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Contents

Invited talks

Maria Axenovich – **Improper colorings of planar graphs**

Karlsruher Institut für Technologie

What happens when the vertices of a planar graph are colored with less than four colors? We know that the coloring might be improper, i.e., contain adjacent vertices of the same color. Can one nevertheless make sure that the color classes induce simple enough graphs, such as vertex-disjoint unions of short paths? We shall present the history of the problem and some recent progress towards its solution.
The talk is based on a joint work with T. Ueckerdt and P. Weiner.

Csilla Bujtás – **Domination and transversal games**

University of Pannonia

The notion of the domination game was introduced by Brešar, Klavžar and Rall only six years ago, but already has a significant literature. This is a two-person competitive optimization game, where the players, Dominator (Fast) and Staller (Slow), alternately select vertices of a graph G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously selected, and the process eventually produces a dominating set of G. Dominator wishes to minimize the number of vertices chosen in the game, while Staller wishes to maximize it. The game domination number of G is the number of vertices chosen when Dominator starts the game and both players play optimally. We survey results on the different versions of domination game and also consider the transversal game defined analogously on hypergraphs.

Kathie Cameron – **Structure and Algorithms for (Cap, Even Hole)-Free Graphs**

Wilfrid Laurier University

A hole is a chordless cycle with at least 4 vertices, and is even if it has an even number of vertices. A cap consists of a hole together with an additional vertex which is adjacent to exactly two vertices of the hole, which themselves are adjacent. A graph is (cap, even hole)-free if it has no induced cap or even hole.

We give an explicit construction of (cap, even-hole)-free graphs. Using this, we prove that every such graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence $\chi(G) \leq \frac{3}{2}\omega(G)$, where $\omega(G)$ denotes the size of the largest clique in G and $\chi(G)$ denotes the chromatic number of G.

We give an $O(nm)$ algorithm for q-colouring these graphs for fixed q and an $O(nm)$ combinatorial algorithm for maximum weight stable set. We also give a polynomial-time algorithm for minimum colouring.

Our algorithms are based our results that (triangle, even hole)-free graphs have treewidth at most 5 and that (cap, even hole)-free graphs without clique cutsets have clique-width at most 48.

This is joint work with Murilo V. G. da Silva, Shenwei Huang and Kristina Vušković.

Daniel Král' – **Graph limits and extremal combinatorics**

University of Warwick

Theory of graph limits has opened new links between analysis, combinatorics, computer science, group theory and probability theory. We start with presenting a connection between dense graph limits and the flag algebra method of Razborov, which has been used to solve many long-standing open problems in extremal combinatorics. We then focus on the structure of graphons, analytic objects representing large dense graphs. Motivated by problems from extremal graph theory, we will focus on graphons that are uniquely determined by finitely many density constraints, so-called finitely forcible graphons, and present counterexamples to several conjectures on the structure of such graphons.

Pavol Hell – **Interval-like Graphs and Digraphs**

Simon Fraser University

In this talk I will focus on forbidden structure characterizations of geometrically defined graph (and digraph) classes. These include interval graphs and digraphs, and circular arc-graphs. I will propose a general class of digraphs that encompasses many of these classes, and present a forbidden structure characterization for it. I will also present a separate forbidden structure characterization for the class of circular-arc graphs. This gives an answer to questions of Klee, and of Hadwiger and Debrunner, from the early 1960's. This is joint with T. Feder, J. Huang, and A. Rafiey for the interval graphs and their generalizations, and with M. Francis and J. Stacho for the circular arc graphs.

Alexandr Kostochka – **On disjoint and longest cycles in graphs**

University of Illinois

We discuss and refine some results on cycle structure of graphs from the sixties. Some results of Dirac and Erdős from 1963 are related to the well-known Corrádi-Hajnal Theorem. Let $V_{\geq t}(G)$ (respectively, $V_{\leq t}(G)$) denote the set of vertices in G of degree at least (respectively, at most) t. The Corrádi-Hajnal Theorem says that for every positive integer k, if $|V(G)| \geq 3k$ and $V_{\geq 2k}(G) = V(G)$, then G has k disjoint cycles. Dirac and Erdős proved that if $k \geq 3$ and G is a graph with $|V_{\geq 2k}(G)| - |V_{\leq 2k-2}(G)| \geq k^2+2k-4$, then G also has k disjoint cycles. They weakened the restriction on $|V_{\geq 2k}(G)| - |V_{\leq 2k-2}(G)|$ to $5k-7$ for planar graphs, and showed graphs G with $|V_{\geq 2k}(G)| - |V_{\leq 2k-2}(G)| = 2k-1$ that do not have k disjoint cycles.

