-free; this is why r -twin-free graphs are also called r -identifiable.

When G is r -twin-free, we denote by $\gamma_r(G)$ the smallest cardinality of an r -identifying code in G . Any r -identifying code C such that $|C| = \gamma_r(G)$ is said to be *optimal*. The search for an optimal r -identifying code in given graphs or families of graphs is an important part of the studies devoted to identifying codes. In general, this problem is NP-hard [1].

2 Main results

In this communication, following [7] and [8] where the notion of “completely different codes” is discussed in the framework of infinite lattices, we are interested in finding graphs which have a large number of different optimal r -identifying codes. Typically, we construct graphs with n vertices which admit $2^{\alpha n}$ different optimal r -identifying codes, and we want to have α as close to 1 as possible. Our results are (see [4] for the proofs and other results on the subject):

- for $r = 1$, there exist graphs with n vertices and about $2^{0.77003n}$ optimal identifying codes;
- for $r > 1$, there exist graphs with n vertices and about $2^{0.664n}$ optimal identifying codes;

Note that if we are interested in the number of r -identifying codes that a graph G can admit (without requiring optimality), then we obtain the upper bound

$$\sum_{i=\lceil \log_2(n+1) \rceil}^n \binom{n}{i} = 2^n - \sum_{i=0}^{\lceil \log_2(n+1) \rceil - 1} \binom{n}{i}.$$

On the other hand, consider the graph G defined as a $q \times q$ square array of vertices in which any two vertices are adjacent if and only if they are on the same row or column. Then we may prove that, if we set $n = q^2$, we have at least

$$2^n \left(1 - \frac{n(\sqrt{n} + 1)}{2\sqrt{n}}\right)$$

1-identifying codes in G .

In comparison, some results are known for 1-dominating codes with respect to *minimality* (for inclusion): it has been proved in [3] that graphs with n vertices exist which admit $2^{0.651n}$ different minimal 1-dominating codes, and that any graph with n vertices admits at most $2^{0.779n}$ different minimal 1-dominating codes; the upper bound is obviously valid also for optimal codes, and the lower bound is obtained by a construction which works for optimal codes too. See also [2]. The clique on n vertices is an obvious example where all the $2^n - 1$ nonempty subsets of vertices are 1-dominating codes.

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Identifying codes and VC-dimension

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EXTENDED ABSTRACT

An identifying code of a graph is a subset of its vertices such that every vertex of the graph is uniquely identified by its closed neighbourhood within the code. We consider lower bounds and approximation algorithms for the minimum size of identifying codes in classes of graphs closed under induced subgraphs.

We first give a dichotomy result for the size of the smallest identifying code in these classes. Our dichotomy is derived from the *Vapnik-Chervonenkis dimension* (VC-dimension for short) of the considered class, defined as follows. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A subset of vertices X is *shattered* if for every subset X' of X , there exists a hyperedge e in \mathcal{E} such that $e \cap X = X'$. The VC-dimension of \mathcal{H} is the maximal size of a shattered set of \mathcal{H} . We define the VC-dimension of a graph G as the VC-dimension of the hypergraph of the *closed* neighbourhoods of G . The VC-dimension of a class of graphs \mathcal{C} is the maximum VC-dimension among the elements of \mathcal{C} . If it is not bounded, we say that \mathcal{C} has infinite VC-dimension.

Theorem 1. *Let \mathcal{C} be a class closed under induced subgraphs. Either*

- 1. \mathcal{C} has infinite VC-dimension and then, for every $k \in \mathbb{N}$, there exists a graph $G_k \in \mathcal{C}$ with more than $2^k - 1$ vertices and an identifying code of size $2k$, or*
- 2. \mathcal{C} has finite VC-dimension d and then, there is a constant c such that no graph $G \in \mathcal{C}$ with n vertices has an identifying code of size smaller than $n^{1/d}$.*

The second part of this theorem allows us to derive lower bounds for several classes of graphs such as graphs with girth at least 5, chordal bipartite graphs, interval graphs, permutations graphs, undirected path graphs. The order of the lower bound can be tight (interval graphs, graphs of girth at least 5) or not (line graphs, planar graphs, permutation graphs).

We then turn to approximation algorithms. Foucaud [1] proved that `MIN ID CODE`, the problem of computing an identifying code of minimum size, is log-APX-hard even restricted to bipartite graphs, split graphs or co-bipartite graphs. In particular, `MIN ID CODE` cannot be approximated to within a $c \cdot \log |V|$ factor (for some $c > 0$) in polynomial time in these three classes. We prove that any class closed under induced subgraphs with infinite VC-dimension must contain one of these three classes.

Theorem 2. *Let \mathcal{C} be a class of graphs closed under induced subgraphs. If \mathcal{C} has infinite VC-dimension, then it contains either the class of bipartite graphs or the class of split graphs or the class of co-bipartite graphs.*

Corollary 3. `MIN ID CODE` is log-APX-hard when the input graph is restricted to a class of graphs closed under induced subgraphs of infinite VC-dimension.

In classes of finite VC-dimension, constant approximation algorithms are known for line graphs [2], planar graphs [5] or unit-interval graphs [3]. We complete this study by giving a constant factor approximation algorithm for interval graphs, answering an open question in [1]. Our algorithm is based on linear programming.

Theorem 4. *There is a polynomial time 6-approximation algorithm for MIN ID CODE on interval graphs.*

However, not all classes of finite VC-dimension admit a constant approximation algorithm. Indeed, we show the class of C_4 -free bipartite graphs which is of VC-dimension 2 has no such algorithm.

Theorem 5. *MIN ID CODE with input restricted to C_4 -free bipartite graphs cannot be approximated to within a $c \cdot \log n$ factor (for some $c > 0$) in polynomial time, unless $NP \subseteq ZTIME(n^{O(\log \log n)})$ (where n is the size of the input).*

This theorem is proved by reducing MIN ID CODE to SET COVER WITH INTERSECTION 1 which cannot be approximated to within a $c \log n$ factor for some $c > 0$ in polynomial time [4].

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Neighbour distinguishing colourings of graphs

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Abstract

Consider a simple graph $G = (V, E)$ and its *proper* edge colouring c with the elements of the set $\{1, 2, \dots, k\}$. The colouring c is said to be *neighbour sum distinguishing* if for every pair of vertices u, v adjacent in G , the sum of colours of the edges incident with u is distinct from the corresponding sum for v . The smallest integer k for which such colouring exists is known as the *neighbour sum distinguishing index* of a graph and denoted by $\chi'_{\Sigma}(G)$. The definition of this parameter, which makes sense for graphs containing no isolated edges, immediately implies that $\chi'_{\Sigma}(G) \geq \Delta$, where Δ is the maximum degree of G . On the other hand, it was conjectured by Flandrin et al. that $\chi'_{\Sigma}(G) \leq \Delta + 2$ for all those graphs, except for C_5 . We shall sketch a proof that this bound is asymptotically correct, as indeed $\chi'_{\Sigma}(G) \leq \Delta(1 + o(1))$. The main idea of the argument confirming this fact relies on a random assignment of the colours, where the choice for every edge is biased by so called *attractors*, randomly assigned to the vertices. The conjecture's exact bound in turn holds for several classes of graphs, as e.g. planar graph of sufficiently large maximum degree. Similar results concerning the total version of the problem, as well as the protoplast of our problem focused on distinguishing neighbours by sets rather than sums shall also be briefly commented on.

EXTENDED ABSTRACT

One of the most elementary facts we learn in the very first lecture of a basic combinatorial course is that every (simple) graph of order at least two contains a pair of vertices of the same degree. This datum gave rise to the natural question studied e.g. by Chartrand, Erdős and Oellermann in [3], on a possible definition of an *irregular graph*, intended as the antonym to the term 'regular graph'. With no convincing individual solution to the problem, Chartrand et al. [2] altered towards measuring the 'irregularity of a graph' instead. Suppose that given a graph $G = (V, E)$ we wish to construct a multigraph with pairwise distinct vertex degrees of it by multiplying some of its edges. The least k so that we are able to achieve such goal using at most k copies of every edge is known as the *irregularity strength* of G and denoted by $s(G)$, see [2]. Alternatively, one may consider (not necessarily proper) edge colourings $c : E \rightarrow \{1, 2, \dots, k\}$ with the *weighted degree*, defined as

$$d_c(v) := \sum_{e \ni v} c(e),$$

of every vertex $v \in V$ distinct from all the remaining ones. Then the least k which permits defining a colouring c with this feature equals $s(G)$. It is straightforward to notice that $s(G)$ is well defined for all graphs containing no isolated edges and at most one isolated vertex. The irregularity strength was studied in numerous papers and was the cornerstone of the later *additive graph labelings*, or more generally – *vertex distinguishing graph colourings*. Many consequential and related graph parameters have been studied ever since its development.

In particular, a *proper* edge colouring $c : E \rightarrow \{1, 2, \dots, k\}$ is said to be *neighbour sum distinguishing* if for every pair of vertices u, v adjacent in G , $d_c(u) \neq d_c(v)$. The least integer k for which such colouring exists is known as the *neighbour sum distinguishing index* of a graph and denoted by $\chi'_{\Sigma}(G)$. The definition of this parameter, which makes sense for graphs containing no isolated edges, immediately implies that $\chi'_{\Sigma}(G) \geq \Delta$, where Δ is the maximum degree of G . On the other hand, Flandrin et al. posed the following daring conjecture.

Conjecture 1 ([4]). *If G is a connected graph of order at least three different from the cycle C_5 , then $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$.*

We shall sketch a proof that this bound is asymptotically correct.

Theorem 2 ([6]). *If G is a connected graph of maximum degree $\Delta \geq 2$, then $\chi'_\Sigma(G) \leq (1 + o(1))\Delta$.*

The main idea of the argument relies on a random assignment of the colours, where the choice for every edge is biased by so called *attractors*, randomly assigned to the vertices. The conjecture's exact bound in turn holds for several classes of graphs. In particular, we prove the following.

Theorem 3 ([1]). *Any planar graph G with $\Delta(G) \geq 28$ and no isolated edge satisfies $\chi'_\Sigma(G) \leq \Delta(G) + 1$.*

Similar results concerning the total version of the problem, as well as the protoplast of our problem focused on distinguishing neighbours by sets rather than sums shall also be briefly commented on. In particular, the latter of these was a great boost to the field, as it triggered and inspired a large number of associated results and new concepts, including the main focus of this talk. The least integer k so that a proper colouring $c : E \rightarrow \{1, 2, \dots, k\}$ exists such that the sets of colours incident with u and v are distinct for every edge uv of G is called the *neighbour set distinguishing index* (or *adjacent strong chromatic index*) of G , denoted by $\chi'_a(G)$. This graph invariant was introduced by Zhang, Liu and Wang in 2002 together with the following challenging conjecture

Conjecture 4 ([7]). *If G is a connected graph of order at least three different from the cycle C_5 , then $\chi'_a(G) \leq \Delta(G) + 2$.*

This was verified for several graph classes. In particular it was studied in a few papers for the case of planar graphs, and confirmed for those of sufficiently large maximum degree. The following asymptotically best result was also proved by Hatami by means of a multistage probabilistic construction.

Theorem 5 ([5]). *If G is a graph with no isolated edges and with maximum degree $\Delta > 10^{20}$, then $\chi'_a(G) \leq \Delta + 300$.*

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The Cost of 2-Distinguishing Graphs

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EXTENDED ABSTRACT

A graph is called *d-distinguishable* if there is a ‘coloring’ of the vertices with the integers $1, \dots, d$ so that no nontrivial automorphism of G preserves the colors. The idea is to break the symmetries of the graph with the minimum number of colors. This concept was introduced by Albertson and Collins [2] and has spawned a wealth of results for both finite and infinite graphs.

Though K_n requires n colors and $K_{m,n}$ requires $\max\{m, n\}$, there are many families of graphs in which each member requires only 2 colors. For finite graphs these include: hypercubes Q_n with $n \geq 4$; Cartesian powers G^n for a connected graph $G \neq K_2, K_3$ and $n \geq 2$ [10]; Kneser graphs $K_{n:k}$ with $n \geq 6, k \geq 2$ [1]; and (with seven small exceptions) 3-connected planar graphs [9]. Families of infinite graphs include: tree-like graphs with not more than continuum many vertices; the hypercube of dimension \aleph_0 [11]; and locally finite, connected graphs whose automorphism group is countably infinite [12].

Interestingly, in many such cases the size of the smaller color class can be extremely small. For example the hypercube Q_{2^k} of dimension 2^k can be 2-distinguished by coloring $k+2$ vertices red and the remaining $2^{2^k} - k - 2$ vertices white [5]. In this talk we consider upper bounds on the minimum size of such a color class for a variety of graph families.

More formally, all a color class in a 2-distinguishing coloring a *distinguishing class* and call the minimum size of a distinguishing class for a 2-distinguishable graph G the *cost of 2-distinguishing G* , denoted by $\rho(G)$. As a tool, we will also use the determining number, $\text{Det}(G)$, the smallest size of a set of vertices with the property that every automorphism of the graph is uniquely determined by its action on this set [3]. This is sometimes called the size of a base for the automorphism group action.

This talk will introduce the concepts above and then sample some of the following results for finite graphs:

- $\rho(K_{2^m-1, 2^m-1}) = m + 1$ [5];
- $\rho(Q_{2^m-2}) = \rho(Q_{2^m-1}) = \rho(Q_{2^m}) = m + 2$ [5];
- $\lceil \log_2 n \rceil + 1 \leq \rho(Q_n) \leq 2\lceil \log_2 n \rceil - 1$ for $n \geq 5$ [4];
- if $\text{Det}(G) \leq k$ and $\max\{2, \text{Det}(G)\} < \text{Det}(G^k)$, then $\rho(G^k) \in \{\text{Det}(G^k), \text{Det}(G^k) + 1\}$ [6];
- if G and H are relatively prime, H is 2-distinguishable with a distinguishing class of size at least as large as $\text{Det}(G)$, then $\rho(G \square H) = \max\{\rho(H), \text{Det}(G)\}$ [8].

This talk will also discuss results for connected, locally finite, infinite graphs with infinite automorphism group, including:

- $\rho(G) < \aleph_0$ if and only if $\text{Aut}(G)$ is countable;
- if $\text{Aut}(G)$ is countable, $\rho(G) < 3\text{Det}(G)$; and
- if G exhibits linear growth c , then $\rho(G) \leq c + 1$ [7].

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Perfect graphs with no balanced skew-partition are 2-clique-colorable

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EXTENDED ABSTRACT

The chromatic number of a graph G is denoted by $\chi(G)$, and the *clique number* of G (i.e. the size of the maximum clique of G) is denoted by $\omega(G)$. A graph G is *perfect* if for all induced subgraphs H of G , $\chi(H) = \omega(H)$. A graph is *imperfect* if it is not perfect. A graph G is *Berge* if neither G nor \overline{G} (the complement of G) contains an induced odd cycle of length at least five. The Strong Perfect Graph Theorem [3] states that a graph is perfect if and only if it is Berge. One direction of this theorem is an easy exercise: it is easy to see that odd cycles of length at least five and their complements are imperfect, and consequently, all perfect graphs are Berge. However, the proof of other direction (“all Berge graphs are perfect”) is long and complicated, and the main ingredient of the proof is a decomposition theorem for Berge graphs [2, 3]. One of the decompositions in this decomposition theorem is the “balanced skew-partition.” A *skew-partition* of a graph G is a partition (X, Y) of $V(G)$ such that $G[X]$ and $\overline{G}[Y]$ are both disconnected. The *length* of a path is the number of edges that it contains, and a path is *odd* if its length is odd. A skew-partition (X, Y) of G is *balanced* if it satisfies the following two conditions:

- G contains no induced odd path of length greater than one whose endpoints belong to Y and all of whose interior vertices belong to X ;
- \overline{G} contains no induced odd path of length greater than one whose endpoints belong to X and all of whose interior vertices belong to Y .

It was shown in [3] that a minimum imperfect Berge graph does not admit a balanced skew-partition, and so this decomposition was “good enough” for the purposes of proving the Strong Perfect Graph Conjecture. However, for other purposes, the balanced skew-partition is notoriously difficult to handle, and this has led to the study of perfect (equivalently: Berge) graphs that do not admit this particular decomposition (see for instance [4, 6, 8]).

A *clique-coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no inclusion-wise maximal clique of G of size at least two is monochromatic. A graph G is k -clique-colorable if it can be clique-colored with at most k colors, and the *clique-chromatic number* of G is the smallest number k such that G is k -clique-colorable. There exist graphs of arbitrarily large clique-chromatic number (this is because for triangle-free graphs, clique-coloring is the same as proper coloring, and there exist triangle-free graphs of arbitrarily large chromatic number [7]), but it is not known whether the clique-chromatic number of perfect graphs is bounded. Progress in this direction has been made in some special cases. For example, Bacsó et al. [1] proved that all {claw, odd-hole}-free graphs are 2-clique-colorable, and Défossez [5] proved that all {bull, odd-hole}-free graphs are 2-clique-colorable. (If \mathcal{H} is a family of graphs, we say that a graph G is \mathcal{H} -free if no induced subgraph of G is isomorphic to a graph from \mathcal{H} . An *odd-hole* is an induced odd cycle of length at least five, and so every perfect graph is odd-hole-free. The *claw* is the complete bipartite graph $K_{1,3}$, and the *bull* is the five-vertex graph consisting of a triangle and two vertex-disjoint pendant edges.) In addition, Bacsó et al. [1] proved that “almost all” perfect graphs are 3-clique-colorable. On the other hand, there exist perfect graphs whose clique-chromatic number is three: one well-known example is the graph obtained from the cycle of length nine by choosing three evenly spaced vertices and adding edges between them. There are currently no known examples of perfect graphs whose clique-chromatic number is greater than three. All this suggests that it might be true that all perfect graphs are 3-clique-colorable. Our main result is the following.

Theorem 1. *Every perfect graph that does not admit a balanced skew-partition is 2-clique-colorable.*

The main tool used in the proof of Theorem 1 is a decomposition theorem for “tame Berge trigraphs” due to Chudnovsky et al. [4]. Roughly speaking, a “trigraph” is a generalization of a graph in which there are three types of adjacency: strongly adjacent pairs (edges), strongly anti-adjacent pairs (non-edges), and semi-adjacent pairs (pairs with “undetermined” adjacency). Obviously, every graph can be seen as a trigraph (a graph is simply a trigraph with no semi-adjacent pairs). A “tame trigraph” is a trigraph with certain restrictions on the occurrence of semi-adjacent pairs (in particular, every graph is a tame trigraph). While there is no such thing as a “perfect trigraph,” it is straightforward to define a “Berge trigraph” as well as the “balanced skew-partition” for trigraphs. For the purposes of certain decompositions, trigraphs are more convenient to work with than graphs; indeed, trigraphs were originally introduced by Chudnovsky [2] for the purposes of studying decompositions of Berge (tri)graphs. The decomposition theorem from [4] is particularly well suited for the study of tame Berge trigraphs that do not admit a balanced skew-partition.

A *realization* of a trigraph G is any graph that can be obtained from G by turning each semi-adjacent pair of G into an edge or a non-edge. (Thus, a trigraph with m semi-adjacent pairs has 2^m realizations.) We generalize the notion of clique-coloring to trigraphs as follows: a clique-coloring of a trigraph G is an assignment of colors to the vertices of G in such a way that this assignment of colors is a clique-coloring of all the realizations of G simultaneously. Using the decomposition theorem from [4], we show that, with the exception of two small trigraphs (each of which contains exactly two semi-adjacent pairs), every tame Berge trigraph that does not admit a balanced skew-partition is 2-clique-colorable. Since every graph is a tame trigraph that does not contain any semi-adjacent pairs, we obtain as an immediate corollary that every Berge (equivalently: perfect) graph is 2-clique-colorable.

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Cores of cubic graphs: structure and oddness

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Abstract

Let G be a bridgeless cubic graph. Consider a list of k 1-factors of G . Let E_i be the set of edges contained in precisely i members of the k 1-factors. Let $\mu_k(G)$ be the smallest $|E_0|$ over all lists of k 1-factors of G . If G is not 3-edge-colorable, then $\mu_3(G) > 0$. In [2] it is shown that if $\mu_3(G) \neq 0$, then $2\mu_3(G)$ is an upper bound for the girth of G . We show that $\mu_3(G)$ bounds the oddness $\omega(G)$ of G as well. We prove that $\omega(G) \leq \frac{2}{3}\mu_3(G)$. We show that for any given oddness there is a cyclically 4-edge-connected cubic graph G with $\omega(G) = \frac{2}{3}\mu_3(G)$. We also show that the difference between $\omega(G)$ and $\frac{2}{3}\mu_3(G)$ can be arbitrarily big. In addition, we prove that for every integer $k \geq 3$, there exists a bridgeless cubic graph G such that $\mu_3(G) = k$.

EXTENDED ABSTRACT

Let G be a cubic graph and S_3 be a list of three 1-factors M_1, M_2, M_3 of G . Let $\mathcal{M} = E_2 \cup E_3$, $\mathcal{U} = E_0$, and $|\mathcal{U}| = k$. The k -core of G with respect to S_3 (or to M_1, M_2, M_3) is the subgraph G_c of G which is induced by $\mathcal{M} \cup \mathcal{U}$; that is, $G_c = G[\mathcal{M} \cup \mathcal{U}]$. If the value of k is irrelevant, then we say that G_c is a core of G . If $M_1 = M_2 = M_3$, then $G_c = G$. A core G_c is *proper* if $G_c \neq G$. If G_c is a cycle, i.e. the union of pairwise disjoint circuits, then we say that G_c is a *cyclic core*.

Cores are introduced in [2], and were used to prove partial results on some hard conjectures which are related to 1-factors of cubic graphs. In particular, the following conjecture of Fan and Raspaud is true for cubic graphs G with $\mu_3(G) \leq 6$.

Conjecture 1 ([1]). *Every bridgeless cubic graph has a cyclic core.*

Let G be a bridgeless cubic graph and G_c a core of G with respect to three 1-factors M_1, M_2, M_3 . G_c is called a *Petersen core* if the following two conditions hold:

- (1) G_c is cyclic.
- (2) If P is a 5-path of G_c , then there exists no pair of edges e_1, e_2 of P and two integers i, j such that $e_1, e_2 \in M_i \cap M_j$ and $1 \leq i < j \leq 3$.

Let G be a bridgeless cubic graph and M a 1-factor of G . Denote by \overline{M} the complement to M and by $|\overline{M}|_{\text{odd}}$ the number on odd circuits of \overline{M} .

Theorem 2. *Let G be a bridgeless cubic graph. If G_c is a k -core of G with respect to three 1-factors M_1, M_2, M_3 , then $|\overline{M}_1|_{\text{odd}} + |\overline{M}_2|_{\text{odd}} + |\overline{M}_3|_{\text{odd}} \leq 2k$. Moreover, if G_c is a k -core such that the equality holds, then G_c is a Petersen core.*

Corollary 3. *If G is a bridgeless cubic graph, then $\omega(G) \leq \frac{2}{3}\mu_3(G)$. Moreover, if $\omega(G) = \frac{2}{3}\mu_3(G)$, then every $\mu_3(G)$ -core is a Petersen core.*

By the following theorem, we show that there exists an infinite class of cubic graph such that the equality $\omega(G) = \frac{2}{3}\mu_3(G)$ holds. Hence the upper bound $\frac{2}{3}\mu_3(G)$ for $\omega(G)$ is best possible.