We present refinements of the Dirac–Erdős results. In particular, we show that each graph G with $|V_{\geq 2k}(G)| - |V_{\leq 2k-2}(G)| \geq 3k$ has k disjoint cycles. This is sharp if we do not impose restrictions on $|V(G)|$. This is joint work with Kierstead and McConvey.

The Turán-type theorems of Erdős and Gallai from 1959 on the most edges in n-vertex graphs that do not have paths/cycles with at least k vertices were sharpened later by Faudree and Schelp, Woodall, and Kopylov. Let $h(n,k,a) = \binom{k-a}{2} + a(n-k+a)$. The strongest result (by Kopylov) was: if $t \geq 2$, $k \in \{2t+1, 2t+2\}$, $n \geq k$, and G is an n-vertex 2-connected graph with at least $\max\{h(n,k,2), h(n,k,t)\}$ edges, then G contains a cycle of length at least k unless $G = H_{n,k,t} := K_n - E(K_{n-t})$. We prove stability versions of these results. In particular, if $n \geq k$ and the number of edges in an n-vertex 3-connected graph G with no cycle of length at least k is greater than $\max\{h(n,k,3, h(n,k,t-1)\}$, then G is a subgraph of $H_{n,k,t}$ or of $H(n,k,2)$. The restriction on $|E(G)|$ is tight. This is joint work with Füredi, Luo and Verstraëte.

Carsten Thomassen – **Chords in longest cycles**

Technical University of Denmark

In 1975 John Sheehan made the conjecture that every 4-regular Hamiltonian cycle has a second Hamiltonian cycle. Around the same time I made the conjecture that every longest cycle in every 3-connected graph has a chord. In this talk I will discuss connections between the two conjectures and to the chromatic polynomial, and also report on recent progress on the chord conjecture.

Contributed talks

Treelike snarks

Marien Abreu [1*], *Tomas Kaiser* [2†], *Domenico Labbate* [1*] and *Giuseppe Mazzuoccolo* [3],

[1] Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata – Potenza, Italy

[2] Department of Mathematics, Institute for Theoretical Computer Science and the European Centre of Excellence NTIS, University of West Bohemia, Czech Republic.

[3] Dipartimento di Informatica, Università degli Studi di Verona, Italy.

Extended Abstract

A *snark* (cf. e.g. [11]) is defined as a bridgeless cubic graph with edge chromatic number equal to four that contains no circuits of length at most four and no non-trivial 3-edge cuts.

Following the terminology introduced in [1], the *excessive index of* G, denoted by $\chi'_e(G)$, is the least integer k such that the edge-set of G can be covered by k perfect matchings. Note that the excessive index is sometimes also called *perfect matching index* (see [6]).

The main source of motivation for the above notion is the conjecture of Berge which asserts that the excessive index of any cubic bridgeless graph is at most 5. As proved recently by the third author [13], this conjecture is equivalent to the famous conjecture of Berge and Fulkerson [7] that the edge-set of every bridgeless cubic graph can be covered by six perfect matchings, such that each edge is covered precisely twice.

As for cubic bridgeless graphs G with $\chi'_e(G) \geq 5$, it was asked by Fouquet and Vanherpe [6] whether the Petersen graph was the only such graph that is cyclically 4-edge-connected. Hägglund [9] constructed another example (of order 34) and asked for a characterisation of such graphs [9, Problem 3]. Esperet and Mazzuoccolo [4] generalised Hägglund's example to an infinite family.

We construct another infinite family of graphs, arising from a generalization in a different direction, called *treelike snarks*. We need some preliminary definitions and remarks before defining this family of snarks.

For a given graph G, the vertex set of G is denoted by $V(G)$, and its edge set by $E(G)$. Each edge is viewed as composed of two *half-edges* (that s are *associated* to each other) and we let $E(v)$ denote the set of half-edges incident with a vertex v.

A *join* in a graph G is a set $J \subseteq E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. In the literature, the terms *postman join* or *parity subgraph* have essentially the same meaning. We will be dealing with cubic graphs, so *joins* will be spanning subgraphs where each vertex has degree 1 or 3.

As usual, e.g., in the theory of nowhere-zero flows, we define a *cycle* in a graph G to be any subgraph $H \subseteq G$ such that each vertex of H has even degree in H. Thus, a cycle need not be connected. A *circuit* is a connected 2-regular graph. In a cubic graph, a cycle is a disjoint union of circuits and isolated vertices. Note that *a subgraph* $H \subseteq G$ *is a cycle in* G *if and only if* $E(G) - E(H)$ *is a join.*

A *cover* (or *covering*) of a graph G is a family $\mathcal{F}$ of subgraphs of G, not necessarily edge-disjoint, such that $\bigcup_{F \in \mathcal{F}} E(F) = E(G)$.

We view each edge of a graph as composed of two half-edges. We now extend the notion of a graph by allowing for *loose* half-edges that do not form part of any edge; the resulting structures will be called *generalised graphs*. A generalised graph is *cubic* if each vertex is incident with three half-edges.

We define a *fragment* F as a generalised cubic graph with exactly five loose half-edges, ordered in a sequence (see Figure 1). The *Petersen fragment* F_0 is the fragment with loose half-edges $(a_1, \ldots, a_5)$ obtained from the Petersen graph (see Figure 2b) as follows:

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- in the Petersen graph, remove a vertex x, keeping the half-edges a_3, a_4, a_5 incident with its neighbours y, z, t, respectively,
- subdivide the two edges incident with y,
- and add half edges a_1, a_2 to the new vertices of the subdivision (see Figures 2a, 2b) in any one of the two ways.

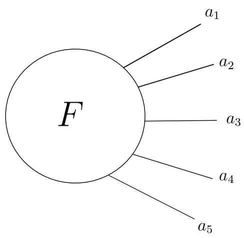
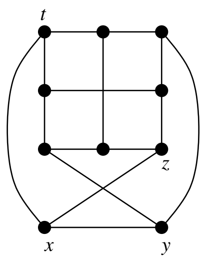
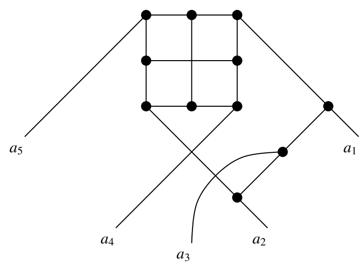


Figure 1: A fragment.



(a) The Petersen graph.



(b) The Petersen fragment F_0.

Figure 2: Constructing the Petersen fragment.

A *Halin graph* is a plane graph consisting of a planar representation of a tree without degree 2 vertices, and a circuit on the set of its leaves. (cf., e.g., [10, 3]).

Let H_0 be a cubic Halin graph consisting of the tree T_0 and the circuit C_0. The *treelike snark* $G(T_0, C_0)$ is obtained by the following procedure:

- for each leaf ℓ of T_0, we add a copy F_0^ℓ of the Petersen fragment F_0 with loose half-edges $(a_1, \ldots, a_5)$ and attach the half-edge a_3 to ℓ,
- for each leaf ℓ of T_0 and its successor ℓ' with respect to a fixed direction of C_0, if F_0^ℓ has loose half-edges $(a_1, \ldots, a_5)$ and $F_0^{\ell'}$ has loose half-edges $(a'_1, \ldots, a'_5)$, then we join a_4 to a'_2 and a_5 to a'_1, obtaining new edges.

If there is no danger of a confusion, we abbreviate $G(T_0, C_0)$ to $G(T_0)$.

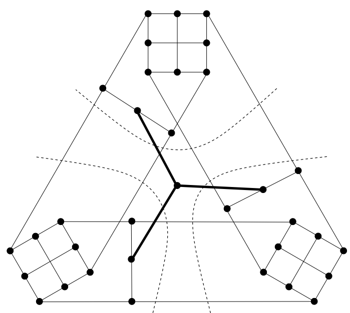
Some small examples of treelike snarks are shown in Figure 3.

We prove that they are indeed snarks and that their excessive index is greater than or equal to five.