Theorem 4. *For every positive integer k , there is a cyclically 4-edge connected cubic graph G of order $36k$ and $\omega(G) = \frac{2}{3}\mu_3(G) = 2k$.*

Next we show that the difference between the oddness of a cubic graph G and $\frac{2}{3}\mu_3(G)$ can be arbitrary big.

Theorem 5. *For any given positive integers k and c , there exists a connected cubic graph G with $\omega(G) = 2k$ and $\mu_3(G) \geq c$.*

Theorem 6. For every positive integer k , there is a cyclically 5-edge connected cubic graph G such that $\omega(G) = 2k$ and $\mu_3(G) = 4k$.

By the following theorem, we give a positive answer to the existence of a cubic graph with any given $\mu_3(G)$.

Theorem 7. For every integer $k \geq 3$, there exists a bridgeless cubic graph G such that $\mu_3(G) = k$.

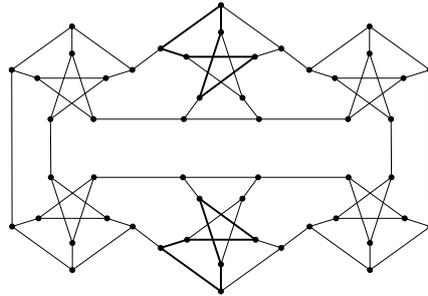


Figure .6: One example for Theorem 4: cubic graph G with $\omega(G) = \frac{2}{3}\mu_3(G) = 4$

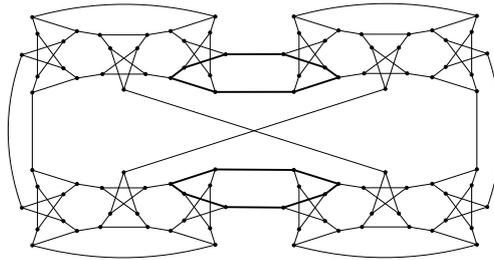


Figure .7: One example for Theorem 6: cubic graph G with $\omega(G) = \frac{1}{2}\mu_3(G) = 4$

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Safe set problem on graphs

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EXTENDED ABSTRACT

Facility location problems involve placing certain facilities on given topologies of living spaces (i.e. cities, buildings, etc.) so that given demands are efficiently served. These are intensively studied in the literature on combinatorial optimization (see [3]). We treat the following variation of the problem: for a given topology of a building, it is required to place temporary accident refuges in addition to business spaces like discussion or conference rooms. Each temporary refuge should be available for the staff in every adjacent business space; hence its capacity must be at least that of each adjacent business space. (To mitigate the space cost, we assume that each temporary refuge will be used by the people in at most one of the adjacent business space.) Subject to the topology of the building being given, how can the temporary refuges be efficiently located so that the amount of business spaces is maximized? To address this problem, we propose the following mathematical model, namely, a *safe set* of a graph.

We use [1] for terminology and notation not defined here. Only finite, simple graphs are considered. We write $|G|$ as shorthand for $|V(G)|$. The subgraph of G induced by the subset $S \subseteq V(G)$ is denoted by $G[S]$. When A and B are vertex-disjoint subgraphs of G , the set of edges that join some vertex of A and some vertex of B is denoted by $E(A, B)$.

A non-empty subset $S \subseteq V(G)$ is a *safe set* if, for every component C of $G[S]$ and every component D of $G - S$, we have $|C| \geq |D|$ whenever $E(C, D) \neq \emptyset$. If $G[S]$ is connected, then S is called a *connected safe set*.

In our model, the graph G describes the topology of the building. The safe sets of G correspond to candidates for locations of the temporary refuges in the building.

As an initial step, let us observe some basic properties in safe sets.

Proposition 1. *Let n be an integer with $n \geq 2$. Any connected graph G of order n has a connected safe set of size at most $\lceil n/2 \rceil$.*

Proof. Let T be a spanning tree of G , and S be any subset of $\lceil n/2 \rceil$ vertices such that $T[S]$ is connected. Clearly, S is a connected safe set. (Q.E.D.)

The following two parameters are therefore well-defined. For a connected graph G , the *safe number* $s(G)$ of G is defined as $s(G) = \min\{|S| : S \text{ is a safe set of } G\}$, and the *connected safe number* $cs(G)$ of G is defined as $cs(G) = \min\{|S| : S \text{ is a connected safe set of } G\}$.

It is easy to see that the path on n vertices has $s(P_n) = cs(P_n) = \lceil n/3 \rceil$, and the cycle on n vertices has $s(C_n) = cs(C_n) = \lceil n/2 \rceil$.

Proposition 2. *Let G be a connected graph. Then $s(G) \leq cs(G) \leq 2s(G) - 1$.*

Proof. It is clear that $s(G) \leq cs(G)$. We prove the second inequality. Let S be a safe set of cardinality $s(G)$, and assume $G[S]$ is not connected.

Let H be the bipartite graph with bipartition (A, B) , where A is the set of components of $G[S]$, B is the set of components of $G - S$, and there is an edge from $a \in A$ to $b \in B$ if there is an edge of G joining a vertex of a to a vertex of b .

The graph H is connected because G is connected. Since $G[S]$ is not connected, $|A| \geq 2$ and hence there exists a vertex of B with degree at least two. Let T be a spanning tree of H . Note that some vertex of B is not a leaf of T .

Let T' be the rooted tree obtained from T by deleting any leaf belonging to B , and choosing as root any remaining vertex $r \in B$. The tree T' has a matching in which every vertex in $b \in B \cap T'$ is

paired with a child vertex in $a_b \in A$. By definition of a safe set, $|b| \leq |a_b|$. Consequently, $V(T')$ is a connected safe set.

Since r is not a leaf of T' , there exists $a \in N_{T'}(r)$ which is not matched with any vertex of B . Thus $|V(T')| < 2s(G)$. (Q.E.D.)

In this work, we showed that the following two decision problems are NP-complete in general and also gave a linear algorithm when the input is restricted to be a tree.

SAFE SET

INSTANCE: A graph G and an integer t .

QUESTION: Does there exist $S \subseteq V(G)$ with $1 \leq |S| \leq t$ such that, for every component C of $G[S]$ and every component D of $G - S$, we have $|C| \geq |D|$ whenever $E(C, D) \neq \emptyset$?

CONNECTED SAFE SET

INSTANCE: A graph G and an integer t .

QUESTION: Does there exist $S \subseteq V(G)$ with $1 \leq |S| \leq t$ such that $G[S]$ is connected and $|S| \geq |C|$ for every component C of $G - S$?

We obtained several results on the safe number of trees. Further results will be presented in this talk. The full paper version of this work is available as [2].

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Colouring graphs with no odd holes

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EXTENDED ABSTRACT

The chromatic number $\chi(G)$ of a graph G is always at least the size of its largest clique (denoted by $\omega(G)$), and there are graphs with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large. On the other hand, the perfect graph theorem asserts that if neither G nor its complement has an odd hole, then $\chi(G) = \omega(G)$. (An "odd hole" is an induced cycle of odd length at least five.)

What happens in between?

With Alex Scott, we have just proved the following, a 1985 conjecture of Gyarfás: For graphs G with no odd hole, $\chi(G)$ is bounded by a function of $\omega(G)$. The proof is short and quite pretty.

Fractional and j -fold colouring of the plane

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EXTENDED ABSTRACT

We present results referring to famous Hadwiger-Nelson problem which asks for the minimum number of colours required to colour the plane in such a way that no two points at distance 1 from each other have the same colour. Nelson first showed that at least 4 colours are needed (see the proof by Mosers [1]) and Isbell was first to prove that 7 colours are enough (this result was published by Hadwiger [2]). For a comprehensive history of the Hadwiger-Nelson problem consult a recent monograph [8]. Exoo [3] considers a more general problem which is colouring of infinite graphs $G_{[a,b]}$, where all points on the plane are vertices and two vertices are neighbours when the distance between them is in the interval $[a, b]$. So the Hadwiger-Nelson problem actually asks for $\chi(G_{[1,1]})$. Exoo proved $\chi(G_{[a,b]}) \geq 5$ for $b > 1.3114 \cdot a$ and conjectured $\chi(G_{[a,b]}) \geq 7$ for $b > a$. We partially answer his conjecture.

Theorem 1. *For any $0 < a < b$ we have $\chi(G_{[a,b]}) \geq 5$.*

The fractional and j -fold chromatic number of the graph $G_{[1,1]}$ was also considered [4]. A j -fold colouring of graph $G = (V, E)$ is an assignment of j -element sets of colours to the vertices of G , in such a way that for any two adjacent vertices the sets assigned to them are disjoint. The smallest number of colours needed for j -fold colouring of a graph G is called j -fold chromatic number and denoted by $\chi_j(G)$. Fractional chromatic number is defined to be: $\chi_f(G) := \inf_{j \in \mathbb{N}} \frac{\chi_j(G)}{j}$. Since 1-fold colouring is classic colouring we have $\chi_f(G) \leq \chi(G)$.

The best upper bound for the fractional chromatic number of the $G_{[1,1]}$ graph are due to Hochberg and O'Donnell [5] and the best lower bound can be found in the book of Scheinerman and Ullman [4] (alongside with a good explanation of the upper bound): $3.555 \leq \chi_f(G_{[1,1]}) \leq 4.36$

We present some bounds for j -fold chromatic number of $G_{[a,b]}$ for small j in particular of $G_{[1,1]}$. The j -fold colouring for small j has strong practical motivation especially in scheduling theory.

Theorem 2. *There exists a j -fold colouring with k colours of the graph $G_{[1,1]}$ where j and k are given in a table*

k	=	7	12	16	25	33	35	37
j	=	1	2	3	4	5	6	7
k/j	≈	7	6	5.33	6.25	6.6	5.83	5.26

Our method of colouring is based on various shiftings of hexagon grid. See Figure 1 for 2-fold colouring with 12 colours. It finds application in more general case, too:

Theorem 3. *There exists a nm -fold colouring with $\lceil (\frac{2b}{\sqrt{3}} + 1) \cdot n \rceil \cdot \lceil (\frac{2b}{\sqrt{3}} + 1) \cdot m \rceil$ colours of the graph $G_{[1,b]}$ e.i. $\chi_f(G_{[1,b]}) \leq \frac{\chi_{nm}(G_{[1,b]})}{nm} \leq \frac{\lceil (2b/\sqrt{3} + 1) \cdot n \rceil \cdot \lceil (2b/\sqrt{3} + 1) \cdot m \rceil}{nm}$.*

(Note that it is enough to consider $G_{[1,b]}$ graphs since $G_{[a,b]} \cong G_{[1,b/a]}$.)

Graph $G_{[1,2]}$ is often used to model hidden conflicts in radio networks, hence its colourings have application in telecommunication [7]. Ivanov [6] showed that 12 colours are enough to colour $G_{[1,2]}$. Theorem 3 for $G_{[1,2]}$ gives us the following values:

k	$=$	12	100	2809	9216	19321
j	$=$	1	9	256	841	1764
k/j	\approx	12	11.1111	10.9726	10.9583	10.9529

Moreover we generalize method from [5] for $G_{[a,b]}$ graphs for fractional colouring.

Theorem 4. *If $b \geq 1$ then $\chi_f(G_{[1,b]}) \leq \frac{\sqrt{3}}{3} \cdot \frac{b + \sqrt{1-x^2}}{x}$ where x is the root of $bx = \frac{\pi}{6} - \arcsin(x)$. See Figure 2 for chart. In particular $\chi_f(G_{[1,2]}) \leq 9.9$.*

The proof is constructive e.i. we give an infinite sequence of j -fold colourings such that $\frac{\chi_j(G_{[a,b]})}{j}$ converges to stated bound with j converging to infinity.

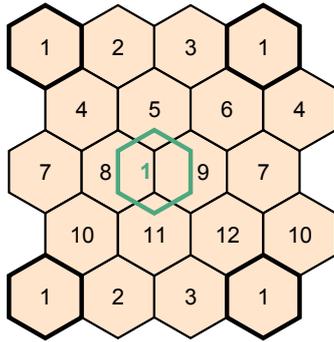


Figure 1: 2-fold colouring of $G_{[1,1]}$

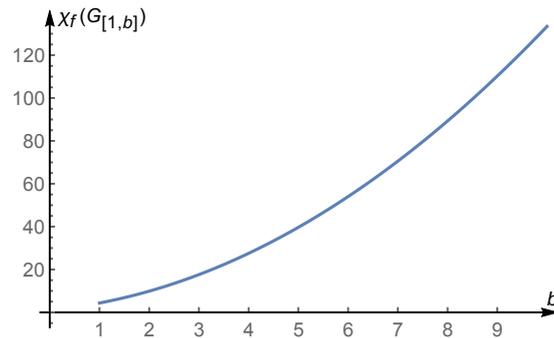


Figure 2: The upper bound for $\chi_f(G_{[1,b]})$

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Splitting cycles in triangulations

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EXTENDED ABSTRACT

A **splitting cycle** on a topological surface is a simple closed curve that cuts the surface into two non-trivial pieces, none of which is homeomorphic to a disk. See Fig. .8. A torus does not have any splitting cycle but any closed surface (orientable or not) of genus at least two admits a splitting cycle. Given a combinatorial surface, that is a cellular embedding of a graph G into a surface Σ , it is natural to ask whether G contains a circuit that is a splitting cycle in Σ . As usual, a circuit in a graph is a closed path without any repeated vertex.

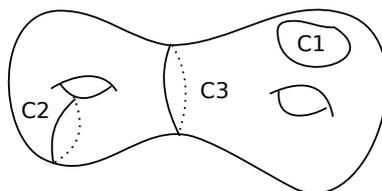


Figure .8: A cycle may be null-homotopic (C1), or null-homologous but non null-homotopic (C2) or neither null-homotopic nor null-homologous (C3). C3 is also called a splitting cycle.

It is known to be NP-hard to decide whether a combinatorial surface contains a splitting circuit or not [2, 3]. However, it was conjectured by Barnette that

Conjecture 1 (Barnette '1982 [7, p. 166]). *Every triangulation of a surface of genus at least 2 has a splitting circuit.*

Here, the graph of a triangulation of a surface Σ is the 1-skeleton of a simplicial complex homeomorphic to Σ . Another relevant parameter for the existence of a splitting circuit is the **face-width** of a graph embedding on Σ (non necessarily triangulated). This is the least number of intersections between the graph and any non-contractible cycle on Σ . It was proved by Zha and Zhao [12] that a graph embedding of face-width at least 6 for Σ orientable and at least 5 in the non-orientable case has a splitting cycle. In the specific case of a genus two surface, those conditions were lowered to face-width 4 for Σ orientable [5] and face-width 3 otherwise [9]. Note that a triangulation has face-width at least three. Turning back to triangulations, Sulanke [11] obtained by brute force computations that every orientable genus two triangulation has a splitting cycle. A formal proof appears in Jennings' thesis [6]. To our knowledge, no progress has been made on Conjecture 1 since then. A stronger version of the Conjecture was later proposed by Mohar and Thomassen.

Conjecture 2 (Mohar and Thomassen '2001 [7, p. 167]). *Let Σ be a surface of genus $g \geq 2$ and let $h \in \mathbb{N}$ such that $1 \leq h < g$. Any triangulation of Σ has a splitting circuit which cuts the surface into 2 punctured surfaces of genera h and $g - h$.*

In this talk we present counter-examples to conjecture 2 obtained with the help of a computer. Let M_{19} be the embedding of K_{19} , the complete graph on 19 vertices, given by Ringel and Youngs [8]. From the Euler characteristic it is easily seen that M_{19} has genus 20. A brute force approach to compute a splitting circuit in the spirit of Sulanke would lead to years of computations. Thanks to a branch and bound heuristic we were able to show that Conjecture 2 fails for M_{19} and for every

$h \in [5, 10]$. In particular, it is not possible to split M_{19} into two pieces of equal genus. We obtain similar results for M_{31} and other orientable or non-orientable embeddings of bigger complete graphs.

Computational approaches are interesting for trying to either prove or disprove the conjectures. With the help of edge contractions, the conjectures can be reduced to the case of irreducible triangulations. For a fixed genus there is only a finite number of them that can be listed following an algorithm by Sulanke [10]. However, there are already more than millions of irreducible triangulations of genus 3 so that Conjecture 1 could only be verified for genus 2 irreducible surfaces in practice. On the other hand, trying to disprove the conjectures, we need to find counter-example candidates. As remarked by Ellingham and Stephen [4], complete graphs are often good candidates. It is possible to test all the 59 non-isomorphic orientable embeddings of K_{12} [1] and all of them confirm the conjectures. Generating larger complete graph embeddings is difficult and we decided to use the specific Ringel and Youngs' examples [8]. There are too many circuits in K_n (namely $2^n - \binom{n+1}{2}$) to just list all of them and check if one is splitting. However, as we explore the vertices of a circuit, we can 2-color their incident edges in K_n depending on their position (left of right) with respect to the circuit. If the circuit is *not* separating this coloration will induce an inconsistent coloring of some vertex star. This simple remark allows to stop exploring most of the circuits after visiting 5 or 6 of their vertices. This leads to an efficient branch and bound heuristic and we were able to check all the circuits of K_{19} , M_{31} or K_{40} and perform some experiments. In particular, our computations invalidate Conjecture 2.

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Avoiding repetitions on the plane

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EXTENDED ABSTRACT

One of the most challenging problems in geometric graph theory is the Hadwiger-Nelson problem (originally proposed by Nelson) about determining the minimal number of colors to color every point on the Euclidean plane \mathbb{R}^2 in such a way that there are no two points in the same color distance one apart. It can be equivalently stated as determining the chromatic number of the so-called unit distance graph on the set \mathbb{R}^2 in which every two points in distance one are adjacent. The classic result states that this number is at least 4 and at most 7. Nothing better in general has been proved for more than 50 years, although it inspired a stream of research and many interesting results have been obtained in various related areas. See for example [7] for an extensive discussion on the history of the question and some related problems.

A sequence $a_1 a_2 \dots a_{rn}$ such that $a_i = a_{i+jn}$ for $i = 1, \dots, n$ and $j = 1, \dots, r-1$ is called an r -repetition. The classic theorem of Thue [8] states that there exists an arbitrarily long sequence avoiding 2-repetitions over just three symbols. This result is one of the foundations of a whole branch of combinatorics called combinatorics on words - area with many interesting results, challenging open problems and various applications. One of the concepts in this field is avoiding repetitions in graphs: the Thue number of a graph G denoted by $\pi(G)$ is the minimal number of colors to color the vertices in such a way that no simple path produce an 2-repetition (where 'produce' corresponds to reading the sequence of colors of vertices on the path).

The Thue-like problems on geometrical structures were studied for example in [1, 3, 6]. We will refer to the Thue number of the unit distance graph on \mathbb{R}^2 as the Thue number of \mathbb{R}^2 (and $\pi(\mathbb{R}^2)$). J. Grytczuk in his survey [4] on many Thue-like problems asked the following question:

Problem 1. *Determine the Thue number of \mathbb{R}^2 .*

Grytczuk observed that $\pi(\mathbb{R}^2)$ has to be infinite. In our work we solve the problem, by proving that countable set of colors is not enough:

Theorem 2. *The Thue number of \mathbb{R}^2 is greater than \aleph_0 .*

This infiniteness of this number leads to relaxing the condition by examining only 'some' paths of the unit distance graph on \mathbb{R}^2 . In [5] authors propose to take into account only *line paths* - sequences of collinear points where consecutive points are in distance one. The coloring of the plane is *line r -nonrepetitive* if colors of no line path determine an r -repetition. The minimal number of colors for a line 2-nonrepetitive coloring of \mathbb{R}^2 is denoted by $\bar{\pi}(\mathbb{R}^2)$. It turns out that this number is finite:

Theorem 3 ([5]).
 $\bar{\pi}(\mathbb{R}^2) \leq 36$.

Our main result in the area of 2-repetitions on the plane is the following strong improvement on the bound. In the proof we use a special rectangle tiling of the plane combined with a help of the classic Thue sequence.

Theorem 4. $\bar{\pi}(\mathbb{R}^2) \leq 18$.

Tools created for the proof of this theorem also give an additional result concerning another concept studied in the theory of combinatorics on words: palindromes. A *palindrome* is a sequence of length greater than 1 such that it is the same if we reverse the order of elements. We will call a coloring of \mathbb{R}^2 *line palindrome-free* if colors of no line path produce a palindrome.

Theorem 5. *There exists a 32-coloring of \mathbb{R}^2 that is line 2-nonrepetitive and line palindrome-free.*

With other authors we also studied r -repetitions. Clearly it should be easier to avoid r -repetitions as the r grows, but how far can we get? We managed to prove that for sufficiently large r there exists a line r -nonrepetitive coloring of the plane using only two colors. The technique used in this result is the weighted version of the Lovasz Local Lemma.

Theorem 6 ([2]).

There exists a 2-coloring of the plane that is 47-nonrepetitive.

It may be interesting to reverse the question: if we are given a natural number k , what is the minimal r such that there exists a line r -nonrepetitive coloring of \mathbb{R}^2 using k colors. In other words: how nonrepetitive on the plane can we get with k colors? We admit that our work in [2] gives bounds for this question.

The 'machinery' developed in this work also gives the following result for greater dimensions.

Theorem 7 ([2]).

For any natural number n there exists a sufficiently large number $r(n)$ such that \mathbb{R}^n admits a line $r(n)$ -nonrepetitive coloring using 2 colors.

Where the *line r -nonrepetitive coloring* of \mathbb{R}^n is the natural generalization of the concept for \mathbb{R}^2 - no sequence of collinear points of consecutive distance one produces an r -repetition.

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Generation and Propagation in Graphs

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EXTENDED ABSTRACT

Introduction. In graph theory, there are too many parameters and numbers with motivation from theory or application which describe the properties of a given graph. Coloring numbers, Domination numbers, Alliance numbers, Clique number, Independence number are a few examples of such parameters. In lack of a hierarchy or a unifying framework, there is no guideline for introducing absent concepts and parameters. In such condition, seeking for unifying frameworks is a demanded area of research in graph theory.

For example Mihok et al. introduced an elegant language for expressing various coloring theorems in a unified framework.[1] Also K. H. Shafique in his PhD thesis introduced a high level description for alliance numbers to study the complexity of some related computational problem.[2]

This work can be seen as another attempt in this direction.

Generation and propagation parameters. Consider a situation in which a quantity, say information or some material, can be generated and propagated in a given network. Let g (g') be the rate of generation in each vertex (edge), and p (p') be the rate of propagation in each vertex (edge). Call the 4-tuple $(g, g'; p, p')$ a *GP-code*.