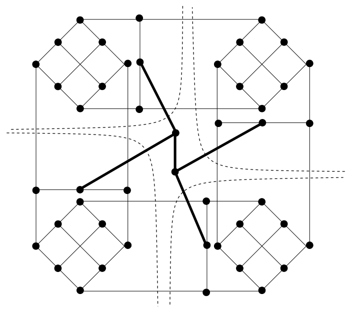
Proposition 1 *Treelike snarks are snarks.*

Theorem 2 *Treelike snarks have excessive index at least* 5.

Hence, as mentioned above, we expand the known family of snarks of excessive index ≥ 5, with a different generalization of Hägglund's example than the one found by Esperet and Mazzuoccolo in [4].



(a) Windmill 1 [4, Figure 7].



(b) Blowup($Prism, C_4$) [9, Figure 7].

Figure 3: Small treelike snarks.

We recall the notion of circular nowhere-zero r-flow, first introduced in [8].

Let $G = (V, E)$ be a graph. Given a real number $r \geq 2$, a *circular nowhere-zero r-flow* (r-CNZF for short) in G is an assignment $f : E \to [1, r-1]$ and an orientation D of G, such that f is a flow in D. That is, for every vertex $x \in V$, $\sum_{e \in E^+(x)} f(e) = \sum_{e \in E^-(x)} f(e)$, where $E^+(x)$, respectively $E^-(x)$, are the sets of edges directed from, respectively toward, x in D.

The *circular flow number* $\phi_c(G)$ of G is the infimum of the set of numbers r for which G admits an r-CNZF. If G has a bridge then no r-CNZF exists for any r, and we define $\phi_c(G) = \infty$.

A circular nowhere-zero *modular-r*-flow (r-MCNZF), is an analogue of an r-CNZF, where the additive group of real numbers is replaced by $\mathbb{R}/r\mathbb{Z}$. We would like to stress that, given an r-MCNZF f, the direction of an edge e can be always reversed and f transformed into another r-MCNZF, where $f(e) \in \mathbb{R}/r\mathbb{Z}$ is replaced by $-f(e) \in \mathbb{R}/r\mathbb{Z}$.

The following result is well-known and implicitly proved also in Tutte's original work on integer flows [15].

Proposition 3 ([15]) *The existence of a circular nowhere-zero r-flow in a graph G is equivalent to that of an r-MCNZF.*

The outstanding 5-Flow Conjecture is equivalent to the statement that the circular flow number of no bridgeless graph is greater than 5. In [5], the authors present some general methods for constructing graphs (in particular snarks) with circular flow number at least 5. By a direct application of the main results in [5], one can deduce that all (few) known snarks with excessive index 5 have circular flow number at least 5. In other words, if a snark is "critical" with respect to Berge-Fulkerson's Conjecture, then it seems to be critical also for the 5-flow Conjecture. The converse is false, as shown by the snark G of order 28, found by Máčajová and Raspaud [12]: it has $\phi_C(G) = 5$ and $\chi'_e(G) = 4$.

We furnish a further element in the direction of the previous observation, by proving that also all treelike snarks have circular flow number at least 5, i.e. treelike snarks are, in a sense, also critical for this conjecture:

Theorem 4 *Treelike snarks have circular flow number at least* 5.

Finally, since it is known that any cubic graph G that is a counterexample to the Cycle Double Cover Conjecture satisfies $\chi'_e(G) \geq 5$, it is natural to ask whether treelike snarks admit cycle double covers. We show that this is indeed the case. In fact, using a new general sufficient condition for the existence of a 5-cycle double cover, we show that treelike snarks satisfy the 5-Cycle Double Cover Conjecture of Preissmann [14] and Celmins [2].

Indeed, we introduce a new general sufficient condition for a cubic graph to admit a 5-cycle double cover (Theorem 5) and we show how it can be easily used to prove the existence of a 5-cycle double cover for every treelike snark (Theorem 7).