For a given graph G and a given *GP-code* consider the following 0, 1-programming problem:

$$\begin{aligned} & \text{Minimize} && \sum_{v \in V(G)} x_v \\ & \text{subject to} && \sum_{v \in [u]} \geq (p'd_u + p)x_u + (g'd_u + g) \\ & && \sum_{v \in V(G)} x_v > 0 \end{aligned}$$

Where $[u]$ denotes the closed neighborhood of vertex u and the first constrain is valid for every vertex u . The optimum value of this problem is called $(g, g'; p, p')$ -parameter. A $(g, g'; p, p')$ -parameter represents a known parameter of a graph if for every graph the result of the above problem is equal to that known parameter. As an evidence for the power of the above model for representing known graph parameters. One can prove the following elementary result:

Theorem. The following statements are true:

$(1, 0; 0, 0)$ represents the Domination number.

$(0, 1; 0, 0)$ represents the 2-packing number.

$(0, 0; 1, 0)$ represents Proper Nearly Perfect number. [3]

In the above theorem we use only 0 and 1 as the components of the *GP-code* while the components can be fractional or negative. In general case many parameters like k -domination number (and some other domination numbers), Independence number, deffensive alliance number are among known parameters which can be represented by *GP-codes*. For example $(0, 0; \frac{1}{2}, 0)$ represents the Deffensive Alliance nubmer.

Now one can ask the reverse questions: what is the parameter which is represented by a given *GP-code*? In this paper we show that many new interesting parameters can be introduced as a *GP-parameter*.

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Improper coloring and density of graphs with a given girth

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EXTENDED ABSTRACT

A *proper k -coloring* of a graph G is a partition of $V(G)$ into k independent sets V_1, \dots, V_k . A (d_1, \dots, d_k) -*coloring* of a graph G is a partition of $V(G)$ into sets V_1, \dots, V_k such that for every $1 \leq i \leq k$, the subgraph $G[V_i]$ of G induced by V_i has maximum degree at most d_i . If $d_1 = \dots = d_k = 0$, then a (d_1, \dots, d_k) -coloring is simply a proper k -coloring. If at least one of d_i is positive, then a (d_1, \dots, d_k) -coloring is called *improper* or *defective*. Several papers on improper colorings of planar graphs with restrictions on girth and of sparse graphs have appeared.

In this talk, we consider the simplest versions of improper colorings, the (j, k) -colorings. Even such colorings are not simple if $(j, k) \neq (0, 0)$. In particular, Esperet, Montassier, Ochem and Pinlou [5] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0, 1)$ -coloring is NP-complete. Since the problem is hard, it is natural to consider related extremal problems.

The maximum average degree, $\text{mad}(G)$, of a graph G is the maximum of $\frac{2|E(H)|}{|V(H)|}$ over all subgraphs H of G . It measures sparseness of G . Kurek and Rucinski [10] called graphs with low maximum average degree *globally sparse*.

In particular, it is an easy consequence of Euler's formula that

$$\text{if } G \text{ is a planar graph of girth } g, \text{ then } \text{mad}(G) < \frac{2g}{g-2}. \quad (1)$$

We will use the following slight refinement of the notion of $\text{mad}(G)$. For $a, b \in \mathbf{R}$, a graph G is (a, b) -*sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. For example, every forest is $(1, 0)$ -sparse, and every graph G with $\text{mad}(G) < a$ is $(a/2, 0)$ -sparse.

Glebov and Zambalaeva [6] proved that every planar graph G with girth at least 16 is $(0, 1)$ -colorable. Then Borodin and Ivanova [1] proved that every graph G with $\text{mad}(G) < \frac{7}{3}$ is $(0, 1)$ -colorable. By (1), this implies that every planar graph G with girth at least 14 is $(0, 1)$ -colorable. Borodin and Kostochka [2] proved that every graph G with $\text{mad}(G) < \frac{12}{5}$ is $(0, 1)$ -colorable, and this is sharp. This implies that every planar graph G with girth at least 12 is $(0, 1)$ -colorable. As mentioned above, Esperet et al. [5] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0, 1)$ -coloring is NP-complete. Dorbec, Kaiser, Montassier, and Raspaud [4] mention that because of these results, the remaining open question is whether all planar graphs with girth 10 or 11 are $(0, 1)$ -colorable.

In this talk, instead of considering planar graphs with given girth, we consider graphs G with given girth that are (a, b) -sparse for small a . Let $F_{j,k}(g)$ denote the supremum of positive a such that there is some (possibly negative) b with the property that every (a, b) -sparse graph G with girth g is (j, k) -colorable. The above mentioned result in [2] implies $F_{0,1}(3) = \frac{12}{5} = 1.2$. It turns out that $F_{0,1}(g)$ does not differ much from $F_{0,1}(3)$ even for large g . In the [9] we prove that for every g , $F_{0,1}(g) \leq 1.25$ and we also find there the exact values of $F_{j,k}(g)$ for all g in the cases $k \geq 2j + 2$ for all j .

Theorem 1 (Kim, Kostochka, and Zhu, '2014+ [9]). *For all $k \geq j \geq 0$ and $g \geq 3$, $F_{j,k}(g) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}$. Moreover, if $k \geq 2j + 2$, then $F_{j,k}(g) = 2 - \frac{(k+2)}{(j+2)(k+1)}$.*

In [8], we prove the exact result: $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$. In fact, we also find the best possible value of b as follow.

Theorem 2 (Kim, Kostochka, and Zhu, '2014 [8]). *If a graph G is triangle-free and $11|A| - 9|E(G[A])| \geq -4$ for all $A \subseteq V(G)$, then G is $(0, 1)$ -colorable. On the other hand, there are infinitely many non- $(0, 1)$ -colorable graphs G with girth 5 such that $11|V(G)| - 9|E(G)| = -5$ and $11|A| - 9|E(G[A])| \geq -4$ for all $A \subsetneq V(G)$*

This theorem yields the following.

Corollary 3 (Kim, Kostochka, and Zhu, '2014 [8]). *Every planar graph with girth at least 11 is $(0, 1)$ -colorable.*

This answers half of the question above by Dorbec et al. [4].

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Extending 3-colorings of a cycle to a triangle-free planar graph

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EXTENDED ABSTRACT

Garey and Johnson [1] proved that deciding if a graph is k -colorable is NP-complete even when $k = 3$. Moreover, deciding if a graph is 3-colorable is still NP-complete when restricted to planar graphs [2]. Therefore, even though planar graphs are 4-colorable by the celebrated Four Color Theorem [3, 4, 5], finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research. A landmark result in this area is Grötzsch’s Theorem [6], which is the following:

Theorem 1 ([6]). *Every triangle-free planar graph is 3-colorable.*

We direct readers to a nice survey by Borodin [7] for more results and conjectures regarding 3-coloring planar graphs.

A graph G is k -critical if it is not $(k - 1)$ -colorable but every proper subgraph of G is $(k - 1)$ -colorable. Critical graphs are important since they are (in a certain sense) the minimal obstacles in reducing the chromatic number of a graph. Numerous coloring algorithms are based on detecting critical subgraphs. Despite its importance, there is no known characterization of k -critical graphs when $k \geq 4$. On the other hand, there has been some success regarding 4-critical planar graphs. See [11] for more details.

Given a graph G and a proper subgraph C of G , we say G is C -critical for k -coloring if for every proper subgraph H of G where $C \subseteq H$, there exists a proper k -coloring of C that extends to a proper k -coloring of H , but does not extend to a proper k -coloring of G . Roughly speaking, a C -critical graph for k -coloring is a minimal obstacle when trying to extend a proper k -coloring of C to a proper k -coloring of the entire graph. Note that $(k + 1)$ -critical graphs are exactly the C -critical graphs for k -coloring with C being the empty graph.

In the proof of Theorem 1, Grötzsch actually proved that any proper coloring of a 4-cycle or a 5-cycle extends to a proper 3-coloring of a triangle-free planar graph. This implies that there are no triangle-free planar graphs that are C -critical for 3-coloring when C is a face of length 4 or 5. This sparked the interest of characterizing triangle-free planar graphs that are C -critical for 3-coloring when C is a face of longer length. Since we deal with 3-coloring triangle-free planar graphs in this paper, from now on, we will write “ C -critical” instead of “ C -critical for 3-coloring” for the sake of simplicity.

The investigation was first done on planar graphs with girth 5. Thomassen [13] and Walls [12] independently characterized C -critical planar graphs with girth 5 when C is a face of length at most 11. The case when C is a 12-face was initiated in [13], but a complete characterization was given by Dvořák and Kawarabayashi in [14]. Moreover, a recursive approach to identify all C -critical planar graphs with girth 5 when C is a face of any given length is given in [14]. Dvořák and Lidický [15] implemented an algorithm and used a computer to generate all C -critical graphs with girth 5 when C is a face of length at most 16. The graphs generated were then used to reveal some structure of 4-critical graphs on surfaces without short contractible cycles.

The situation for planar graphs with girth 4, which are triangle-free planar graphs, is more complicated since the list of C -critical graphs is not finite when C has size at least 6. We already mentioned that there are no C -critical triangle-free planar graphs when C is a face of length 4 or 5. An alternative proof of the case when C is a 5-face was given by Aksionov [8]. Gimbel and Thomassen [16] not only showed that there exists a C -critical triangle-free planar graph when C is a 6-face, but also characterized all of them. Aksenov, Borodin, and Glebov [17] independently

proved the case when C is a 6-face using the discharging method, and also characterized all C -critical triangle-free planar graphs when C is a 7-face in [18]. Dvořák and Lidický [19] used properties of nowhere-zero flows to give simpler proofs of the case when C is either a 6-face or a 7-face, and also characterized C -critical triangle-free planar graphs when C is an 8-face. The case where C is a 7-face was used in [11].

We push the project further and characterize all C -critical triangle-free planar graphs when C is a face of length 9.

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Between treewidth and cliquewidth

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EXTENDED ABSTRACT

Many hard problems can be solved efficiently when restricted to graphs of bounded treewidth, and more generally to graphs of bounded clique-width. But there is a price to be paid for this generality, exemplified by the four problems MAXCUT, GRAPH COLORING, HAMILTONIAN CYCLE and EDGE DOMINATING SET that are all FPT parameterized by treewidth but none of which can be FPT parameterized by clique-width unless the Exponential Time Hypothesis fails, as shown by Fomin et al [3]. We therefore seek a structural graph parameter that shares some of the generality of clique-width without paying this price.

Based on splits, branch decompositions and the work of Vatshelle [6] on maximum matching-width, we consider the graph parameter sm-width which lies between treewidth and clique-width. Some dense graph classes of unbounded tree-width, like distance-hereditary graphs, have bounded sm-width. We show that MAXCUT, GRAPH COLORING, HAMILTONIAN CYCLE and EDGE DOMINATING SET are all FPT parameterized by sm-width.

We accomplish this by two main results. Firstly, given a graph G with sm-width k we use split decompositions of graphs and an approximation algorithm for width parameters defined by submodular functions due to Oum and Seymour [4] to provide a $O^*(8^k)$ time algorithm returning a branch decomposition of sm-width at most $O(k^2)$. Secondly, we give dynamic programming algorithms solving each of the four problems in FPT time when parameterized by the width of this decomposition.

1 Results

We give some definitions and a sketch of the approximation algorithm for sm-width. We use split decompositions of graphs, see [4, 1, 5], that are based on the notion of a *split* of a connected graph G , which is a partition of $V(G)$ into two sets V_1, V_2 such that $|V_1| \geq 2$, $|V_2| \geq 2$ and every vertex in V_1 with a neighbor in V_2 has the same neighborhood in V_2 . We also use branch decompositions over the vertex set of a graph, and a cut function that assigns a value to every cut of a graph, to define graph parameters, see [4]. A *cut* (A, \bar{A}) for $A \subseteq V(G)$ is the bipartite graph with the same vertices as G but with edge set $\{uv \in E(G) : u \in A, v \in \bar{A}\}$.

Vatshelle [6] defined the maximum-matching-width $\text{mmw}(G)$ of a graph G based on the cut function mm defined for any graph G and $A \subseteq V(G)$ by letting $\text{mm}(A)$ be the cardinality of a maximum matching of the bipartite graph (A, \bar{A}) .

Theorem 1 ([6]). *Let G be a graph, then $\frac{1}{3}(\text{tw}(G) + 1) \leq \text{mmw}(G) \leq \text{tw}(G) + 1$*

We define the split-matching-width $\text{smw}(G)$ of a graph G based on branch decompositions over vertices and the cut function sm defined for any graph G and $A \subseteq V(G)$ by:

$$\text{sm}(A) = \begin{cases} 1 & \text{if } (A, \bar{A}) \text{ is a split of } G \\ \text{mm}(A) = \max\{|M| : M \text{ is a matching of } (A, \bar{A})\} & \text{otherwise} \end{cases}$$

A cut function $f : 2^{V(G)} \rightarrow \mathbb{N}$ is said to be submodular if for any $A, B \subseteq V(G)$ we have $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. The following very general result of Oum and Seymour will be useful to us since we show that the cut function mm is symmetric and submodular.

Theorem 2 ([4]). *For symmetric submodular cut-function f and graph G of optimal f -width k , a branch decomposition of f -width at most $3k + 1$ can be found in $\mathcal{O}^*(2^{3k+1})$ time*

Our algorithm that given a graph G finds a branch decomposition of G having sm-width $O(\text{smw}(G)^2)$, in time FPT parameterized by $\text{smw}(G)$, has four steps:

1. Find a split decomposition of G into prime graphs G_1, G_2, \dots, G_q .
2. For each G_i find a branch decomposition (T_i, δ_i) of sm-width $O(\text{smw}(G_i))$.
3. For each G_i restructure (T_i, δ_i) into (T'_i, δ'_i) having the property that any cut of G_i , induced by an edge of (T'_i, δ'_i) and having split-matching value k , is lifted, by the split decomposition of G , to a cut of G having split-matching value $O(k^2)$.
4. Combine all (T'_i, δ'_i) into a branch decomposition of G of sm-width $O(\text{smw}(G)^2)$.

For step 1 there exists a well-known polynomial-time algorithm by Cunningham [4]. For step 2 we are dealing with a prime graph G_i , which by definition has no non-trivial splits and hence $\text{sm}(V_i) = \text{mm}(V_i)$ for all $V_i \subseteq V(G_i)$ meaning that $\text{mmw}(G_i) = \text{smw}(G_i)$. Furthermore, as the cut function defining mmw is submodular we can apply the algorithm of Oum and Seymour from Theorem 2 to accomplish the task of step 2. Steps 3 and 4 require more work.

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Random Free Limits of Tree-Semilattices

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EXTENDED ABSTRACT

The study of limits of graphs recently gained much interest (see [7]). A sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is said *left-convergent* if, for every graph F , the probability that a random map $f : V(F) \rightarrow V(G_n)$ is a *homomorphism* (that is an adjacency preserving map) converges as n grows to infinity. The limit object can be represented by means of a *graphon*, that is a symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$. A graphon W is *random-free* if it is almost everywhere $\{0, 1\}$ -valued. A random-free graphon is essentially the same (up to isomorphism mod 0) as a *Borel graph* — that is a graph having a standard Borel space V as its vertex set and a Borel subset of $V \times V$ as its edge set — equipped with a non-atomic probability measure on V . A class of graph \mathcal{C} is said to be *random-free* if every left-convergent sequence of graphs in \mathcal{C} has a random-free limit. For instance, Janson derived from a structural characterization of random-free hereditary classes of graphs given by Lovász and Szegedy [8] that the class of cographs is random-free [5].

For k -regular hypergraphs, left-limits have been constructed by Elek and Szegedy [2] using ultraproducts as measurable functions $W : [0, 1]^{2^k-2} \rightarrow [0, 1]$ (called *hypergraphons*), and have also been studied by Hoover [4], Aldous [1], and Kallenberg [6] in the setting of exchangeable random arrays. Relational structures are a natural generalization of k -uniform hypergraphs. Recall that a σ -structure \mathbf{A} is defined by its *domain* A , its *signature* σ (which is a set of symbols of relations and functions with their arities, which contains no function symbols in the case of a *relational structure*), and the interpretation in A of all the relations and functions in σ .

A general notion of convergence driven by a fragment X of first-order logic has been introduced for σ -structures by Nešetřil and the third author [10]: a sequence $(\mathbf{A}_n)_{n \in \mathbb{N}}$ of σ -structures is *X -convergent* if, for every formula $\phi \in X$, the probability $\langle \phi, \mathbf{A}_n \rangle$ that ϕ is satisfied for a random assignment of elements of A_n to the free variables converges as n grows to infinity. In particular, denoting QF the fragment of quantifier-free formulas, QF-*convergence* is equivalent (for sequences of graphs with increasing orders) to left-convergence.

We extend the notion of Borel graph to σ -structures: A *Borel σ -structure* is a σ -structure \mathbf{A} such that the domain A of \mathbf{A} is a standard Borel space, such that for every relational (resp. functional) symbol R (resp. f) with arity k , the interpretation $R^{\mathbf{A}}$ (resp. $f^{\mathbf{A}}$) of R (resp. f) is a Borel subset of A^k (resp. a Borel function $A^k \rightarrow A$). We mainly consider pairs (\mathbf{A}, ν) of Borel σ -structures \mathbf{A} together with a probability measures ν . Being almost surely the left limit of a sequence of random induced subgraphs with growing orders, a Borel graph \mathbf{G} is the left limit of a sequence of finite graphs. However, in presence of functional symbols, particularly when substructures generated by finitely many elements can be infinite, the situation is less clear. Extending the notion of random-free classes of graphs, we say that a class \mathcal{C} of finite σ -structures is *random free* if every QF-convergent sequence of σ -structures in \mathcal{C} has a Borel σ -structure (with suitable probability measure) QF-limit.

Theorem 1. *The class of weighted colored tree-semilattices is a random-free class. Precisely, for every QF-convergent sequence $(\mathbf{T}_n, \mu_n)_{n \in \mathbb{N}}$ of finite weighted colored tree-semilattices there exists a Borel colored tree-semilattice \mathbf{T} and a probability measure μ on T , such that (\mathbf{T}, μ) is a QF-limit of the sequence $(\mathbf{T}_n, \mu_n)_{n \in \mathbb{N}}$.*

Conversely, for every Borel colored tree-semilattice \mathbf{T} and every probability measure μ on T there exists a QF-convergent sequence $(\mathbf{T}_n, \mu_n)_{n \in \mathbb{N}}$ of finite weighted colored tree-semilattices with QF-limit (\mathbf{T}, μ) .

Noteworthy, although QF-convergence does not in general preserve structural properties, the Borel structures we obtain as limits in Theorem 1 are still tree-semilattices.

The m -partite cographs — introduced in [3] by generalizing the construction of cographs through cotrees — are defined by means of quantifier-free formulas from tree-semilattices: A graph G is an m -partite cograph if there exists a colored tree-semilattice \mathbf{T} such that $V(G)$ is the set of maximal elements of \mathbf{T} , each element v of $V(G)$ is assigned a color $\gamma(v) \in \{1, \dots, m\}$, each other element of T is assigned a symmetric function $\tau(v) : \{1, \dots, m\} \times \{1, \dots, m\} \rightarrow \{0, 1\}$, and two vertices u, v are adjacent exactly if $\tau(u \wedge v)(\gamma(u), \gamma(v)) = 1$. Note that every graph is an m -partite cograph for some m . This allows to derive from Theorem 1 an explicit characterization of left limits of m -partite cographs as interpretations of Borel tree-semilattices.

The possibility to approximate a large graph (or a graphon) by a small graph of order $N(\epsilon)$ — where ϵ measures the distance between the graph and its approximation — is an essential consequence of Szemerédi's regularity lemma. The order of the approximation is, in general, very fast growing with ϵ^{-1} . Nevertheless, the number of parts in a regular partition of a graph G with parameter ϵ can be bounded by $O(1/\epsilon^k)$ for stable classes of graphs [9]. It is thus expected that graphs in a random-free hereditary class, like the class of m -partite cographs, can be ϵ -approximated by relatively small graphs (of order polynomial in ϵ^{-1}), and that such a property extends to random-free classes of σ -structures. We prove that this holds for the class of tree-semilattices.

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Perfect weight matchings versus maximum weight matchings: the ratio η for some graph classes

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EXTENDED ABSTRACT

Motivated by applications in computer graphics, where we seek to convert a triangulation into a quadrangulation, Brazil et al. [1] studied the ratio η between the maximum weight of a perfect matching and the maximum weight of a general matching. Due to the computer graphics application, their study focused mostly on cubic graphs. The parameter η can be defined for any graph which admits a perfect matching. Therefore, it is natural to consider other graph classes such as grids and regular bipartite graphs. Surprisingly, the determination of η is often non-trivial even for such restricted types of graphs. Moreover, the original study left an open question with respect to nonhamiltonian bipartite cubic graphs, which we answer herein.

Let $G = (V, E)$ be a connected undirected graph. A *matching* in G is a set $M \subset E$ such that no two edges in M share a common vertex. Given a matching M in a graph G , we say that M *saturates* a vertex v and that vertex v is *M -saturated*, if some edge of M is incident to v . A matching P is *perfect* if $|P| = |V|/2$. A matching is *maximal* if it is not a subset of any other matching and is *maximum* if it has maximum cardinality.

Let $w : E \rightarrow \mathbb{R}^+$ be the *weight* of the edges. Given a subset $E' \subseteq E$, we refer to the quantity $w(E') = \sum_{e \in E'} w(e)$ as the *weight* of E' . A *maximum weight matching* is a matching $M^*(G)$ of maximum possible weight in G . A *maximum weight perfect matching* is a perfect matching $P^*(G)$ of maximum possible weight (among all perfect matchings of G). Given a graph G which admits a perfect matching, the parameter $\eta(G)$ is defined as follows:

$$\eta(G) = \min_{w: E \rightarrow \mathbb{R}^+} \frac{w(P^*(G))}{w(M^*(G))}$$

Let \mathcal{G} be a graph class and consider a graph $G \in \mathcal{G}$. Since $\eta(G)$ is only defined if G admits a perfect matching, we assume that all graphs in \mathcal{G} admit perfect matchings. Notice that for different graphs $G, G' \in \mathcal{G}$ we may have $\eta(G) \neq \eta(G')$. In the worst case scenario, in terms of the motivation of approximating a maximum weight matching with a perfect matching, there exists a graph $G \in \mathcal{G}$ for which the value of $\eta(G)$ is very small. Therefore, we define the value of $\eta(\mathcal{G})$ in terms of this worst case behavior:

$$\eta(\mathcal{G}) = \min_{G \in \mathcal{G}} \eta(G)$$

Sometimes, when the graph G or the graph class \mathcal{G} is clear from the context, we refer to $\eta(G)$ or $\eta(\mathcal{G})$ simply as η .

An immediate consequence of this definition is that given two graph classes $\mathcal{G}_1 \subseteq \mathcal{G}_2$, we have $\eta(\mathcal{G}_1) \geq \eta(\mathcal{G}_2)$. Therefore, in order to prove bounds on η that apply to as many graph classes as possible, it is useful to obtain lower bounds on η for “large” graph classes and upper bounds on η for “small” graph classes. It is important to notice that in many applications is not possible to know a priori all graphs that will be given as input, thus the worst case scenario is very useful information. Therefore, we will treat the upper bounds as valuable knowledge about the whole graph class even when it is proved only for a single example.

In this work, we present the following two general results that help us to determine bounds of the parameter η .

Lemma 1. Let G be a graph with minimum degree $\delta \geq 2$, containing a perfect matching and such that there exists a maximal matching that does not saturate a vertex v of degree δ . Then, $\eta(G) \leq \frac{\delta-1}{\delta}$.

Lemma 2. Let $G = (V, E)$ be a graph and let $G_1 = (V, E_1), G_2 = (V, E_2), \dots, G_k = (V, E_k)$ be subgraphs of G with minimum degree $\delta \geq 1$, such that each edge of G is in at least one of E_1, E_2, \dots, E_k and such that each subgraph G_i admits a perfect matching. If $\eta(G_i) \geq c$ for $i = 1, \dots, k$, then $\eta(G) \geq c/k$.

Furthermore, we establish the exact value of the parameter η for rectangular grid graphs and provide bounds for toroidal grids, cylindrical grids, and cubic bipartite graphs that are dual of 4, 8-meshes. The proofs of these results use the previous theorems. Due to space constraints, the latter result has been omitted.

A *rectangular grid* of width $w \geq 2$ and height $h \geq 2$ is a graph $G = (V, E)$ where $V = \{(i, j) : i = 1, \dots, w; j = 1, \dots, h\}$ and $E = \{(i, j)(i+1, j) : i = 1, \dots, w-1; j = 1, \dots, h\} \cup \{(i, j)(i, j+1) : i = 1, \dots, w; j = 1, \dots, h-1\}$. A *cylindrical grid* is a graph obtained from a rectangular grid by adding edges $(w, j)(1, j)$ for $j = 1, \dots, h$. A *toroidal grid* is a graph obtained from a cylindrical grid by adding edges $(i, h)(i, 1)$ for $i = 1, \dots, w$.

Theorem 3. Let \mathcal{G}_{EE} and \mathcal{G}_{EO} be the classes of rectangular grid graphs with even width and even height respectively with even width and odd height. Then $\eta(\mathcal{G}_{EE}) = 1/2$ and $\eta(\mathcal{G}_{EO}) = 1/3$.

Theorem 4. Let $\mathcal{C}_{EE}, \mathcal{C}_{EO}$, and \mathcal{C}_{OE} be the classes of cylindrical grid graphs with even width and even height, respectively with even width and odd height, respectively with odd width and even height. Then $1/2 \leq \eta(\mathcal{C}_{EE}) \leq 2/3$, $1/3 \leq \eta(\mathcal{C}_{EO}) \leq 1/2$, and $1/3 \leq \eta(\mathcal{C}_{OE}) \leq 2/3$.

Theorem 5. Let \mathcal{T}_{EE} and \mathcal{T}_{EO} respectively be the classes of toroidal grid graphs with even width and even height and with even width and odd height. Then $1/2 \leq \eta(\mathcal{T}_{EE}) \leq 3/4$ and $1/3 \leq \eta(\mathcal{T}_{EO}) \leq 2/3$.

For the case of even width and odd height, we obtain a lower bound of $\eta(\mathcal{G}_{EO})$ which approaches $1/2$ as the height increases. Since a trivial upper bound for all grid graphs is $1/2$, it follows that $\eta(\mathcal{G}_{EO})$ tends to $1/2$ for large grid graphs.

Theorem 6. Let \mathcal{G}_{EO} be the class of rectangular grid graphs with even width and odd height. Then, for $G \in \mathcal{G}_{EO}$ of height h , we have $\eta(G) \geq \frac{1}{2} - \frac{1}{h+1}$.

Finally, we prove that for all $\Delta \geq 2$, the class \mathcal{B} of Δ -regular bipartite graphs satisfies $\eta(\mathcal{B}) = \frac{1}{\Delta}$. Moreover, we present a nonhamiltonian bipartite cubic graph G in Figure .9 that satisfies $\eta(G) \leq 1/3$, improving a former result of [1] stating that $1/3 \leq \eta(\mathcal{G}) \leq 1/2$, where \mathcal{G} is the class of nonhamiltonian bipartite cubic graphs.

Theorem 7. Let \mathcal{B} be the class of Δ -regular bipartite graphs. Then, $\eta(\mathcal{B}) = \frac{1}{\Delta}$.

Theorem 8. Let \mathcal{G} be the class of nonhamiltonian bipartite cubic graphs. Then $\eta(\mathcal{G}) = 1/3$.

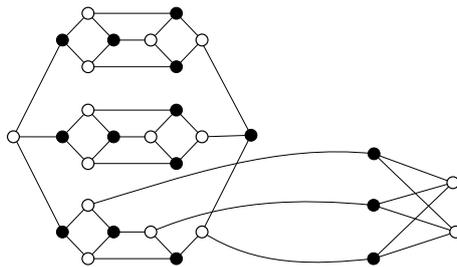


Figure .9: Nonhamiltonian bipartite cubic graph G with $\eta(G) = 1/3$.

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Extremal Graphs of Girth 5

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EXTENDED ABSTRACT

In this paper we consider the problem of extremal graphs of girth 5, that is, the problem of determining the maximum number of edges in a graph where the length of the smallest cycle it could contain is 5. Triangles and squares are called the forbidden subgraphs. Let n be the number of vertices in a graph. The maximum number of edges in the graph containing no such forbidden subgraphs is denoted by $ex(n; 4)$. This problem was started by Erdős in 1975. The most recent results in this problem are stated in the following theorems.

Theorem 1. [1] $ex(31; 4) = 80$.

Furthermore, Balbuena et al. constructed the graph of 32 vertices with 85 edges and proved that it is the best possible.

Theorem 2. [1] $ex(32; 4) = 85$.

Therefore, there are only 33 exact values known at this moment. In all other cases, only the lower bound and the upper bound are known (see Table .1 for the summary of known results). In this paper, we improve the upper bounds for some values of n .

Theorem 3. $ex(33; 4) \neq 90$.

This result improves the upper bound for $n = 34, 35$ and 37 . Besides, we also improve the upper bound for infinitely many values of n in the form $n = d^2 + 1$, given in Theorem 4.

Theorem 4. For $d \geq 4, d \neq 7, 57, ex(d^2 + 1; 4) < \frac{d^3+d}{2}$.

We conclude this paper with several open problems in this research area.

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n	0	1	2	3	4	5	6	7	8	9
0	0	0	1	2	3	5	6	8	10	12
10	15	16	18	21	23	26	28	31	34	38
20	41	44	47	50	54	57	61	65	68	72
30	76	80	85	87-90	90-95	95-100	99-105	104-111	109-115	114-120
40	120-124	124-129	129-134	134-139	139-144	145-149	150-154	156-159	162-164	168-169
50	175	176-180	178-185	181-191	185-196	188-202	192-207	195-213	199-218	203-224
60	207-230	212-236	216-242	221-248	226-253	231-260	235-266	240-272	245-278	251-284
70	257-290	263-297	269-303	275-309	281-316	288-322	294-329	301-335	307-342	314-348
80	320-355	326-362	332-369	339-375	346-382	352-389	359-396	366-403	373-410	381-417
90	388-424	395-431	402-438	409-446	417-453	424-460	432-467	440-475	449-482	452-490
100	455-497	459-505	462-512	466-520	469-527	475-535	482-543	489-550	497-558	505-566
110	512-574	520-582	528-589	536-597	544-605	552-613	560-621	567-630	575-638	583-646
120	591-654	599-662	607-671	615-679	623-687	631-695	640-704	649-712	658-721	661-729
130	665-738	668-746	672-755	675-764	683-772	691-781	700-790	708-798	717-807	725-816
140	734-825	743-834	752-843	761-852	770-860	779-870	788-879	797-888	806-897	815-906
150	825-915	834-924	843-933	853-943	863-952	872-961	882-971	892-980	902-989	912-999
160	923-1008	934-1018	945-1027	948-1037	951-1046	954-1056	957-1066	961-1075	964-1085	968-1095
170	972-1105	975-1114	979-1124	983-1134	987-1144	992-1154	997-1164	1002-1174	1006-1184	1011-1194
180	1016-1204	1023-1214	1033-1224	1043-1234	1052-1244	1061-1254	1071-1264	1080-1275	1089-1285	1099-1295
190	1108-1306	1118-1316	1128-1326	1138-1337	1148-1347	1158-1358	1168-1368	1178-1379	1188-1389	1198-1400
200	1208-1410	1218-1421	1228-1431	1238-1442	1248-1453	1258-1463	1268-1474	1278-1485	1288-1496	1298-1507

Table .1: Summary of known values $ex(n; \{C_3, C_4\})$, $ex_l(n; \{C_3, C_4\})$ and $ex_u(n; \{C_3, C_4\})$, for $n \leq 210$.

On the Structure of Extremal Singular Graphs

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EXTENDED ABSTRACT

A graph G is singular of nullity $\eta(G) \geq 1$, if the nullspace of its 0–1–adjacency matrix \mathbf{G} has dimension η . Such a graph contains η cores corresponding to the vectors in a basis for the nullspace of \mathbf{G} (termed kernel eigenvectors). These cores are subgraphs induced in G by the non-zero entries of the kernel eigenvectors of the chosen nullspace basis.

The weight of a vector $\mathbf{u} \in \mathbb{R}^n$ is the number of non-zero entries of \mathbf{u} . The sum $\sum_{i=1}^{\eta} \text{wt}(\mathbf{u}_i)$ of weights $\text{wt}(\mathbf{u}_i)$ in a basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_\eta$, written in non-decreasing weight order, is said to be the weight sum of the basis. We found it convenient to work with a nullspace basis of minimum weight referred to as a minimal basis B_{min} . A fundamental system of cores of G corresponds to a minimal basis B_{min} [8]. For a catalogue of all cores up to order six see [7].

We observe:

(i) Although various B_{min} may be possible, the weight sequence $\{\text{wt}(\mathbf{u}_1), \text{wt}(\mathbf{u}_2), \dots, \text{wt}(\mathbf{u}_\eta)\}$ is an invariant for $\ker(A)$ and therefore for G .

(ii) Moreover for any basis $B = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_\eta\}$, $\text{wt}(\mathbf{u}_i) \leq \text{wt}(\mathbf{w}_i)$ [8].

(iii) A basis B for the nullspace can be transformed into another basis B' by linear combinations of the vectors of B . However the union of the collections of the positions of the non-zero entries in the basis vectors is the same for all bases. The vertices corresponding to a zero entry in all vectors in a nullspace basis are said to be *core-forbidden vertices* and the remaining vertices are termed *core vertices*. Thus the partition of the vertices into core vertices and core-forbidden vertices, for a singular graph, is independent of the choice of basis used for the nullspace [5, 6].

(iv) Characterization of singular graphs can be reduced to the non-trivial solutions of a system of linear homogeneous equations $\mathbf{G}\mathbf{x} = \mathbf{0}$ for the 0-1 adjacency matrix \mathbf{G} . The existence of non-trivial solutions $\mathbf{x} \neq \mathbf{0}$ corresponds to a linear transformation \mathbf{G} that is not invertible corresponding to a singular graph.

(v) There are several applications for singular graphs not only in other areas of mathematics [11, 13, 14, 19] but also in related sciences including molecular chemistry [12], social networks [1], bioinformatics [4] and computer science [3]. Once \mathbf{G} is determined for a particular model, the eigenvectors \mathbf{x} satisfying $\mathbf{G}\mathbf{x} = \mathbf{0}$ are easily calculated. A more challenging problem, and one which is discussed in [10], is to determine the properties of the possible linear transformations \mathbf{A} that satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$ for a feasible non-zero \mathbf{x} .

The adjacency matrix \mathbf{G} that yields a non-trivial solution of $Gx = 0$ is the key to discover why a graph is singular. A feasible vector \mathbf{x} determines a core (F, \mathbf{x}_F) which can be grown by adding vertices (adjacent to the core-vertices) to a singular configuration $SC(F, \mathbf{x}_F)$ of nullity one. Graphs of nullity one are identified so that substructures corresponding to linearly independent kernel eigenvectors do not *mask* one another [9].

A singular configuration $SC(F, \mathbf{x}_F)$ is found as an induced substructure in a graph kernel eigenvector \mathbf{x} . A singular configuration that is also a core graph is called a *nut graph*. Nut graphs with at least one edge exist on seven or more vertices. They are not bipartite and have no pendent edges [2].

Necessary and sufficient conditions for a graph to be singular in terms of admissible induced subgraphs are determined [10]. There exists a set of η distinct vertices representing the singular configurations. We ask the questions: How does the nullity control the size of the singular substructures [18]? To what extent can vertices be added to a non-singular graph so that it reaches maximum nullity, without allowing duplicate vertices? Do graphs of maximum nullity have characteristic structures? There are indications that graphs of maximum nullity have a nut graph as one of its induced singular configurations.

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Spanning trees in random series-parallel graphs

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EXTENDED ABSTRACT

A graph is series-parallel (or SP for short) if it does not contain the complete graph K_4 as a minor. Edge-maximal SP graphs are exactly the class of 2-trees and conversely every subgraph of a 2-tree is SP. Subsequently, all graphs are considered to be labelled. By a random object of a given family we mean an object chosen uniformly at random from all the elements of the same size (i.e. same number of vertices).

It can be shown easily that the number of edges of a n -vertex 2-tree is precisely $2n - 6$. In the same context, Moon [6] showed that the number of 2-trees on n vertices is equal to $\binom{n}{2}(2n - 3)^{n-4}$. The enumeration of SP graphs is more involved: Bodirsky, Giménez, Kang, and Noy proved in [2] that the number of connected SP graphs on n vertices is asymptotically of the form $cn^{-5/2}\rho^{-n}n!$, where $c \approx 0.0067912$ and $\rho \approx 0.11021$ are computable constants. In the same paper they showed that the number of edges in a random connected SP graph is asymptotically normally distributed with mean asymptotically equal to κn and variance asymptotically equal to λn , where $\kappa \approx 1.61673$ and $\lambda \approx 0.2112$ are again computable constants.

In this work we study enumerative properties of spanning trees defined on SP graphs. The main result of our paper is the precise asymptotic estimate for the expected number of spanning trees in a random connected SP graph on n vertices.

Theorem 1. *Let X_n denote the number of spanning trees in a random connected SP graph on n vertices. Then, the expected value μ_n of X_n is asymptotically equal to $s\rho^{-n}$, where $s \approx 0.09063$ and $\rho^{-1} \approx 2.08415$ are computable constants.*

The proof of Theorem 1 is based on singularity analysis of generating functions. The main idea can be roughly described as follows. A *network* is defined as a simple graph with two distinguished vertices, that are called *poles*, such that adding an edge between the two poles creates a 2-connected multigraph. Using network terminology [7], a network of connectivity at most 2 is either trivial, series, or parallel. For the purpose of counting spanning trees in networks, we need apart from the class of networks with a distinguished spanning tree, the utility class of networks with a distinguished spanning forest with two components, each of which contains one of the poles. Analogously, series and parallel subclasses of these two classes are defined. Using the Symbolic Method [4] we get a system of equations for these exponential generating functions. By the so-called Drmota–Lalley–Woods Theorem [4], we know that these generating functions have the same unique singularity.

Now, by using the Dissymmetry Theorem for Trees [1] and its extension to tree-decomposable classes [3], we can determine the singular expansion of the exponential generating function B of all 2-connected SP graphs carrying a distinguished spanning tree. The relation between B and the exponential generating function C of all connected SP graphs with a distinguished spanning tree is given by $xC'(x, y) = x \exp(B'(xC'(x, y), y))$, where x and y mark vertices and edges, respectively. Setting $y = 1$ in order to avoid analyzing the number of edges, we now determine the singularity R of C by solving the equation $\bar{\tau}B''(\bar{\tau}, 1) = 1$ with respect to $\bar{\tau}$ and computing $\bar{\rho} = \bar{\tau}/\exp(B'(\bar{\tau})) \approx 0.05288$ (see [5] for similar technical arguments). The singular expansion of $C(x, 1)$ at x equal to $\bar{\rho}$ is then of the form

$$C(x, 1) = C_0(1) + C_2(1)X^2 + C_3(1)X^3 + O(X^4),$$

where we write $X = \sqrt{1 - x/\bar{\rho}}$. Finally, by Transfer Theorems for Singularity Analysis [4], the number of connected SP graphs on n vertices carrying a distinguished spanning tree is asymptotically

equal to $\frac{C_3(1)}{\Gamma(-3/2)}n^{-5/2}\rho^{-n}$. Dividing by the number of connected SP graphs finishes the proof.

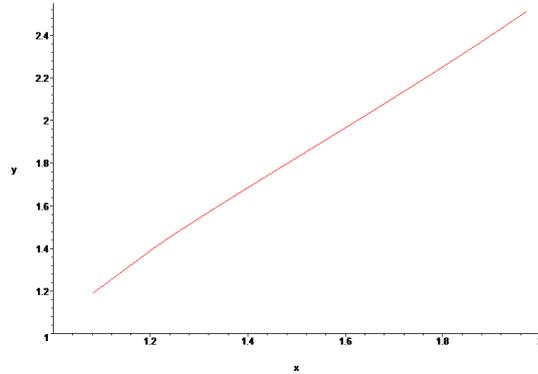


Figure 10: The growth constant $\bar{R}(\mu)$ (y in the plot) of the expected number of spanning trees in random 2-connected SP graphs with edge density $\mu \in [1.08, 1.97]$ (x in the plot).

Instead of setting $y = 1$ in the last steps in the proof of Theorem 1, one can also set y at some other value, thus fixing the edge density μ of the random (connected or 2-connected) SP graph, where the relation (see e.g. [5]) between y and μ is given by $-y\bar{R}'(y)/\bar{R}(y) = \mu$. Figure 1 shows the growth constant $\bar{R}(\mu)$ of the expected number of spanning trees in a random 2-connected SP graph with edge density μ . When $\mu \rightarrow 1$, then $\rho(\mu) \rightarrow 1$ since SP graphs of edge density equal to 1 are precisely the cycles. When we fix the edge density of SP graphs at its maximum, we get the class of 2-trees and $\bar{R}^{-1} = 2.55561$.

The nice structure of 2-trees allows us to provide an alternative, shorter proof for the expected number of spanning trees in random 2-trees. Here, the main ingredients for the proof are again the Symbolic Method, the extension of the Dissymmetry Theorem to tree-decomposable classes, and the singularity analysis of generating functions.

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Proof of a conjecture of Henning and Yeo on vertex disjoint directed cycles

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EXTENDED ABSTRACT

M.A. Henning and A. Yeo conjectured in [2] that a digraph of minimum out-degree at least 4, contains 2 vertex disjoint cycles of different lengths. In this paper we prove this conjecture. The main tool, is a new result asserting that in a digraph D of minimum out-degree at least 4, there exist two vertex-disjoint cycles C_1 and C_2 , a path P_1 from a vertex x of C_1 to a vertex z not in $V(C_1) \cup V(C_2)$, and a path P_2 from a vertex y of C_2 to z , such that $V(P_1) \cap (V(C_1) \cup V(C_2)) = \{x\}$, $V(P_2) \cap (V(C_1) \cup V(C_2)) = \{y\}$, and $V(P_1) \cap V(P_2) = \{z\}$ (we say then that D has property (\mathcal{P})).

For proving that every digraph of minimum out-degree at least 4 has property (\mathcal{P}) , we begin by proving it for non-oriented digraphs of minimum out-degree at least 4. Next for oriented graphs of minimum out-degree at least 4, we proceed by induction on the order $n \geq 9$ of such an oriented graph. We prove first that the assertion is true when $n = 9$, that is when D is a 4-regular tournament. We suppose that the assertion is true up to the row $n - 1$, $n \geq 10$, and then we prove it for n (assertion $P(n)$). For this we use induction on the size k , $4n \leq k \leq \frac{n(n-1)}{2}$ of an oriented graph of minimum out-degree at least 4 and of order n . We prove first that the assertion $P(n)$ is true when $k = 4n$. So, we consider an oriented graph D of minimum out-degree at least 4 and of size $a(D) = 4n$. We observe that necessarily, D is 4-out-regular, and for the sake of a contradiction, we suppose that D does not have property (\mathcal{P}) . We prove then that D is strongly connected and arc-dominated (which means that for every arc (x, y) of D , there exists a vertex z such that (z, x) and (z, y) are arcs of D). The fact that D is arc-dominated implies that for every vertex x of D , the induced sub-digraph $D[N^-(x)]$ is of minimum out-degree at least 1, and then it contains a cycle. So for every vertex x of D we may choose a cycle $C(x)$ of $D[N^-(x)]$ of minimum order (so, $C(x)$ is an induced cycle). We point out that by a result of Thomassen (see [3], if m is the maximum number of vertex-disjoint cycles of D , it holds $m \geq 2$). We consider then a set \mathcal{C} of m vertex-disjoint induced cycles C_1, C_2, \dots, C_m . We consider the digraph D' defined in the following way

- The vertex set of D' is \mathcal{C} .
- The arcs of D' are the couples (C, C') of distinct cycles of \mathcal{C} , such that there exist at least one path from C to C' , internal disjoint with every cycle of \mathcal{C} .

We prove that D' is a directed cycle of length m . Without loss of generality we suppose that this directed cycle is $D' = (C_1, \dots, C_m, C_1)$. For i , $1 \leq i \leq m$, we consider the set A_i of the vertices of $V(D) \setminus (V(C_1) \cup \dots \cup V(C_m))$ having an in-neighbor in C_i . We prove that the sets A_i are vertex-disjoint and that for $1 \leq i, j \leq m$ with $i \neq j$, there are no arcs from a vertex of A_i to a vertex of A_j (for otherwise D would have property \mathcal{P}). We prove also that the union of the pairwise disjoint sets $V(C_i) \cup A_i$ is $V(D)$. We establish several properties relative to the sets $V(C_i) \cup A_i$, and next by considering the cases $m \geq 3$ and $m = 2$, we get a contradiction. Then by pursuing our induction on the size, we prove that the assertion $P(n)$ is true, and then any oriented graph of minimum out-degree at least 4 has property \mathcal{P} . It follows that any digraph of minimum out-degree at least 4 has property \mathcal{P} . From this result we easily prove the conjecture of Henning and Yeo.

Thomassen proved in [3] the existence for every $k \geq 1$, of a minimum integer $f(k)$ such that any digraph of minimum out-degree at least $f(k)$ contains k vertex-disjoint cycles. Alon proved that $f(k) \leq 64k$ (see [1]). Motivated by these results and by our proof of the conjecture of Henning and Yeo, we conjecture that for every integer $k \geq 2$, there exists a minimum integer $g(k)$ such that any digraph of minimum out-degree at least $g(k)$ contains k vertex-disjoint cycles of different lengths. We proved this conjecture for $k = 2$ (and since Henning and Yeo proved in [2] that there exists

a digraph of minimum out-degree 3 which does not contain two vertex-disjoint cycles of different lengths, we have $g(2) = 4$). We finish by proving that if $g(k)$ exists, we have $g(k) \geq \frac{k^2+5k-2}{2}$, which means that a possible upper bound on $g(k)$ is not linear.

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Leaf-critic graphs

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EXTENDED ABSTRACT

The *minimum leaf number* $\text{ml}(G)$ of a connected graph G is defined as the minimum number of leaves of the spanning trees of G if G is not Hamiltonian and 1 if G is Hamiltonian. We study nonhamiltonian graphs, whose vertex-deleted subgraphs have the same minimum leaf number. The deletion of a vertex obviously may increase the minimum leaf number of the graph, but there are vertices (e.g. leaves of an optimum spanning tree) whose deletion does not increase or even decrease $\text{ml}(G)$. However, it is easy to see that by deleting a vertex of a connected graph G , $\text{ml}(G)$ can be decreased by at most one. Thus if G is nonhamiltonian and $l = \text{ml}(G - v)$ does not depend on v , then either $\text{ml}(G - v) = \text{ml}(G) - 1$ for each $v \in V(G)$ or $\text{ml}(G - v) = \text{ml}(G)$ for each $v \in V(G)$. Such graphs will be called $(l + 1)$ -leaf-critic and l -leaf-stable, respectively.

At first sight it is not obvious whether leaf-critic and leaf-stable graphs exist. The 2-leaf-critic graphs turn out to be the so-called hypohamiltonian graphs. A graph G is hypohamiltonian [3] if G is not Hamiltonian, but every vertex-deleted subgraph of G is Hamiltonian. The smallest hypohamiltonian graph is the Petersen graph [3] and it is known that hypohamiltonian graphs on n vertices exist for every $n \geq 18$ [3] and even planar hypohamiltonian graphs on n vertices exist for a sufficiently large n [9] (the best known bound is $n \geq 42$ [5]). Planarity of hypohamiltonian and hypotractable graphs is an important issue since 1973, when Chvátal asked whether such graphs exist [1].

The 3-leaf-critic graphs are the so-called hypotractable graphs. A graph is hypotractable [3] if G is not traceable, but every vertex-deleted subgraph of G is traceable. The existence of hypotractable graphs was an open problem till 1975, when Horton found such a graph on 40 vertices [7]. Actually, even the existence of graphs without concurrent longest paths was an open question from 1966 to 1969 (raised by Gallai [2] and settled by Walther [8]). The smallest known hypotractable graph (having 34 vertices) is due to Thomassen [6], who also proved that hypotractable graphs on n vertices exist for every $n \geq 39$ [7]. It is also known that planar hypotractable graphs on n vertices exist for a sufficiently large n [9] (the best known bound is $n \geq 156$ [5]). First we show that l -leaf-stable and l -leaf-critic graphs exist for every integer $l \geq 2$. Using the construction it is also easy to prove that for n sufficiently large, cubic planar l -leaf-stable and l -leaf-critic graphs exist on n vertices. Then we prove some structural theorems concerning leaf-critic graphs of connectivity 2. These are interesting because not much is known about the structure of hypohamiltonian and hypotractable graphs, i.e. all known hypotractable graphs are constructed using hypohamiltonian graphs, and it is not even known whether there exists a hypohamiltonian graph without a vertex of degree 3. (All leaf-critic graphs are easily seen to be 3-edge-connected and Thomassen proved that all planar hypohamiltonian graph contains a vertex of degree 3.)

For the construction we use the so-called *J-cells* [4]. A pair of vertices (a, b) of a graph G is said to be good if there exists a Hamiltonian path of G between them. A pair of pairs $((a, b), (c, d))$ is good if there exists a spanning subgraph of G consisting of two vertex disjoint paths, one between a and b and the other one between c and d . The quintuple (H, a, b, c, d) is a J-cell if H is a graph and $a, b, c, d \in V(H)$, such that the pairs (a, d) , (b, c) are good in H , none of the pairs (a, b) , (a, c) , (b, d) , (c, d) , $((a, b), (c, d))$, $((a, c), (b, d))$ are good in H , and for each $v \in V(H)$ there is a good pair in $H - v$ among (a, b) , (a, c) , (b, d) , (c, d) , $((a, b), (c, d))$, $((a, c), (b, d))$. It is worth mentioning that flip-flops used by Chvátal to obtain many hypohamiltonian graphs [1] are special J-cells. J-cells can also be obtained by deleting two adjacent vertices of degree 3 from a hypohamiltonian graph, as pointed out by Thomassen, who used them to construct 3-connected hypotractable graphs [7]. Our construction is basically a generalization of this construction.

Let now $F_i = (H_i, a_i, b_i, c_i, d_i)$ be J-cells for $i = 1, 2, \dots, k$. We define the graphs G_k as follows. G_k consists of vertex-disjoint copies of the graphs H_1, H_2, \dots, H_k , the edges $(b_i, a_{i+1}), (c_i, d_{i+1})$ for all $i = 1, 2, \dots, k-1$, and the edges $(b_k, a_1), (c_k, d_1)$.

Theorem 1. G_{2l+1} is $(l+1)$ -leaf-critic, G_{2l} is l -leaf-stable.

Let G be a non-complete graph with connectivity k and $X = \{x_1, x_2, \dots, x_k\}$ be a cut of G . Let furthermore H be one of the components of $G - X$. Then $G[H \cup X]$ is called a k -fragment of G , and X is called the vertices of attachment of H . Now we characterize 2-fragments of leaf-critic graphs generalizing a lemma of Thomassen [6].

Theorem 2. Let G be a graph, $a, b \in V(G)$ and $l = ml(G - a)$. G is a 2-fragment of a leaf-critic graph with vertices of attachment a and b if and only if the following hold:

1. $ml(G) \geq l$, moreover G has no spanning tree with at most l leaves where a or b is a leaf and G has no spanning forest with at most $l+1$ leaves consisting of two trees, such that a is a leaf of one of the trees and b is a leaf of the other tree.
2. For any $v \in V(G)$, $G - v$ has a spanning tree with at most $l-1$ leaves or a spanning tree with at most l leaves, where a or b is a leaf or $G - v$ has a spanning forest with at most $l+1$ leaves consisting of two trees, such that a is a leaf of one of the trees and b is a leaf of the other tree.

Theorem 3. Let G be a graph of connectivity 2 and $\{a, b\}$ a cut in G . If both 2-fragments G_1 and G_2 of G with vertices of attachment a, b are leaf-critic 2-fragments, then G is l -leaf-critic, where $l = ml(G_1 - a) + ml(G_2 - a) - 1$.

Corollary 4. If G is an l -leaf-critic graph of connectivity 2, then it contains an $\lfloor \frac{l+3}{2} \rfloor$ -leaf-critic 2-fragment.

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Decomposing graphs into locally irregular subgraphs: allowing K_2 's helps a lot

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EXTENDED ABSTRACT

A graph G is *locally irregular* if every two adjacent vertices of G have distinct degrees. We say that an edge-colouring c of G is *locally irregular* if every colour class of c induces a locally irregular subgraph. The least number of colours of a locally irregular edge-colouring of G (if any) is called the *irregular chromatic index* of G , and is denoted $\chi'_{irr}(G)$.

The notion of locally irregular edge-colouring was introduced by Baudon, Bensmail, Przybyło and Woźniak in [1] as a tool to deal with the well-known 1-2-3 Conjecture posed by Karoński, Łuczak and Thomason [4]. The point being that, in specific situations, a graph admitting a locally irregular edge-colouring directly agrees with the 1-2-3 Conjecture. In [1], the authors first noted that not all graphs admit locally irregular edge-colourings. Such graphs, called *exceptions*, are fortunately easy to recognize since the family of exceptions is made up of odd length paths and cycles, and another class of graphs obtained by connecting vertex-disjoint triangles in a specific way. Regarding graphs which are not exceptions, the following conjecture was raised.

Conjecture 1 ([1]). *For every graph G which is not an exception, we have $\chi'_{irr}(G) \leq 3$.*

Conjecture 1 was notably verified for trees, complete graphs, Cartesian products of graphs with irregular chromatic index at most 3, and regular graphs with degree at least 10^7 (see [1]). The latter result, which was proved by using probabilistic tools, is the most interesting as regular graphs are in some sense the “least locally irregular” graphs. It is also worth mentioning that Conjecture 1, if true, would be tight since e.g. $\chi'_{irr}(C_6) = 3$. However, an easy classification of graphs with irregular chromatic index at most 2 should not exist (unless $P=NP$) due to the NP-completeness of the associated problem (this was proved in [2] by Baudon, Bensmail and Sopena).

Towards Conjecture 1, no good upper bound on the irregular chromatic index of non-exception graphs is known in general. The only such upper bound given in [1] is the following.

Theorem 2 ([1]). *For every graph G which is not an exception, we have $\chi'_{irr}(G) \leq \lfloor \frac{|E(G)|}{2} \rfloor$.*

The upper bound given in Theorem 2 is clearly not satisfying as we are rather interested in upper bounds involving no graph invariant, recall Conjecture 1. Actually $\lfloor \frac{|E(G)|}{2} \rfloor$ is the worst possible upper bound on $\chi'_{irr}(G)$ as the smallest non-trivial (i.e. with edges) locally irregular graph is P_3 , which has size 2. However, the proof of Theorem 2 is, though not complicated, not trivial.

Improving Theorem 2 to better upper bounds (even involving graph parameters) does not seem to be an easy task, mainly due to the fact that a locally irregular edge-colouring can clearly not induce a component isomorphic to K_2 . This fact makes disputable the success of any “naive” colouring procedure where one would basically construct the induced locally irregular subgraphs inductively (i.e. by augmenting them in a locally irregular way as long as possible). This notably has implications on bipartite graphs, for which we still do not know whether Conjecture 1 (or even a weaker constant version of it) is true, since this problem makes improbable the success of any recursive colouring scheme.

In the complete version of the current paper [3], we consider the consequences on this problem of allowing components isomorphic to K_2 , or more generally regular components, in locally irregular edge-colourings. We say that an edge-colouring c of G is *regular-irregular* if every colour class of c induces a subgraph whose components are either regular or locally irregular. The *regular-irregular*

chromatic index of G , denoted $\chi'_{reg-irr}(G)$, is the least number of colours used by a regular-irregular edge-colouring of G . Note that, in this context, there is no need for a notion of exception as notably every proper edge-colouring is also regular-irregular. So $\chi'_{reg-irr}(G) \leq \chi'(G) \leq \Delta(G) + 1$, where χ' and Δ denote the standard chromatic index and maximum degree parameters. Since, by definition, a locally irregular edge-colouring is also regular-irregular (but the contrary may obviously not hold), we clearly also have $\chi'_{reg-irr}(G) \leq \chi'_{irr}(G)$. So, according to Conjecture 1, the regular-irregular chromatic index of non-exception graphs should be upper-bounded by 3. Actually, investigations on small graphs even suggest that the following stronger should be true.

Conjecture 3. *For every graph G , we have $\chi'_{reg-irr}(G) \leq 2$.*

Our first result concerning regular-irregular edge-colouring deals with bipartite graphs. For this family, we prove the following constant upper bound on $\chi'_{reg-irr}$, which has no constant analogue when only locally irregular components are allowed in an edge-colouring.

Theorem 4. *For every bipartite graph G , we have $\chi'_{reg-irr}(G) \leq 6$.*

Theorem 4 is basically proved by showing that G can always be edge-partitioned into one forest and one bipartite graph whose all components are Eulerian, which we show have regular-irregular chromatic index at most 2 and 4, respectively. Theorem 4 in turns implies the following result, which gives a general upper bound on $\chi'_{reg-irr}$ which is better than every known upper bound on χ'_{irr} (in particular Theorem 2). In this result, χ denotes the standard chromatic number parameter.

Corollary 5. *For every graph G , we have $\chi'_{reg-irr}(G) \leq 6 \log_2(\chi(G))$.*

Corollary 5 is proved by showing that G can be edge-partitioned into $\log_2(\chi(G))$ bipartite graphs. Using at most 6 colours to obtain a regular-irregular edge-colouring of each of these resulting subgraphs (such colourings exist according to Theorem 4), a regular-irregular $(6 \log_2(\chi(G)))$ -edge-colouring of G is then obtained. Corollary 5 notably confirms that allowing regular components (in particular K_2) helps a lot when edge-partitioning graphs into locally irregular subgraphs (though the two colouring notions are slightly different).

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A note on non-repetitive colourings

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EXTENDED ABSTRACT

A vertex-colouring c of a graph G is *non-repetitive* if for any path $P = v_1v_2 \dots v_{2k}$ in G the colour sequence $(c(v_1), c(v_2), \dots, c(v_k)) \neq (c(v_{k+1}), c(v_{k+2}), \dots, c(v_{2k}))$. The Thue chromatic number of a graph G , denoted by $\pi(G)$, is the least integer k such that G has a non-repetitive colouring using k colours. As two neighbouring vertices form a path of length 1, each non-repetitive colouring is a proper colouring. The study of non-repetitive (edge) colourings was initiated by Alon et al. in [1]. Since then non-repetitive colourings have intensively been studied and many variations were introduced [4, 3].

Let G be a graph. A supergraph H of G is called *non-repetitive supergraph* of G if $V(G) = V(H)$ and for each odd path $P = v_1v_2 \dots v_{2k}$ of G at least one of the edges $v_1v_{k+1}, v_2v_{k+2}, \dots, v_kv_{2k}$ is an edge of H . We prove the following result.

Theorem 1. *Thue chromatic number of a graph G equals the minimum over all chromatic numbers of non-repetitive supergraphs of G .*

One of the natural modifications of the notion of a non-repetitive colouring is its fractional version. A (p, q) -*non-repetitive colouring* c of a graph G is a colouring of vertices of G that uses p colours and each vertex is coloured by exactly q colours such that for each odd path $P = v_1v_2 \dots v_{2k}$ of G and for all $c_1 \in c(v_1), c_2 \in c(v_2), \dots, c_{2k} \in c(v_{2k})$ we have $(c_1, c_2, \dots, c_k) \neq (c_{k+1}, c_{k+2}, \dots, c_{2k})$. The *fractional Thue chromatic number* is defined as

$$\pi_f(G) = \inf\{p/q : \text{there exists a } (p, q)\text{-non-repetitive colouring of } G\}.$$

Fractional non-repetitive colourings were introduced by Keszegh, Patkós, and Zhu [4] and studied further by Zhong and Zhu [5]. Theorem 1 also extends to the fractional Thue chromatic number.

Theorem 2. *Fractional Thue chromatic number of a graph G equals the minimum over all fractional chromatic numbers of non-repetitive supergraphs of G .*

As fractional chromatic number is attained and rational and the number of non-repetitive supergraphs is finite, we have the following corollary.

Corollary 3. *Fractional Thue chromatic number is attained and rational.*

Theorem 2 provides an algorithm to determine the fractional Thue chromatic number.

We can reformulate Theorem 1 for list colourings and fractional list colourings. Thue choice number and fractional Thue choice number (which can be defined analogously to Thue chromatic number where the lists of available colours are prescribed for each vertex). Alon, Tuza, and Voigt [2] showed that the fractional choice number equals the fractional chromatic number for every graph. Together with our result this gives the following corollary:

Corollary 4. *Fractional Thue choice number of a graph equals its fractional Thue chromatic number.*

Similarly, as for fractional Thue colourings, we can define circular Thue colourings—we add a constraint that colours at a vertex must be consecutive modulo p . A circular variant of Theorem 1 translates many known results about circular colourings to circular Thue colourings.

Corollary 5. *Circular Thue chromatic number is attained and rational. The ceiling of circular Thue chromatic number of a graph G is the Thue chromatic number of G .*

We use the developed methods to show that the fractional Thue chromatic number of Petersen graph is 5. The fractional Thue chromatic number of all circuits, except for C_{10} , C_{14} , C_{17} was determined by Zhong and Zhu [5]. We determined fractional Thue chromatic numbers of C_{10} , C_{14} , C_{17} . Our approach relies on computer search based on an idea from [5]. We describe our algorithms.

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Rainbow connection and forbidden induced subgraphs

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EXTENDED ABSTRACT

An edge-colored connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G , denoted by $\text{rc}(G)$, is the minimum number of colors such that G is rainbow-connected.

The concept of rainbow connection in graphs was introduced by Chartrand et al. in [2]. An easy observation is that if G has n vertices then $\text{rc}(G) \leq n - 1$, since one may color the edges of a given spanning tree of G with different colors and color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [2]. The rainbow connection number has been studied for further graph classes and for graphs with fixed minimum degree, see [5] for a survey.

There are various applications for such edge colorings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e. g., [3]).

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1. *Let G be a connected graph of order n . Then*

- $1 \leq \text{rc}(G) \leq n - 1$,
- $\text{rc}(G) \geq \text{diam}(G)$,
- $\text{rc}(G) = 1$ if and only if G is complete,
- $\text{rc}(G) = n - 1$ if and only if G is a tree,
- if G is a cycle of length $n \geq 4$, then $\text{rc}(G) = \lceil \frac{n}{2} \rceil$.

Note that the difference $\text{rc}(G) - \text{diam}(G)$ can be arbitrarily large. For $G = K_{1,n-1}$ we have $\text{rc}(K_{1,n-1}) - \text{diam}(K_{1,n-1}) = (n - 1) - 2 = n - 3$. Especially, each bridge requires a single color.

Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free if G does not contain an induced subgraph isomorphic to a graph from \mathcal{F} . Specifically, for $\mathcal{F} = \{X\}$ we say that G is X -free, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y) -free. The members of \mathcal{F} will be referred to in this context as *forbidden induced subgraphs*.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, in general, there is no upper bound on $\text{rc}(G)$ in terms of $\text{diam}(G)$, and, in bridgeless graphs, $\text{rc}(G)$ can be quadratic in terms of $\text{diam}(G)$ (see [1]), it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\text{rc}(G)$.

Namely, we will consider the following question.

For which families \mathcal{F} of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being \mathcal{F} -free implies $\text{rc}(G) \leq \text{diam}(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Theorem 2, and for $|\mathcal{F}| = 2$ in Theorem 3.

Theorem 2 (Holub, Ryjáček, Schiermeyer, Vrána [4]). *Let X be a connected graph. Then there is a constant k_X such that every connected X -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem 3 (Holub, Ryjáček, Schiermeyer, Vrána [4]). *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y) -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$, $r \geq 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N .*

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Finite families of forbidden subgraphs for rainbow connection in graphs

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EXTENDED ABSTRACT

Let G be an edge-colored graph (the coloring need not be proper). A path in G is said to be *rainbow* if its edges have pairwise distinct colors, and G is *rainbow-connected* if any two distinct vertices of G are connected by a rainbow path. The *rainbow connection number* $\text{rc}(G)$ of G is the minimum number k of colors such that G is rainbow-connected for some edge coloring using k colors.

The concept of rainbow connection was introduced by Chartrand et al. [2]. For a survey of recent results see e.g. [4], [5]. Specifically, it is easy to observe that, for a connected graph G of diameter $\text{diam}(G)$, $\text{rc}(G) \geq \text{diam}(G)$, and the difference $\text{rc}(G) - \text{diam}(G)$ can be arbitrarily large (as can be seen e.g. for $G = K_{1,r}$, for which $\text{rc}(G) = r$ while $\text{diam}(G) = 2$). However, in bridgeless graphs, it is known [1] that $\text{rc}(G) \leq \text{rad}(G)(\text{rad}(G) + 2) \leq \text{diam}(G)(\text{diam}(G) + 2)$ (where $\text{rad}(G)$ denotes the radius of G). Note that this upper bound on $\text{rc}(G)$ is still quadratic in terms of $\text{diam}(G)$.

Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free if G does not contain an induced subgraph isomorphic to a graph from \mathcal{F} . Specifically, for $\mathcal{F} = \{X\}$ we say that G is X -free, and for $\mathcal{F} = \{X_1, \dots, X_k\}$ we say that G is (X_1, \dots, X_k) -free. The members of \mathcal{F} will be referred to in this context as *forbidden induced subgraphs*.

If X_1, X_2 are graphs, we write $X_1 \stackrel{\text{IND}}{\subset} X_2$ if X_1 is an induced subgraph of X_2 (not excluding the possibility that $X_1 = X_2$). If $\mathcal{F}_1, \mathcal{F}_2$ are finite families of graphs, we write $\mathcal{F}_1 \stackrel{\text{IND}}{\subset} \mathcal{F}_2$ if $|\mathcal{F}_1| = |\mathcal{F}_2|$ and there is a bijection $\varphi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that $F \stackrel{\text{IND}}{\subset} \varphi(F)$ for every $F \in \mathcal{F}_1$. Obviously, if $\mathcal{F}_1 \stackrel{\text{IND}}{\subset} \mathcal{F}_2$, then every \mathcal{F}_1 -free graph is also \mathcal{F}_2 -free.

It turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\text{rc}(G)$. Namely, we will consider the following question:

For which finite families $\mathcal{F} = \{X_1, X_2, \dots, X_k\}$ ($k \geq 3$ an integer) of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being \mathcal{F} -free implies $\text{rc}(G) \leq \text{diam}(G) + k_{\mathcal{F}}$?

In [3], a complete answer was given for $1 \leq |\mathcal{F}| \leq 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 1 (Holub, Ryjáček, Schiermeyer, Vrána, 2014 [3]). *Let X be a connected graph. Then there is a constant k_X such that every connected X -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem 2 (Holub, Ryjáček, Schiermeyer, Vrána, 2014 [3]). *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y) -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_{XY}$, if and only if either $\{X, Y\} \stackrel{\text{IND}}{\subset} \{K_{1,r}, P_4\}$ for some $r \geq 4$, or $\{X, Y\} \stackrel{\text{IND}}{\subset} \{K_{1,3}, N\}$.*

We continue this study by giving a complete characterization for larger finite families of forbidden subgraphs. For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex of three vertex disjoint paths of lengths i, j, k , $N_{i,j,k}$ the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex disjoint paths of lengths i, j, k , and let K_t^h denote the graph obtained by attaching a pendant edge to every vertex of a complete graph K_t .

Theorem 3. Let $X, Y, Z \neq P_3$ be connected graphs such that $\{K_{1,3}, N\} \not\subset \{X, Y, Z\}$ and $\{K_{1,r}, P_4\} \not\subset \{X, Y, Z\}$ ($r \geq 4$). Then there is a constant k_{XYZ} such that every connected (X, Y, Z) -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_{XYZ}$, if and only if the triple $\{X, Y, Z\}$ satisfies one of the following conditions:

- (i) $\{X, Y, Z\} \stackrel{\text{IND}}{\subset} \{K_{1,3}, K_s^h, N_{1,p,q}\}$, $s > 3, p + q > 2, 1 \leq p \leq q$,
- (ii) $\{X, Y, Z\} \stackrel{\text{IND}}{\subset} \{K_{1,r}, K_s^h, P_\ell\}$, $r > 3, s > 3, \ell > 4$,
- (iii) $\{X, Y, Z\} \stackrel{\text{IND}}{\subset} \{K_{1,r}, S_{1,p,q}, N\}$, $r > 3, p + q > 2, 1 \leq p \leq q$.

Theorem 4. Let $X, Y, Z, W \neq P_3$ be connected graphs such that $\{K_{1,3}, N\} \not\subset \{X, Y, Z, W\}$, $\{K_{1,r}, P_4\} \not\subset \{X, Y, Z, W\}$ ($r > 3$), $\{K_{1,r}, K_s^h, P_\ell\} \not\subset \{X, Y, Z, W\}$ ($r > 3, s > 3, \ell > 4$), $\{K_{1,3}, K_s^h, N_{1,j,k}\} \not\subset \{X, Y, Z, W\}$ ($s > 3, j + k > 2, j \leq k$), and $\{K_{1,r}, N, S_{1,j,k}\} \not\subset \{X, Y, Z, W\}$ ($r > 3, j + k > 2, j \leq k$). Then there is a constant k_{XYZW} such that every connected (X, Y, Z, W) -free graph G satisfies $\text{rc}(G) \leq \text{diam}(G) + k_{XYZW}$, if and only if

$$\{X, Y, Z, W\} \stackrel{\text{IND}}{\subset} \{K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}}\}, \quad r > 3, s > 3, j + k > 2, 1 \leq j \leq k, \\ \bar{j} + \bar{k} > 2, 1 \leq \bar{j} \leq \bar{k}.$$

We summarize these observations in the following theorem, showing that there are no other finite families than those listed in Theorems 1, 2, 3 and 4. For this, we first introduce some notation. We set:

$$\begin{aligned} \mathcal{F}_1 &= \{P_3\}, \\ \mathcal{F}_2 &= \{K_{1,3}, N\}, \\ \mathcal{F}_3 &= \{K_{1,r}, P_4\}, r \geq 4, \\ \mathcal{F}_4 &= \{K_{1,3}, K_s^h, N_{1,j,k}\}, s > 3, 1 \leq j \leq k, j + k > 2, \\ \mathcal{F}_5 &= \{K_{1,r}, K_s^h, P_\ell\}, r > 3, s > 3, \ell > 4, \\ \mathcal{F}_6 &= \{K_{1,r}, S_{1,j,k}, N\}, r > 3, 1 \leq j \leq k, j + k > 2, \\ \mathcal{F}_7 &= \{K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}}\}, r > 3, s > 3, 1 \leq j \leq k, j + k > 2, 1 \leq \bar{j} \leq \bar{k}, \bar{j} + \bar{k} > 2. \end{aligned}$$

Theorem 5. Let \mathcal{F} be a finite family of connected graphs. Then there is a constant $k_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph satisfies $\text{rc}(G) \leq \text{diam}(G) + k_{\mathcal{F}}$, if and only if \mathcal{F} contains a subfamily \mathcal{F}' such that $\mathcal{F}' \stackrel{\text{IND}}{\subset} \mathcal{F}_i$ for some $i, 1 \leq i \leq 7$.

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Nonrepetitive coloring of geometric graphs

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EXTENDED ABSTRACT

A sequence $R = r_1 r_2 \dots r_{2t}$ is called a *repetition* if $r_i = r_{i+t}$ for all $i = 1, 2, \dots, t$. A *segment* in a sequence S is a subsequence consisting of consecutive terms of S . A sequence S is *nonrepetitive* if none of its segments is a repetition. In 1906 Thue [10] proved that there exist arbitrarily long nonrepetitive sequences over the set of three symbols. This result has lots of interesting applications and generalizations (cf. [4], [6], [7]). In particular, one may consider *nonrepetitive colorings* of graphs in which a color sequence of every simple path is nonrepetitive. The least number of colors in such coloring of a graph G is denoted by $\pi(G)$ and called the *Thue chromatic number* of G . It was proved that $\pi(G)$ is bounded for graphs of bounded degree [1] and for graphs of bounded treewidth [5], but the following conjecture remains open.

Conjecture 1. *There is a constant C such that every planar graph G satisfies $\pi(G) \leq C$.*

Here we study a geometric variant of nonrepetitive colorings inspired by this conjecture. Let L be a *line arrangement* consisting of a finite set of lines in the plane. Let $P = P(L)$ denote the set of all intersection points of these lines. A *nonrepetitive coloring* of L is a coloring of the set P such that a sequence of colors determined by consecutive points on every line in L is nonrepetitive. Our main result reads as follows.

Theorem 2. *Every line arrangement has a nonrepetitive coloring using at most 405 colors.*

The proof is based on the following theorem of Alon and Marshall [2].

Theorem 3. (Alon and Marshall [2]) *Let k be a positive integer. There exists a graph H_k on at most $5k^4$ vertices with k -colored edges such that every planar graph G whose edges are colored arbitrarily with k colors embeds homomorphically into H_k .*

We naturally expect that the bound of 405 from Theorem 2 is not optimal. We propose the following (risky) conjecture.

Conjecture 4. *Every line arrangement has a nonrepetitive coloring with at most 4 colors.*

The famous Hadwiger-Nelson problem asks for the chromatic number of the plane $\chi(\mathbb{R}^2)$, defined as the least number of colors needed to color the plane such that every pair of points at distance one apart is colored differently. We formulate a natural analog of this question in the spirit of nonrepetitive colorings of line arrangements. A finite sequence of points P_1, P_2, \dots, P_n in the plane is called *nice* if the points are collinear and the distance between each pair P_i and P_{i+1} is one. A coloring of the plane is *straight-nonrepetitive* if a sequence of colors determined by every nice sequence of points is nonrepetitive. Let $\bar{\pi}(\mathbb{R}^2)$ denote the least number of colors needed for a nonrepetitive coloring of the plane. We prove that this number is finite.

Theorem 5. $\bar{\pi}(\mathbb{R}^2) \leq 36$.

We also expect that this is not optimal, but this time we do not pose the conjecture that $\bar{\pi}(\mathbb{R}^2) \leq 4$ (or do we?).

The above discussion leads to the following general problem for geometric graphs. Recall that a *geometric graph* is a graph drawn on the plane such that each vertex corresponds to a point and every edge is a closed line segment connecting two vertices but not passing through a third. A *straight path* in a geometric graph G is a path whose vertices are collinear. A *straight-nonrepetitive coloring* of a geometric graph is a coloring of its vertices such that the sequence of colors on any straight path is nonrepetitive. Let $\bar{\pi}(G)$ denote the least number of colors needed in such coloring of G .

Conjecture 6. *There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every geometric graph G satisfy $\bar{\pi}(G) \leq f(\chi(G))$.*

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The duality between matchings and vertex covers in balanced hypergraphs

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EXTENDED ABSTRACT

In this talk we present our investigations on matchings and vertex covers in the class of balanced hypergraphs. As usual, we define a hypergraph to be a pair $H = (V, E)$, where V is a finite set and E a multiset of subsets of V . A sequence $v_0 e_1 v_1 e_2 \cdots e_l v_l$ of vertices $v_0, \dots, v_l \in V$ with $v_0 = v_l$, and edges $e_1, \dots, e_l \in E$ is called a cycle if e_i contains v_{i-1} and v_i for all i and v_0, \dots, v_{l-1} are pairwise distinct. It is called strong if none of its edges e_i contains three vertices from v_0, \dots, v_{l-1} . Berge [1] defined a hypergraph H to be balanced, if H contains no strong cycles of odd length and studied them as one possible generalization of bipartite graphs. Just like bipartite graphs, balanced hypergraphs have a lot of interesting coloring properties which were also found by Berge in [2].

For a subset $F = \{f_1, \dots, f_k\} \subset E$ we call the hypergraph $H' := H(F) = \left(\bigcup_{i=1}^k f_i, F \right)$ the partial hypergraph generated by the set F . We assume that two weight functions $d : E \rightarrow \mathbb{N}$ and $b : V \rightarrow \mathbb{N}$ are given. Then, we define the weight of a partial hypergraph H' of H as

$$w(H') := \sum_{e \in E(H')} d(e) - \sum_{v \in V(H')} (\deg_{H'}(v) - 1)b(v).$$

Consider the optimization problem of maximizing $w(H')$ over all partial hypergraphs $H' \subseteq H$. Intuitively, we want to maximize the sum of the edge weights of H' , but put a penalty on points which are covered more than once. If the penalty is large, we are looking for maximum weight matchings. Let $x : V \rightarrow \mathbb{N}$. Then x is called a d -vertex cover if the inequality $\sum_{v \in e} x(v) \geq d(e)$ holds for every edge $e \in E$. By means of the penalty function b , we define the set

$$X := X(H, d, b) := \{x \mid x \text{ is a } d\text{-vertex cover and } 0 \leq x(v) \leq b(v) \text{ for all } v \in V\}.$$

Berge and Las Vergnas [3] and Fulkerson et al. [5] proved what may be called König's Theorem for balanced hypergraphs, namely

Theorem 1 (Berge and Las Vergnas '70 [3], Fulkerson et al. '74 [5]). *Let a balanced hypergraph $H = (V, E)$ with weight function $d : E \rightarrow \mathbb{N}$ be given. Then*

$$\max_{M \text{ is a matching of } H} w(M) = \min_{x \in X(H, d, \infty)} \sum_{v \in V} x(v)$$

(the value of a maximum weighted matching equals the sum of weights of a minimum d -vertex cover).

In 1996, Conforti et al. [4] generalized Hall's Theorem to balanced hypergraphs. By Linear Programming methods, they proved

Theorem 2 (Conforti et al. '96 [4]). *Let a balanced hypergraph $H = (V, E)$ be given. H has a perfect matching, i.e., a matching covering all the vertices, if and only if H satisfies the following condition: If some vertices of H are colored red and blue, and if there are more blue than red vertices in total, then there is an edge containing more blue than red vertices.*

Conforti et al. asked for a combinatorial proof which was given in [6]. We present the following Min-Max Theorem which generalizes the Theorems 1 and 2.

Theorem 3. *Let $H = (V, E)$ be a balanced hypergraph and $d : E \rightarrow \mathbb{N}$ and $b : V \rightarrow \mathbb{N}$, such that for all $e \in E : \sum_{v \in e} b(v) \geq d(e)$. Then the following minimax-relation holds:*

$$\max_{H' \subseteq H} w(H') = \min_{x \in X(H, d, b)} \sum_{v \in V} x(v).$$

In particular, we obtain a defect version of the generalized Hall Theorem. Our proof is purely combinatorial and based on the coloring properties mentioned above. However, it is possible to obtain the result by similar reasoning as in [4].

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Two Domination Problems Parameterized by Neighborhood Diversity

5

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EXTENDED ABSTRACT

Introduction. The field of parameterized complexity is concerned with relating the running time of an algorithm not just with the size of the input, but also either with the solution size or some structural property of the input. Specifically in the case of graphs, the parameterized perspective has led to much research in structural parameters such as treewidth [1] or cliquewidth [2].

A new parameter called *neighborhood diversity* (n.d. for short in the following text) was introduced by Lampis in 2009 [3]. The importance of this parameter is in that it is simple, incomparable with treewidth (see Figure .11) and not restricted by some lower-bounds affecting treewidth [4] (i.e., the time complexity of MSO_1 model checking on graphs with bounded n.d. is only singly-exponential in n.d.).

Since n.d. is incomparable to treewidth (i.e., there are graph classes with unbounded treewidth and constant n.d. and vice versa), it is desirable to know which problems that are hard w.r.t. one parameter become easy as we move to the other and vice versa. Here we look at two domination-style problems: CAPACITATED DOMINATING SET and VECTOR DOMINATING SET (CDS and VDS for short) that have been shown to be W[1]-hard w.r.t. treewidth (the latter quite recently) [5] [6]. Using an old result from the 80s about the fixed-parameter tractability of integer linear programming with respect to the dimension [7], we show both of these problems to be fixed-parameter tractable with respect to n.d.

Neighborhood diversity. A graph with n.d. k can be viewed as a k -partite graph where first, between any two partitions there is either a complete bipartite graph or no edges at all, and second, where every partition is either a clique or an empty graph. We call the partites *types*.

Capacitated Dominating Set. In the CDS problem we are given a graph along with non-negative integers c_v called *domination capacities* of vertices. The goal is to find $D \subseteq V$ as small as possible along with an assignment $\varphi : V \setminus D \rightarrow D$ which respects the capacities, i.e. $|f^{-1}(v)| \leq c_v$ for every $v \in D$.

Key Ideas. There are three key ingredients to the proof showing that the CDS problem is FPT w.r.t. n.d. We will present them here as informal proposition and two lemmas.

Proposition 1. *For every optimal CDS solution there is a solution such that the vertices within every n.d. type are present ordered by their capacities. We will call solutions of this form fundamental solutions.*

This proposition means that a solution can be described simply by saying *how many* vertices to select from every type, instead of *which* vertices they are.

Lemma 2. *Let G be a graph with n.d. k . Then there is an integer convex program using $k + k^2$ variables such that every fundamental CDS solution corresponds to a feasible solution of the program and vice versa.*

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The program we mention is not linear, but has constraints containing some convex functions f_i (one for every type). With a small trick, these can be turned into a $O(n)$ linear constraints that transcribe the same region (see Figure 2).

Lemma 3. *There is an integer linear program describing the same convex space as the integer convex program mentioned above.*

Theorem 4. *The CDS problem is fixed-parameter tractable with respect to $n.d.$*

We also observe a corollary relating to the field of approximation algorithms:

Corollary 5. *Solving the aforementioned linear program without the requirement of integrality, using a very simple rounding procedure we get a CDS solution of size at most $OPT + k.$*

Vector Dominating Set. The VDS problem is very similar to the CDS problem both in description and in how we solve it. Instead of vertex *capacities* here we have vertex *demands* determining how many neighboring vertices have to be in the dominating set for a given vertex to be satisfied. The solution is similar because we can again create an integer convex program. This was not obvious since proving hardness for the two problems w.r.t treewidth required two different approaches. Luckily, when converting the convex program for CDS to the one for VDS, we only have to change a *sum* function to a *max* function, which is again convex.

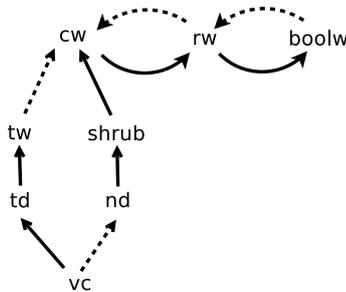


Figure .11: Hierarchy of graph parameters

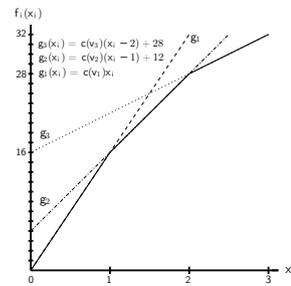


Figure .12: The functions f_i

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Short cycle covers of cubic graphs

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EXTENDED ABSTRACT

A *cycle* is a graph that has all vertices of even degree. The *length of a cycle* is the number of edges contained in the cycle. A *cycle cover* of a graph G is a collection of cycles in G such that each edge from $E(G)$ is contained in at least one of the cycles. The *length of a cycle cover* is a sum of all lengths of cycles in the cover. We are interested in finding cycle covers such that their length is as short as possible.

Conjecture 1 (Shortest Cycle Cover Conjecture). [1] *Every bridgeless graph on m edges has cycle cover of length at most $1.4m$.*

Conjecture 1 relates to several other well known conjectures, such as cycle double cover conjecture [5]. Jamshy, Raspaud, and Tarsi [4] showed that graphs with nowhere-zero 5-flow have a cycle cover of length at most $1.6m$ (their result extends to matroids with nowhere-zero 5-flow). Hence a nowhere-zero 5-flow conjecture implies a cycle cover of length at most $1.6m$. Máčajová, Raspaud, Tarsi, and Zhu [7] showed that the existence of Fano-flow using at most 5 lines of Fano plane on bridgeless cubic graphs implies a short cycle cover of length at most $1.6m$ (this is a consequence of Fulkerson Conjecture).

The best known general result on short cycle covers is due to Alon and Tarsi [1] and Bermond, Jackson, and Jaeger [2]: every bridgeless graph with m edges has a cycle cover of total length at most $5m/3$. There are numerous results on short cycle covers for special classes of graphs. We refer the reader to the monograph of Zhang [8] where whole chapter is devoted to short cycle covers. Significant attention has been devoted to cubic graphs. The best result for cubic graphs up to date is by Kaiser et al. [6]: every bridgeless cubic graph has a cycle cover of length at most $34m/21 \approx 1.619m$.

The main obstacle in approach of Kaiser et al. [6] are the circuits of length 5 contained in the graph. Hou and Zhang [3] proved that if G is a bridgeless cubic graph with no circuits of length 5, then G has a cycle cover of length at most $1.6m$. Moreover, if all circuits of length 5 are disjoint, then the length of the cover is at most $351/225 \cdot m \approx 1.6044m$.

We slightly refine the approach of Kaiser et al. [6] and combine it with a technique for avoiding certain number of 5-circuits in 2-factors of cubic graphs (There is a 2-factor such that at least half of the vertices that are not in certain special subgraphs are not in 5-circuits of the 2-factor). We improve the bound for cubic graphs as follows.

Theorem 2. *Every bridgeless cubic graph on m edges has a cycle cover of length at most $1.6m$.*

We can restate our results in terms of the number of circuits of length 5 that are contained in the graph.

Theorem 3. *Every bridgeless cubic graph on m edges with at most k circuits of length 5 has a cycle cover of length at most $14/9m + 1/9k$.*

Theorem 3 improves the known results for cubic graphs with restrictions on circuits of length 5.

Corollary 4. *Every bridgeless cubic graph on m edges without circuits of length 5 has a cycle cover of length at most $14/9 \cdot m \approx 1.556m$.*

Corollary 5. *Every bridgeless cubic graph on m edges with all circuits of length 5 disjoint has a cycle cover of length at most $212/135 \cdot m \approx 1.570m$.*

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Roman k -tuple dominating function on graphs

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EXTENDED ABSTRACT

Let $G = (V, E)$ be a simple graph. For an integer $k \geq 1$, a function $f : V \rightarrow \{0, 1, 2\}$ is a Roman k -tuple dominating function if for any vertex v with $f(v) = 0$, there exist at least k vertices w in its neighborhood with $f(w) = 2$, and for any vertex v with $f(v) \neq 0$, there exist at least $k - 1$ vertices w in its neighborhood with $f(w) = 2$. The weight of a Roman k -tuple dominating function f of G is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman k -tuple dominating function of G is its Roman k -tuple domination number, denoted by $\gamma_{\times kR}(G)$. In this talk, we state some of our recent results on the Roman k -tuple domination number of a graph.

Theorem 1. *Let G be a graph of order n with minimum degree at least $k - 1 \geq 1$. Then $2k \leq \gamma_{\times kR}(G) \leq 2n$, and $\gamma_{\times kR}(G) = 2k$ if and only if $G = K_k$ or $G = H \circ_k K_k$ for some graph H .*

Note that if $k \geq 2$ and G is $(k - 1)$ -regular, then $\gamma_{\times kR}(G) = 2n$. We will show that its converse holds only for $k = 2$. For $k \geq 3$, for example, let G be a graph which is obtained from the complete bipartite graph $K_{k,k}$ minus a matching of cardinality $k - 1$. Then, obviously, $\gamma_{\times kR}(G) = 4k$ while G is not $(k - 1)$ -regular.

Proposition 2. *Let G be a graph of order n without isolated vertex. Then $\gamma_{\times 2R}(G) = 2n$ if and only if $G = \ell K_2$ for some $\ell \geq 1$.*

Theorem 3. *For any graph G with $\delta(G) \geq k - 1 \geq 1$,*

$$\gamma_{\times k}(G) + k \leq \gamma_{\times kR}(G) \leq 2\gamma_{\times k}(G),$$

and the lower bound is sharp.

Proposition 4. *Let $k \geq 2$, and let G be a graph which is not isomorphic to K_k or $H \circ_k K_k$, for some graph H . Then $\gamma_{\times kR}(G) = 2k + 1$ if and only if $G = \mathcal{A}_k$.*

Proposition 5. *Let $f = (V_0, V_1, V_2)$ be any $\gamma_{\times kR}$ -function on G . Then the following statements hold.*

- (a) $V_1 \cup V_2$ is a k -tuple dominating set of G .
- (b) V_2 is a k -tuple dominating set of $G[V_0 \cup V_2]$.
- (c) For $k \geq 2$, V_2 is a $(k - 1)$ -tuple total dominating set of G .
- (d) Every vertex of degree $k - 1$ belongs to $V_1 \cup V_2$.
- (e) $G[V_1]$ has maximum degree 1.
- (f) Every vertex of V_1 is adjacent to precisely $k - 1$ vertices of V_2 .
- (g) Each vertex of V_0 is adjacent to at most two vertices of V_1 .
- (h) If G is a k -tuple Roman graph and $V_1 = \emptyset$, then $\gamma_{\times k}(G) = |V_2|$.

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Edge-colorings of graphs avoiding rainbow-triangles

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EXTENDED ABSTRACT

The problem studied in this paper has been motivated by a series of developments which followed a question raised by Erdős and Rothschild [3] regarding edge-colorings of graphs avoiding a given monochromatic subgraph. As usual, a graph G is said to be F -free if it does not contain F as a subgraph. Given a graph F , the well-known Turán problem [6] associated with F asks for the maximum number $\text{ex}(n, F)$ of edges over all F -free n -vertex graphs and for the graphs that achieve this maximum, which are called F -extremal. Instead of looking for F -free n -vertex graphs, Erdős and Rothschild asked for n -vertex graphs that admit a large number of *edge-colorings* such that *every color class is F -free*. (We observe that edge-colorings are not necessarily proper.) Formally, an F -free r -coloring of a graph G is an r -edge-coloring of G such that the graphs induced by the edges of each color are F -free. In particular, Erdős and Rothschild conjectured that the number of K_ℓ -free 2-colorings is maximized by the $(\ell - 1)$ -partite Turán graph on n -vertices. Note that F -extremal graphs are natural candidates, as any r -coloring is trivially F -free, which leads to $r^{\text{ex}(n, F)}$ such colorings.

Yuster [7] provided an affirmative answer to this conjecture for K_3 and any $n \geq 6$, while Alon, Balogh, Keevash and Sudakov [1] showed that, for $r \in \{2, 3\}$ and $n \geq n_0$, where n_0 is a constant depending on r and ℓ , the respective Turán graph is also optimal for the number of K_ℓ -free r -colorings. However, this is not the case for any $r \geq 4$, where extremal graphs are not yet known unless $r = 4$ and $F \in \{K_3, K_4\}$ [6].

Balogh [2] added a twist to this problem by considering edge-colorings of a graph avoiding a copy of F with a *prescribed coloring*. Naturally, given a number r of colors and a graph F , an r -*pattern* P of F is a partition of its edge set into r (possibly empty) classes, and an edge-coloring (not necessarily proper) of a graph G is said to be (F, P) -free if G does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P . Regarding this problem, Balogh proved that, for $r = 2$ colors and any 2-color pattern of K_ℓ , the $(\ell - 1)$ -partite Turán graph on $n \geq n_0$ vertices once again yields the largest number of 2-colorings with no forbidden pattern of K_ℓ . However, he also remarked that, if we consider $r = 3$ and a rainbow-colored triangle (i.e., a triangle partitioned into three classes with one edge in each), the complete graph on n vertices allows $3 \cdot 2^{\binom{n}{2}} - 3$ colorings, by just choosing two of the three colors and coloring the edges of K_n arbitrarily with these two colors. This is more than $3^{n^2/4}$, which is an upper bound on the number of 3-colorings of the bipartite Turán graph. It was recently proved that the complete graph is indeed optimal in this case [3].

We also considered forbidden rainbow-triangles, and proved that the Turán graph is again optimal if more colors are used, more precisely:

Theorem 1. *There exists r_0 such that, for any fixed number $r \geq r_0$ and n sufficiently large, the Turán graph for $F = K_3$ on n vertices yields the largest number of r -colorings avoiding rainbow-triangles among all n -vertex graphs.*

The current value of r_0 that we have is $r_0 = 49$, but we are convinced that this number may be improved.

With the regularity lemma we can show the following.

Theorem 2. *Let $r \geq 4$ be fixed. Let G be an n -vertex graph where n is sufficiently large. The number of r -edge colorings of G avoiding rainbow-triangles is at most*

$$r^{\frac{n^2}{4}(1+o(1))}.$$

Thus, the number of rainbow-triangle-free colorings of the balanced complete bipartite graph is at least close to the optimum.

Moreover, for forbidden rainbow complete graphs K_ℓ (i.e., the edge set of K_ℓ is partitioned into $\binom{\ell}{2}$ classes each containing one edge) we have.

Theorem 3. *For fixed $\ell \geq 3$ there is an $r_0(\ell)$ such that for any fixed $r \geq r_0(\ell)$ and n sufficiently large the following holds.*

The number of r -edge colorings of any graph G on n vertices avoiding rainbow- K_ℓ is at most

$$r^{\frac{\ell-2}{2(\ell-1)}n^2(1+o(1))}.$$

The calculations show that $r_0(\ell)$ can be chosen to be approximately $\binom{\ell}{2}^{\ell-1}$. Also note that the Turán graph for K_ℓ on n vertices allows $r^{\frac{\ell-2}{2(\ell-1)}n^2}$ distinct r -edge colorings with no rainbow- K_ℓ .

These results suggest that, for any forbidden rainbow- K_ℓ , where $\ell \geq 3$ and any fixed $r \geq r_0(\ell)$, the Turán graph for $F = K_\ell$ might yield the largest number of r -colorings without creating a rainbow- K_ℓ as long as n is sufficiently large. This may even be the case for any fixed forbidden rainbow- F with chromatic number $\chi(F) \geq 3$.

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Edge-colorings of graphs avoiding fixed graphs with given color pattern

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EXTENDED ABSTRACT

We consider edge-colorings of graphs that satisfy a certain property. Given a number r of colors and a graph F , an r -pattern P of F is a partition of its edge set into r (possibly empty) classes. An edge-coloring (not necessarily proper) of a graph H is said to be (F, P) -free if H does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P . If at most r colors are used, we call it an (F, P) -free r -coloring of H . For example, if the pattern of F consists of a single class, no monochromatic copy of F should arise in H . We ask for the n -vertex host graphs H (among all n -vertex graphs) which allow the largest number of (F, P) -free r -colorings.

Questions of this type have been first considered by Erdős and Rothschild [3], who asked whether considering edge-colorings avoiding a monochromatic copy of F would lead to extremal configurations that are substantially different from those of the Turán problem. Indeed, F -free graphs on n -vertices are natural candidates for admitting a large number of colorings, since any r -coloring of their edge set obviously does not produce a monochromatic copy of F (or a copy of F with any given pattern, for that matter), so that (Turán) F -extremal graphs admit $r^{\text{ex}(n, F)}$ such colorings, where, as usual, $\text{ex}(n, F)$ is the maximum number of edges in an n -vertex F -free graph. We should also mention that Balogh [2] was the first to consider colorings avoiding fixed patterns that are not monochromatic.

Yuster [7] proved that, for $r = 2$ colors and $F = K_3$, the n -vertex Turán graph H for K_3 does indeed admit the largest number of 2-colorings with no monochromatic K_3 for all $n \geq 6$. Later, Alon, Balogh, Keevash, and Sudakov [1] extended this result for $r \in \{2, 3\}$ and any fixed monochromatic complete graph K_ℓ by showing that the $(\ell - 1)$ -partite Turán graph H on $n \geq n_0$ vertices yields the largest number of r -colorings, where n_0 is a constant for ℓ and r . However, for $r \geq 4$ colors, the Turán graph for K_ℓ is no longer optimal, and the situation becomes much more complicated; in fact, extremal configurations are not known unless $r = 4$ and $F \in \{K_3, K_4\}$ (see [6]). A similar phenomenon, in which (Turán) extremal graphs admit the largest number of r -colorings if $r \in \{2, 3\}$, but do not for $r \geq 4$, has been observed for a few other classes of graphs and hypergraphs, such as the 3-uniform Fano plane [5].

As it turns out, monochromatic stars $F = S_t$ with $t \geq 3$ edges were the first instances for which it was shown [4] that (Turán) F -extremal graphs (in this case, $(t - 1)$ -regular graphs for n even) do not admit the largest number of r -colorings with no monochromatic copy of F for any fixed $r \geq 2$. For example, for $r = 2$, vertex-disjoint copies of complete bipartite graphs with both vertex classes of size a little less than $2t$ yield many more colorings than $(t - 1)$ -regular graphs. However, extremal n -vertex graphs are not yet known for all $r \geq 2$ and $t \geq 3$.

Here, we consider forbidden rainbow-colored stars S_t , that is, stars such that every edge is in a different class of the pattern (in particular, $r \geq t$). For $t = 2$ and any given number of colors $r \geq 2$, a matching of size $n/2$ (n even) yields the largest number of r -colorings with no rainbow S_2 , as this restriction implies that any such coloring has monochromatic components. The same extremal configuration had been observed for monochromatic S_2 when $r = 2$, but not for larger values of r . (Note that the set of r -colorings avoiding a monochromatic S_2 is precisely the set of proper r -edge-colorings of a graph, and hence this problem consists of finding n -vertex graphs with the largest number of proper colorings.)

As in the monochromatic case, (Turán) extremal graphs are not optimal for any $t \geq 3$ and $r \geq 2$. However, the situation in the case of rainbow-colored stars is rather different, since we prove that the complete graph K_n , admits the largest number of colorings in all such cases. The following is our main result.

Theorem 1. *Let $r \geq 2$ and $t \geq 3$. Then, for n sufficiently large, among all n -vertex graphs, the complete graph K_n yields the largest number of r -colorings with no rainbow S_t .*

A similar behavior may be observed in other cases. Indeed, given $r \geq 3$, the complete graph K_n is extremal for $n \geq n_0$ if we forbid stars S_{2t} , $t \geq 3$, where the $2t$ edges are grouped in t pairs, and every pair of edges in a group is colored the same, but edges in different groups are colored differently. The same holds for forbidden rainbow-matchings I_ℓ , $\ell \geq 3$. However, for $\ell = 2$, the (Turán) extremal graph (in the case of I_2 , the n -vertex star) is extremal for any number $r \geq 2$ of colors, as was the case for S_2 .

In general, applying the (colored version) of the Regularity Lemma shows that, for any $r \geq 3$ and any given rainbow- F such that $\chi(F) = 2$, the complete graph K_n is close to being extremal. However, the picture changes in the case $\chi(F) \geq 3$, where, for $r \geq r_0$, the Turán graph for F on $n \geq n_0$ vertices yields more rainbow- F free r -colorings than the complete graph K_n . Two interesting open questions would be to characterize the bipartite instances of F such that the complete graph is extremal, and the instances of non-bipartite graphs for which Turán graphs are extremal for large (constant) r . It would be also nice to find other extremal configurations for particular choices of F and r .

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Packing coloring of coronas of paths, coronas of cycles and caterpillars

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EXTENDED ABSTRACT

Abstract. *The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets V_1, \dots, V_k , in such a way that every two vertices in V_i are at distance greater than i in G for every $i, 1 \leq i \leq k$. We determine the packing chromatic number of coronas of paths and cycles and give sufficient and/or necessary conditions for a caterpillar to have packing chromatic number at most 4, 5 and 6.*

Keywords: *packing chromatic number, cross product, corona, caterpillar.*

1 Introduction

All the graphs we considered are undirected, finite and connected. We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its set of edges. The *distance* $d_G(u, v)$, or simply $d(u, v)$, between vertices u and v in G is the length of a shortest path joining u and v . The *diameter* $\text{diam}(G)$ of G is the maximum distance between two vertices of G . We denote by P_n the path of order n and by C_n , $n \geq 3$, the cycle of order n .

A *packing k -coloring* of G is a mapping $\pi : V(G) \rightarrow \{1, \dots, k\}$ such that $\pi(u) \neq \pi(v)$ whenever $d(u, v) \leq \pi(u)$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k -coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets V_1, \dots, V_k , in such a way that every two vertices in V_i are at distance greater than i in G for every $i, 1 \leq i \leq k$.

Packing coloring has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [2, 3] under the name *broadcast coloring* and has been studied by several authors in recent years.

Fiala and Golovach [1] proved that determining the packing chromatic number is an NP-hard problem for trees. Determining the packing chromatic number of special subclasses of trees is thus an interesting problem. The exact value of the packing chromatic number of trees with diameter at most 4 was given in [3]. In the same paper, it was proved that $\chi_\rho(T_n) \leq (n+7)/4$ for every tree T_n of order $n \neq 4, 8$, and this bound is tight, while $\chi_\rho(T_n) \leq 3$ if $n = 4$ and $\chi_\rho(T_n) \leq 4$ if $n = 8$, these two bounds being also tight.

The *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbor to every vertex of G . We call such a degree-one neighbor a *pendant vertex* or a *pendant neighbor*.

A *caterpillar of length ℓ* is a tree whose set of internal vertices (vertices with degree at least 2) induces a path of length ℓ , called the *central path*.

2 Coronas of paths and cycles

The packing chromatic numbers of paths and cycles have been determined by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [3]: (i) $\chi_\rho(P_n) = 2$ if $n \in \{2, 3\}$, (ii) $\chi_\rho(P_n) = 3$ if $n \geq 3$, (iii) $\chi_\rho(C_n) = 3$ if $n = 3$ or $n \equiv 0 \pmod{4}$, and (iv) $\chi_\rho(C_n) = 4$ if $n \geq 5$ and $n \equiv 1, 2, 3 \pmod{4}$.

Our first result determines the packing chromatic number of coronas of paths. Note that any corona $P_n \odot K_1$ is also a caterpillar of length n .

Theorem 1. *The packing chromatic number of the corona graph $P_n \odot K_1$ is given by:*

$$\chi_\rho(P_n \odot K_1) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \in \{2, 3\}, \\ 4 & \text{if } 4 \leq n \leq 9, \\ 5 & \text{if } n \geq 10. \end{cases}$$

Recently, William, Roy and Rajasingh proved that $\chi_\rho(C_n \odot K_1) \leq 5$ for every even $n \geq 6$ [5]. We complete their result as follows:

Theorem 2. *The packing chromatic number of the corona graph $C_n \odot K_1$ is given by:*

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

3 Caterpillars

With each caterpillar CT_ℓ with central path $x_1 \dots x_\ell$ we associate the word $w(CT_\ell) = a_1 \dots a_\ell$ on the alphabet \mathbb{N} , where a_i is the number of pendant neighbors of x_i for every i , $1 \leq i \leq \ell$. In order to describe families of caterpillars, we will use the symbol x to mean any integer, as well as the standard operators from Formal Language Theory. For instance, each path P_n , $n \geq 4$, is described by the expression $10^{n-2}1$, the family $\{P_n \odot K_1, n \geq 1\}$ is described by the expression 1^+ and the family of all caterpillars having a central path with end-vertices of degree 2 by the expression $1x^*1$.

The packing chromatic number of caterpillars was first studied by Sloper [4] who proved the following: If CT_ℓ is a caterpillar of length ℓ then (i) $\chi_\rho(CT_\ell) \leq 6$ if $\ell \leq 34$, and (ii) $\chi_\rho(CT_\ell) \leq 7$ if $\ell \geq 35$.

Our goal is to refine this result by characterizing caterpillars with small packing chromatic number. Clearly, caterpillars with packing chromatic number 2 are all stars $K_{1,n}$ with $n \geq 1$. Goddard, Hedetniemi, Hedetniemi, Harris and Rall [3] characterized caterpillars with packing chromatic number at most 3: a caterpillar CT has packing chromatic number at most 3 if and only if $w(CT) = x(0x)^*$.

In this work, we give preliminary results in order to characterize caterpillars with packing chromatic number at most 4, 5 and 6.

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Harmonious Coloring of Hypergraphs

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EXTENDED ABSTRACT

Let $H = (V, \mathcal{E})$ be a hypergraph, and let $c : V \rightarrow \{1, 2, \dots, r\}$ be a *strong coloring* of the vertices of H , which means that no two vertices of the same edge have the same color.

For every edge $X \in \mathcal{E}$, let $p(X)$ denote a *color pattern* of X , that is, the set of colors appearing on X . A coloring c is called *harmonious* if $p(X) \neq p(Y)$ for every pair of distinct edges $X, Y \in \mathcal{E}$.

We denote by $h(H)$ the least number of colors needed for a harmonious coloring of H , and call it occasionally the *harmonious number* of H . This notion arose as a natural generalization of harmonious coloring of graphs introduced in [6].

The case of multisets is a natural generalization of *legitimate* coloring of projective planes studied by Alon and Füredi [1].

Actually there are other graph coloring problems related to this topic. Indeed, if H is a hypergraph dual to a simple graph G , then the notion of harmonious coloring of H corresponds to the vertex-distinguishing edge coloring of G , introduced by Burriss and Schelp in [4]. Thus, $h(H) = \chi'_s(G)$, where $\chi'_s(G)$ is the related chromatic parameter.

Let us concentrate on connected graphs with at least two edges (we will call them nice). Burriss and Schelp made two conjectures concerning $\chi'_s(G)$:

Conjecture 1. *Every nice graph satisfies $\chi'_s(G) \leq |V(G)| + 1$.*

This conjecture has been proved by Bazgan et al. in [5]. For the second conjecture we need some notations. Let n_d be the number of vertices of degree d in G . Let $r = r(G)$ be the least integer satisfying $\binom{r}{d} \geq n_d$ for every $d = 1, 2, \dots, \Delta(G)$.

Conjecture 2. *Every nice graph satisfies $\chi'_s(G) \leq r(G) + 1$.*

That one is widely open. Curiously, it is not trivial even in the case $\Delta = 2$ (cf. [2]). It is not known if the weaker version (with any additive constant) is true even for cubic regular graphs.

The following theorem is proved in [3].

Theorem 3. *Every graph G on n vertices with $\Delta(G) \geq \sqrt{2n} + 4$ and $\delta(G) \geq 5$ satisfies Conjecture 2.*

The simplest unknown case not covered by the above result is that of cubic graphs. In the following result we apply the entropy compression argument to get improved estimates for regular graphs with a perfect matching.

Theorem 4. *Let G be a d -regular graph on n vertices. Assume that G has a perfect matching. Then there is a vertex-distinguishing edge coloring of G using at most $K \sqrt[d]{n}$ colors, where K is a constant depending only on d given by the following formula:*

$$K = (2^{2d-1}(d-1)!(d-1))^{\frac{1}{d}} + \left(\frac{2^{2d-1}(d-1)!}{(d-1)^{d-1}} \right)^{\frac{1}{d}}.$$

For cubic graphs the above theorem gives $6\sqrt[3]{2n}$ which is currently best bound.

An analog of Conjecture 2 for hypergraphs is not possible, that is, there is no hope for a general bound of the form $h(H) \leq r(H) + c$, where $r(H)$ is defined analogously to $r(G)$, as the least integer r satisfying $\binom{r}{d} \geq m_d$, where m_d is the number of edges of size d in H . Indeed, a simple star S with m edges (considered as a 2-uniform hypergraph) satisfies $h(S) = m + 1$ while $r(H)$ is of order $m^{1/2}$. But perhaps an analog of Conjecture 1 for hypergraphs could be true.

Conjecture 5. *Every k -uniform hypergraph H with m edges satisfies $h(H) \leq m + k - 1$.*

The following result is obtained using entropy compression argument.

Theorem 6. *Let k and Δ be fixed positive integers. Then there exists $m_0 = m_0(k, \Delta)$ such that every k -uniform hypergraph H of maximum degree Δ with m edges satisfies $h(H) \leq \frac{k}{k-1} \sqrt[k]{k \cdot k! \Delta m}$, provided $m \geq m_0$.*

It confirms Conjecture 5 for many cases of uniform hypergraphs.

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On the Partition Dimension of Circulant Graphs

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EXTENDED ABSTRACT

For a vertex v of a connected graph $G(V, E)$ and a subset S of V , the distance between v and S is defined by $d(v, S) = \min\{d(v, x) : x \in S\}$. For an ordered k -partition $\Pi = \{S_1, S_2 \dots S_k\}$ of V , the representation of v with respect to Π is the k -vector $r(v|\Pi) = (d(v, S_1), d(v, S_2) \dots d(v, S_k))$. The k -partition Π is a resolving partition if the k -vectors $r(v|\Pi)$, $v \in V$ are distinct. The minimum k for which there is a resolving k -partition of V is the *partition dimension* of G . In this paper we obtain the partition dimension of circulant graphs $G = C(n, \pm\{1, 2 \dots j\})$, $1 \leq j < \lfloor \frac{n}{2} \rfloor$, $n \geq (j+k)(j+1)$, $n \equiv k \pmod{2j}$ and k and $2j$ are co-primes as,

$$\begin{aligned} pd(G) &= j+1 && \text{when } j \text{ is even and all } k = 2m-1, 1 \leq m \leq j \\ pd(G) &= j+1 && \text{when } j \text{ is odd and all } k = 2m, 1 \leq m \leq j. \end{aligned}$$

Salman et al.[2] claimed that partition dimension of a class of circulant graph $C(n, \pm\{1, 2\})$, for all even $n \geq 6$ is four and it is 3 when n is odd. While this is correct in most cases, in the case of $n \equiv 2 \pmod{4}$ we proved that the partition dimension is three [1].

In this paper we prove a lower bound for $C(n; \pm\{1, 2 \dots j\})$ and show that this bound is sharp.

Theorem 1. Let $G = C(n; \pm\{1, 2 \dots j\})$, $1 < j < \lfloor n/2 \rfloor$ where j is odd, be a circulant graph of order n . If G is such that, $n \geq (j+k)(j+1)$, $n \equiv k \pmod{2j}$ and k and $2j$ are co-primes. Then for all $k = 2m-1$, $1 \leq m \leq j-1$; $pd(G) = j+1$.

A rigorous attempt is made to solve the partition dimension problem of $C(n; \pm\{1, 2 \dots j\})$, $1 < j < \lfloor n/2 \rfloor$.

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Algebraic connectivity of k -connected graphs

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EXTENDED ABSTRACT

Introduction. Let G be a k -connected graph with $k \geq 2$. A hinge is a subset of k vertices whose deletion from G yields a disconnected graph. We consider the algebraic connectivity and Fiedler vectors of such graphs, paying special attention to the signs of the entries in Fiedler vectors corresponding to vertices in a hinge, and to vertices in the connected components at a hinge. The results extend those in [1], [2] and [3].

Labeling the vertices of G by v_1, v_2, \dots, v_n and denoting a Fiedler vector by $y = [y_i]$, the coordinates of y can be assigned naturally to the vertices of G : the coordinate y_i labels the vertex v_i . This assignment has been called a *characteristic valuation* and Fiedler noticed that it induces partitions of the vertices of G that are naturally connected clusters, important for applications and for characterizing the graph structure. As an example, Pothen, Simon and Liu [4] suggested a spectral graph partitioning algorithm based on the entries of a Fiedler vector.

Structural Results. Our main result is an extension of that in [2] to graphs without cut vertices, describing the structure of the partition arising from a characteristic valuation. In particular, we show how a set of $k > 1$ vertices that disconnect G may induce connected subgraphs having vertex valuation of the same sign, introducing a natural partition of vertices in G . Before stating our main Theorem, we need some definitions.

Let \widehat{H} be a hinge in G and consider a connected component C of the graph $G \setminus \widehat{H}$. We say that the hinge \widehat{H} or the component C is *null*, *positive*, *nonnegative*, or *mixed* if the valuation of each of its vertices is zero, positive, nonnegative, or mixed, respectively.

Theorem 1. *Let G be connected graph and $y = [y_i]$ a characteristic valuation of G . Let \widehat{H} be a hinge of G and let C_0, C_1, \dots, C_r be the connected components of $G \setminus \widehat{H}$. Label the vertices of \widehat{H} as l_1, \dots, l_k , where $y_{l_1} \leq y_{l_2} \leq \dots \leq y_{l_k}$. If \widehat{H} is null and there exists a mixed component C_i , then it is the only mixed component and all the other components are null. If \widehat{H} is null and there is no mixed component, then each component is either null, positive or negative. If \widehat{H} is non-negative, then only one component has vertices with negative valuation. All the remaining vertices v_s satisfies $y_{l_1} < y_s$.*

Characterizing the Algebraic Connectivity. Also, in this study we use our structure theorem to compute the algebraic connectivity for graphs with special hinges. The results are natural extensions of those in [3], where graphs with cut vertices were considered.

Let G be a graph and L its Laplacian matrix. For a hinge \widehat{H} of G , denote the connected components of $G \setminus \widehat{H}$ by C_0, C_1, \dots, C_r . For each component, let $L(C_i)$ be the principal submatrix of L , corresponding to the vertices of C_i . The *Perron value* of C_i is the Perron value of the positive matrix $L^{-1}(C_i)$ and we say C_j is a *Perron component at \widehat{H}* if its Perron values is the largest among all components C_0, C_1, \dots, C_r .

Some of our characterizations for the algebraic connectivity are stated below.

Theorem 2. *Let G be a connected graph and $y = [y_i]$ a characteristic valuation of G . If there exists a null hinge \widehat{H} , such that $G \setminus \widehat{H}$ has no mixed component, then there are two or more Perron components at \widehat{H} . In this case, $a(G) = \frac{1}{\rho(L(C)^{-1})}$ for each Perron component C at \widehat{H} .*

Theorem 3. *Let G be a graph and $y = [y_i]$ a characteristic valuation of G . Let \widehat{H} be a nonnegative (nonpositive) or null hinge of G . For each positive (or negative) component C at \widehat{H} , there is a*

nonnegative matrix M of rank at most 1, and a scalar $\gamma > 0$, such that $\rho(L(C)^{-1} + \gamma M) = \frac{1}{a(G)}$. Furthermore, $M = 0$ if and only if \hat{H} is null.

One question we want to address is whether or not the set of Fiedler vectors identifies the same null hinge. More precisely, the following result constructs the set of all Fiedler vectors that identifies \hat{H} as a null hinge.

Theorem 4. Let G be a graph and $y = [y_i]$ a characteristic valuation of G . Suppose there exists a null hinge \hat{H} , such that $G \setminus \hat{H}$ has no mixed component, and for $t \geq 2$, let C_1, C_2, \dots, C_t be the set of Perron components of $G \setminus \hat{H}$. Assume the Laplacian matrix is in the form (??). Let $y^{(1)}, y^{(2)}, \dots, y^{(t)}$ be the set of Perron vectors for the set of matrices $L(C_1)^{-1}, L(C_2)^{-1}, \dots, L(C_t)^{-1}$ such that $\mathbf{1}^T y^{(i)} = 1$. Define, for $i = 2, \dots, t$, the vector

$$f_i = \begin{cases} y^{(1)}(v) & v \in C_1, \\ -y^{(i)}(v) & v \in C_i, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then f_2, f_3, \dots, f_t is a set of linearly independent eigenvectors associated with $a(G)$ and each Fiedler vector that identifies \hat{H} as a null hinge is a linear combination of f_i , therefore it has no mixed component.

Bounding the algebraic connectivity. We introduce some concepts which will help us to better understand how the algebraic connectivity is bounded. More specifically, want to relate the algebraic connectivity as a function of the number of edges between a hinge and its components, as an attempt to generalize the well known fact (see [1]) that, for a k -connected graph, $a(G) \leq k$.

Let \hat{H} be a hinge of the graph G and let C be a component at \hat{H} . Let v_1, v_2, \dots, v_t be the vertices in the component C . We shall denote by $d_{\hat{H}}(v_i)$ the number of edges connecting v_i to the vertices of \hat{H} . Similarly, for each vertex u of \hat{H} , we let $d_C(u)$ denote the number of vertices in C that are adjacent to u .

Further, we define the quantity $\mathcal{S}_C = \max_{v_i \in C} \{d_{\hat{H}}(v_i)\}$ which shall be named *strength* of the component C . Denoting the set of components at \hat{H} by H , we define the quantity $\mathcal{S}_{\hat{H}} = \max_{C \in H} \{\mathcal{S}_C\}$ which shall be named *strength* of the hinge \hat{H} .

Theorem 5. Let G be a graph and let \hat{H} be a hinge of G . For each $j = 0, \dots, r$, let the component C_j have p_j vertices. Then we have the following conclusions.

- $a(G) \leq \mathcal{S}_{\hat{H}}$.
- If $a(G) = \mathcal{S}_{\hat{H}}$, then each vertex of $G \setminus \hat{H}$ is adjacent to $\mathcal{S}_{\hat{H}}$ vertices in the hinge \hat{H} and for each $i, j = 0, \dots, r$ and each $u \in \hat{H}$ we have $p_i d_{C_j}(u) = p_j d_{C_i}(u)$.
- If each vertex of $G \setminus \hat{H}$ is adjacent to $\mathcal{S}_{\hat{H}}$ vertices in the hinge \hat{H} and for each $i, j = 0, \dots, r$ and each $u \in \hat{H}$ we have $p_i d_{C_j}(u) = p_j d_{C_i}(u)$, then $\mathcal{S}_{\hat{H}}$ is a Laplacian eigenvalue of G . In this case, the multiplicity of $\mathcal{S}_{\hat{H}}$ as an eigenvalue is at least r .

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Automorphisms of Neighborhood Sequence of a Graph

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EXTENDED ABSTRACT

The identifying codes were first introduced by Karpovsky, Chakrabarty, and Levitin in [4]. Furthermore, they have formed a fundamental basis for a wide variety of theoretical work and practical applications. If we settle that every vertex v of a graph G only exhibits the messages from some neighbors of v in G , then we can get a code with size $\leq M(G)$ where $M(G)$ is the minimum cardinality of an identifying code in a graph G . We call such code an *identifying set* of a graph G . If two graphs have the same neighborhood sequence, then they have the same minimum cardinality of an identifying code [4] and the choice identification number [1].

Here we introduce some definitions used in the paper. Let G be a graph, u be a vertex of G , and $B(u)$ (or $B_G(u)$) be the set of u with all its neighbors in G . And $N(u)$ (or $N_G(u)$) is the set of all neighbors of u in G . Then $B(u) = N(u) \cup \{u\}$. A sequence (B_1, B_2, \dots, B_n) of subsets of an n -set S is a *neighborhood sequence* if there exist a graph G with a vertex set S and a permutation (v_1, v_2, \dots, v_n) of S such that $B(v_i) = B_i$ for $i = 1, 2, \dots, n$.

An automorphism f of a graph G is a permutation of vertex set $V(G)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(G)$. The collection $Aut(G)$ of all automorphisms of G is a group by a composition operator. Define $Aut(B_1, B_2, \dots, B_n)$ as the set $\{f : f \text{ is a permutation of } V(G) \text{ and } (f(B_1), f(B_2), \dots, f(B_n)) \text{ is a permutation of } B_1, B_2, \dots, B_n\}$ where $f(S) = \{f(x) : x \in S\}$ for $S \subseteq V(G)$. Such permutation is called a (B_1, B_2, \dots, B_n) -*automorphism*. It is immediate that every automorphism f of G is also an element of $Aut(B_1, B_2, \dots, B_n)$ where (B_1, B_2, \dots, B_n) is the neighborhood sequence of a graph G .

In this paper, we first prove that, for every finite group Γ , there exists a neighborhood sequence (B_1, B_2, \dots, B_n) such that Γ is isomorphic to $Aut(B_1, B_2, \dots, B_n)$. We also get that, for each finite group Γ , there exists a neighborhood sequence (B_1, B_2, \dots, B_n) such that, for each subgroup H of Γ , H is group isomorphic to $Aut(E_1, E_2, \dots, E_t)$ for some neighborhood sequence (E_1, E_2, \dots, E_t) where $E_i \subseteq B_{j_i}$ and $j_1 < j_2 < \dots < j_t$. In the last section, we give some classes of graphs G with neighborhood sequence (B_1, B_2, \dots, B_n) satisfying $Aut(G)$ and $Aut(B_1, B_2, \dots, B_n)$ are different, and construct non-isomorphic graphs with the same neighborhood sequence having different automorphism groups. The main results are in the following.

Let $[n]$ be the set $\{1, 2, \dots, n\}$ and B_1, B_2, \dots, B_n be subsets of an n -set S . Then we say (B_1, B_2, \dots, B_n) has an *adjacent SDR* if there exist $v_i \in B_i$ for $i = 1, 2, \dots, n$ such that v_1, v_2, \dots, v_n are distinct and $v_j \in B_i$ if and only if $v_i \in B_j$.

Theorem 1. *Let B_1, B_2, \dots, B_n be subsets of an n -set S . Then (B_1, B_2, \dots, B_n) is a neighborhood sequence if and only if (B_1, B_2, \dots, B_n) has an adjacent SDR.*

Theorem 2. *For each finite group Γ , there exists a 2-connected graph with its neighborhood sequence (B_1, B_2, \dots, B_n) such that Γ is group isomorphic $Aut(B_1, B_2, \dots, B_n)$.*

Theorem 3. *Let Γ be a finite group. Then there exists a neighborhood sequence (B_1, B_2, \dots, B_n) such that, for each subgroup H of Γ , H is group isomorphic to $Aut(E_1, E_2, \dots, E_t)$ for some neighborhood sequence (E_1, E_2, \dots, E_t) where $E_i \subseteq B_{j_i}$ and $j_1 < j_2 < \dots < j_t$.*

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Automorphism Groups of Geometrically Represented Graphs

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EXTENDED ABSTRACT

The study of symmetries of geometrical objects is an ancient topic in mathematics and its precise formulation led to group theory. The symmetries of a graph X are described by its automorphism group $\text{Aut}(X)$. Every automorphism of X is a permutation of its vertices which preserves adjacencies and non-adjacencies. The famous result of Frucht [5] says that every finite group is isomorphic to the automorphism group of some graph X .

Most graphs are asymmetric, i.e., have only the identity automorphism. However, many results in combinatorics and graph theory rely on highly symmetrical graphs. Automorphism groups are important for studying large objects, since these symmetries allow one to simplify and understand the objects. This algebraic approach is together with recursion and counting arguments the only technique known for working with big objects.

For a class \mathcal{C} of graphs, let $\text{Aut}(\mathcal{C})$ denote its automorphism groups, i.e.,

$$\text{Aut}(\mathcal{C}) = \{\text{Aut}(X) : X \in \mathcal{C}\}.$$

The oldest non-trivial result concerning automorphism groups of restricted graph classes is for trees (**TREE**) by Jordan [7] from 1869. He proved that $\text{Aut}(\text{TREE})$ contains precisely the groups obtained from the trivial group by a sequence of two operations: the direct product and the wreath product with a symmetric group. The direct product constructs the automorphisms that act independently on non-isomorphic subtrees and the wreath product constructs the automorphisms that permute isomorphic subtrees.

Geometric Representations. In our paper [8] we study automorphism groups of geometrically represented graphs. The main question is how the geometry influences their automorphism groups. For instance, the geometry of the sphere translates to 3-connected planar graphs which have unique embeddings [9]. Thus, their automorphism groups are so called spherical groups which are automorphism groups of tilings of the sphere. For general planar graphs, the automorphism groups are more complex and they were described by Babai [1] using semidirect products of spherical and symmetric groups.

We focus on intersection representations. An *intersection representation* \mathcal{R} of a graph X is a collection $\{R_v : v \in V(X)\}$ such that $uv \in E(X)$ if and only if $R_u \cap R_v \neq \emptyset$; the intersections encode the edges. To obtain nice graph classes, one typically restricts the sets R_v to a specific type of geometrical objects. We show that a well-understood structure of all intersection representations allows one to determine the automorphism group. In particular, we study interval graphs and circle graphs, and our technique can be applied to other similar graph classes.

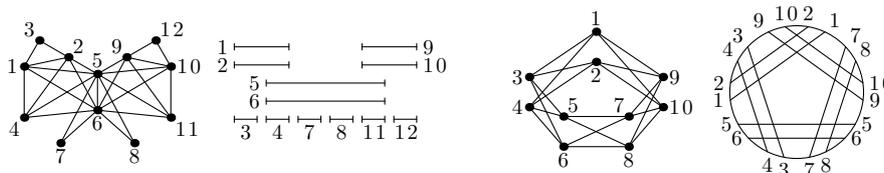


Figure .13: On the left, an interval graph and one of its interval representations. On the right, a circle graph and one of its circle representations.

When every set R_v is a closed interval of the real line, then we get an *interval representation*. In a *circle representation*, the sets R_v are chords of a circle. A graph is an *interval* (resp. *circle*) *graph* if it has an interval (resp. circle) representation; see Fig. .13 for examples. We denote these classes by INT and CIRCLE, respectively.

Our Results. A *pseudotree* is a connected graph with at most one cycle. Their automorphism groups can be constructed from the automorphism groups of trees by semidirect products with cyclic and dihedral groups, which constructs the automorphisms rotating/reflecting the unique cycles. *Pseudoforests* (PSEUDOFORREST) are graphs for which every connected component is a pseudotree.

We prove the following main result [8]:

Theorem 1. *The classes INT and CIRCLE have the automorphism groups*

- (i) $\text{Aut}(\text{INT}) = \text{Aut}(\text{TREE})$,
- (ii) $\text{Aut}(\text{CIRCLE}) = \text{Aut}(\text{PSEUDOFORREST})$.

Concerning (i), this equality is not well known. It was stated by Hanlon [6] without a proof in the conclusion of his paper from 1982 on enumeration of interval graphs. Our structural analysis is based on PQ-trees [2] which combinatorially describe all interval representations of an interval graph. It explains this equality and further solves an open problem of Hanlon: for a given interval graph, to construct a tree with the same automorphism group. Without PQ-trees, this equality is very surprising since these classes are very different.

Using PQ-trees, Colbourn and Booth [3] give a linear-time algorithm to compute permutation generators of the automorphism group of an interval graph. In comparison, our description allows to construct an algorithm which outputs the automorphism group in form of group products what reveals the structure of the group.

Concerning (ii), we are not aware of any results on automorphism groups of circle graphs. One inclusion is trivial since $\text{PSEUDOFORREST} \subsetneq \text{CIRCLE}$. The other one is based on split-trees which describe all representations of circle graphs. The semidirect product with a cyclic or dihedral group corresponds to the rotations/reflections of the central vertex of a split-tree. Geometrically, it corresponds to the rotations/reflections of the entire circle representation.

Our results are constructive and lead to polynomial-time algorithms computing automorphism groups of interval and circle graphs. They output these groups in terms of group products, thus revealing their structure. With a careful implementation, the best running times for these classes can be likely matched.

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Radial bipartite Moore graphs

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EXTENDED ABSTRACT

The ‘bipartite Moore bound’, that is, the maximum number $B_{\Delta,D}$ of vertices in a bipartite graph of maximum degree Δ and diameter D , was given by Biggs [5] as

$$B_{2,D} = 2D \quad \text{and} \quad B_{\Delta,D} = \frac{2(\Delta - 1)^D - 1}{\Delta - 2} \quad \text{if } \Delta > 2.$$

It is interesting to note that the bipartite Moore bound represents not only an upper bound on the number of vertices of a bipartite graph of maximum degree at most Δ and diameter at most D but it is also a lower bound on the number of vertices of a regular graph G of degree Δ and girth $g = 2D$ [5]. A (Δ, g) -cage of order $B_{\Delta,D}$ is therefore a bipartite Moore graph if $g = 2D$.

Bipartite graphs satisfying the equality are called *bipartite Moore graphs*. For degrees 1 or 2, bipartite Moore graphs are K_2 and the $2D$ -cycles, respectively. When $\Delta \geq 3$ the possibility of the existence of bipartite Moore graphs was settled by Feit and Higman [?] in 1964 and independently by Singleton [?] in 1966. They proved that bipartite Moore graphs exist only if the diameter is 2, 3, 4 or 6.

For $D = 2$ and each $\Delta \geq 3$ bipartite Moore graphs of degree Δ are the complete bipartite graphs of degree Δ . For $D = 3, 4, 6$ bipartite Moore graphs of degree Δ have been constructed only when $\Delta - 1$ is a prime power [?]. Furthermore, Singleton [?] proved that the existence of a bipartite Moore graph of diameter 3 is equivalent to the existence of a projective plane of order $\Delta - 1$.

On the other hand, for $D = 3$, there are values of Δ with no corresponding bipartite Moore graphs. The question of whether or not bipartite Moore graphs of diameter 3, 4 or 6 exist for other values of Δ is open, and represents one of the most famous and difficult problems in combinatorics.

In view of the scarcity of bipartite Moore graphs, we next turn our attention to bipartite graphs which are in some sense ‘close’ to being bipartite Moore graphs. The first such approximations have been done by considering the degree/diameter problem for bipartite graphs, which means relaxing the number of vertices by allowing a (hopefully small) defect, while keeping the degree and diameter as given.

In this paper we consider a new approach which is motivated by the idea of radial Moore graphs and digraphs, due to Knor [6]. We define a radial Moore bipartite graph to be a graph of maximum degree Δ , radius k , diameter at most $D = k + 1$ and the number of vertices equal to $B_{\Delta,D}$. As our main result we prove

Theorem 1. For every radius $k > 1$ there exists a positive integer Δ_m such that for all $\Delta \geq \Delta_m$ there exists a radial Moore bipartite graph of radius k and degree Δ .

To prove our main theorem we give a (fairly technical) construction and we will make use of the result by Baker *et al.* [4] concerning the gaps between primes, namely, that for any sufficiently large x there is a prime p such that $x - x^{0.525} \leq p \leq x$.

The paper concludes with several new open problems.

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Searching for defective elements using queries of fixed size

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EXTENDED ABSTRACT

Consider a set X of n elements. We wish to identify a particular subset Y containing at most d unknown (defective) elements. To this end, we perform a series of experiments with the following property: when testing a subset $A \subseteq X$, we receive a *positive* result if and only if A contains at least one of these d unknown elements. In practice, we often have the additional constraint that $|A| \leq k$, and we desire to minimize the total number of queries while yet determining Y exactly, i.e. we consider a test function $\tau : X^{(k)} \rightarrow [0, 1]$, where $X^{(k)}$ denotes the k -sized subsets of X , such that

$$\tau(A) = \begin{cases} 0 & Y \cap A = \emptyset \\ 1 & Y \cap A \neq \emptyset. \end{cases}$$

We call a sequence of (k -)sets $\mathcal{A} = \{A_1, A_2, \dots, A_q\}$ a *search algorithm (of query size k)*, if there exists an evaluation function $\phi : [0, 1]^q \rightarrow [n]^{(d)}$ such that for all test functions τ

$$\phi(\tau(A_1), \dots, \tau(A_q)) = Y.$$

A search algorithm is *adaptive* if $A_i \in X^{(k)}$ may depend on $\tau(A_j)$ for $j < i$. The minimum number of tests required to exactly determine Y is denoted by $t(n, d, k)$. If all queries are specified in advance, we call the algorithm *non-adaptive*. Its minimum is denoted by $q(n, d, k)$. In this case, a successful family \mathcal{A} of such queries is often also referred to as a (d -)separating family.

This question was first posed by A. Renyi in 1961 [6]. In 1966, G. O. H. Katona solved the adaptive case for $d = 1$, where there is a single defective element [5].

Theorem 1 (Katona '66 [5]). *Let n, k be integers, such that $n < k$, then*

$$t(n, 1, k) = \left\lceil \frac{n}{k} \right\rceil - 2 + \left\lceil \log \left(n - k \left\lceil \frac{n}{k} \right\rceil + 2k \right) \right\rceil.$$

Additionally, he also provided upper and lower estimates for the non-adaptive case. In 2013, . Hosszu, J. Tapolcai and G. Wiener simplified the proof and strengthened the theorem [4].

Theorem 2 (Hosszu, Tapolcai, Wiener '13 [4]). *For $k < n/2$, $q(n, 1, k)$ is the least number q for which there exist positive integer $j \leq q - 1$ and $a < \binom{q}{j+1}$ such that*

$$\sum_{i=0}^j i \cdot \binom{q}{i} + a(j+1) \leq kq,$$

$$\sum_{i=0}^j \binom{q}{i} + a = n.$$

As a corollary, they obtain bounds for the minimal size of a separating family for small values of k .

Corollary 3 (Katona '66 [5], Hosszu, Tapolcai, Wiener '13 [4]). *If $n \geq \binom{k}{2} + 1$, then*

$$q(n, 1, k) = \left\lceil 2 \frac{n-1}{k-1} \right\rceil.$$

Using some of their ideas, we obtain similar results for general d . First, we generalise the rather easy adaptive case where we obtain a result which is in fully consistent with the case of where Y is a singleton.

Theorem 4. *Let k, n, d be positive integers with $k < n/2$, then*

$$t(n, d, k) = \left\lceil \frac{n}{k} \right\rceil - 2 + d \left\lceil \log \left(n - k \left\lceil \frac{n}{k} \right\rceil + 2k \right) \right\rceil.$$

More importantly, we also provide new (and to our knowledge the first non-trivial) upper and lower bounds in the non-adaptive case, which require more insight into the problem.

Theorem 5. *Let k, n, d be positive integers with k small enough, then*

$$q(n, d, k) \leq \left\lceil \frac{nd}{k} \right\rceil.$$

In particular,

$$q(n, 2, k) = \left\lceil 2 \frac{n}{k} \right\rceil$$

and

$$\left\lceil \frac{3n+1}{k+3} \right\rceil \leq q(n, 3, k) \leq \left\lceil 3 \frac{n}{k} \right\rceil.$$

Following Hosszu, Tapolcai and Wiener, we consider the related problem of constructing the binary matrix $M = M(\mathcal{A})$ whose rows are the characteristic vectors of our queries $A \in \mathcal{A}$. Interpreting M as the incidence matrix of a (hyper-)graph, we derive a forbidden-subgraph condition which suffices to ensure separability.

Lemma 6. *If a d -uniform k -regular linear hypergraph of order q has girth at least 5, then its associated set system is d -separating.*

Previous work on girth conditions for graphs by Erdős and Sachs [3] and very recently for hypergraphs by Ellis and Linial [2] shows that such graphs exist for sufficiently larger order and we rely on these results to show that our constructions are possible.

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The Complete List of Ramsey $(2K_2, K_4)$ –Minimal Graphs

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EXTENDED ABSTRACT

Let F, G and H be non-empty graphs. The notation $F \rightarrow (G, H)$ means that if all edges of F are arbitrarily colored by red or blue then either the red subgraph of F contains a graph G or the blue subgraph of F contains a graph H . A graph F satisfying $F \rightarrow (G, H)$ and $(F - e) \not\rightarrow (G, H)$ for every $e \in E(F)$ is called a Ramsey (G, H) –minimal graph. The set of all Ramsey (G, H) –minimal graphs is denoted by $\mathfrak{R}(G, H)$.

Burr et al. [7] have described a graph G_r in $\mathfrak{R}(2K_2, K_n)$, for $n \geq 4$, constructed from the complete graph K_{n+1} , as follow. Let $V(K_{n+1}) = R \cup S$ be a partition of the vertices of K_{n+1} , and denote the cardinality of R by r . To each edge $e = xy$ with $\{x, y\} \subseteq R$ or $\{x, y\} \subseteq S$, associate a vertex v_e not in K_{n+1} and let v_e be adjacent to each vertex of K_{n+1} except for x and y . For $1 \leq r \leq \lfloor (n+1)/2 \rfloor$, denote this graph by G_r , with $|V(G_r)| = n+1 + \binom{r}{2} + \binom{n+1-r}{2}$ and $|E(G_r)| = \binom{n+1}{2} + (n-1) \left(\binom{r}{2} + \binom{n+1-r}{2} \right)$. For examples, Graphs G_1 and G_2 as depicted in Figure .14 are the graphs in $\mathfrak{R}(2K_2, K_4)$. Recently, Wijaya and Baskoro [16] gave some necessary and sufficient conditions for graphs in $\mathfrak{R}(2K_2, H)$, for any graph H .

Theorem 1 (Wijaya and Baskoro '2013 [16]). *Let H be any graph. $F \in \mathfrak{R}(2K_2, H)$ if and only if the following conditions are satisfied:*

- (i) for every $v \in V(F)$, $F - v \supseteq H$,
- (ii) for every K_3 in F , $F - E(K_3) \supseteq H$,
- (iii) for every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \not\supseteq H$ or $(F - e) - E(K_3) \not\supseteq H$.

They gave all connected graphs of order at most 8 and a graph of order 9 in $\mathfrak{R}(2K_2, K_4)$, namely $2K_4, K_6, F_1, F_2$, where the graphs F_1 and F_2 as depicted in Figure .14.

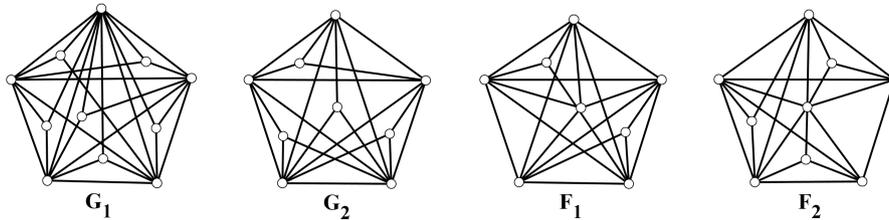


Figure .14: The graphs $G_1, G_2, F_1, F_2 \in \mathfrak{R}(2K_2, K_4)$.

In this paper, we determine all non-isomorphic connected graphs of order at least 9 in $\mathfrak{R}(2K_2, K_4)$. These results will complete the previous results ([7] and [16]) on the characterization of all graphs in $\mathfrak{R}(2K_2, K_4)$. Additionally, we also give a class of graphs in $\mathfrak{R}(2K_2, K_n)$, for any $n \geq 4$.

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