Theorem 5 *Let G be a cubic graph. If G admits a connected join with a congruent 3-edge-colouring, then it has a 5-cycle double cover.*

Corollary 6 *Let G be a cubic graph. If G admits a 3-edge-colourable connected join J with a connected complement in G, then G has a 5-cycle double cover.*

Theorem 7 *Treelike snarks admit a 5-cycle double cover.*

We conclude this section by pointing out a reformulation of Theorem 5 in terms of nowhere-zero flows:

Corollary 8 *Let G be a cubic graph admitting a connected join J. Let G' be the graph obtained from G by contracting each circuit of the complement of J to a vertex. If G' admits a nowhere-zero 4-flow, then G admits a 5-cycle double cover.*

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Independent $[1, 2]$-domination of grids via min-plus algebra

Sahar Aleid, *José Cáceres* and *María Luz Puertas*

Department of Mathematics, University of Almería, Spain

Extended Abstract

Domination of grids has been proved to be a demanding task and with the addition of independence it becomes more challenging. It is known that no grid with $m, n \geq 5$ has a perfect code, that is an independent vertex set such that each vertex not in it has exactly one neighbor in that set. So it is interesting to study the existence of an independent dominating set for grids that allows at most two neighbors, such a set is called independent $[1, 2]$-set. In this paper we develop a dynamic programming algorithm using min-plus algebra that computes the minimum cardinality of an independent $[1, 2]$-set for the grid $P_m \Box P_n$.

Keywords: Domination, independence, grids, min-plus algebra.

1 Introduction

Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is called a dominating set of G if every $v \in V \backslash S$ has at least one neighbor in S. Recall that the grid $P_m \Box P_n$ is the cartesian product of paths P_m and P_n. Domination in grids has been extensively studied and the problem of determining the domination number $\gamma(P_m \Box P_n)$, which is the minimum size of a dominating set of $P_m \Box P_n$ was opened for almost 30 years, since it was first studied in [?]. In his 1992 Ph.D thesis Chang [?] proved that the domination number in grids is bounded by $\gamma(P_m \Box P_n) \leq \lfloor \frac{(n+2)(m+2)}{5} \rfloor - 4$, for $m, n \geq 8$. Chang also conjectured that equality holds for $16 \leq m \leq n$. In 1998 $\gamma(P_m \Box P_n)$ was computed for $m \leq 19$ and every n (see [?]). Finally in 2011, the problem was completely solved in [?] as authors were able to adapt the ideas in [?] to confirm Chang's conjecture.

Independence is a property closely related to domination. A set S of vertices is called *independent* if no two vertices in S are adjacent. The independent domination number is denoted by $i(G)$, which is the minimum cardinality of an independent dominating set for the graph G. Recently in [?] the independent domination number has been computed for all grids.

A *perfect code* is an independent dominating set such that every vertex not in it has a unique neighbor in the set. It was proved in [?] that there exists no perfect code for grids $P_m \Box P_n$, except in cases $m = n = 4$ and $m = 2, n = 2k + 1$. This leads us to work with independent $[1, 2]$-sets, that is an independent vertex set S such that every $v \in V \backslash S$ is adjacent to at least one but not more than 2 vertices in S (see [?]). We solve the open problem proposed in [?] about the existence of independent $[1, 2]$-set in grids and we also compute the independent $[1, 2]$-number $i_{[1,2]}(P_m \Box P_n)$ which is the minimum cardinality of such a set.

2 Algorithm

We present a dynamic programming algorithm to obtain $i_{[1,2]}(P_m \Box P_n)$, following the ideas in [?, ?]. Consider $P_m \Box P_n$ as an array with m rows, n columns and vertex set $\{v_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Let S be an independent $[1, 2]$-set of $P_m \Box P_n$. We define a labeling of vertices of $P_m \Box P_n$ associated to S as follows

$$l(v_{ij}) = \begin{cases} 0 & \text{if } v_{ij} \in S \\ 1 & \text{if } v_{ij} \notin S \text{ and } |\{v_{i(j-1)}, v_{(i-1)j}, v_{(i+1)j}\} \cap S| = 1 \\ 2 & \text{if } v_{ij} \notin S \text{ and } |\{v_{i(j-1)}, v_{(i-1)j}, v_{(i+1)j}\} \cap S| = 2 \\ 3 & \text{if } v_{ij} \notin S \text{ and } |\{v_{i(j-1)}, v_{(i-1)j}, v_{(i+1)j}\} \cap S| = 0 \end{cases}$$

Given an independent $[1, 2]$-set, we can identify each vertex with its label so we obtain an array of labels with m rows and n columns. Hereinafter, given an independent $[1, 2]$-set of $P_m \Box P_n$, the columns of the grid are words of length m in the alphabet $\{0, 1, 2, 3\}$ and the number of zeros in the array is the cardinality of the independent $[1, 2]$-set. The algorithm considers all the arrays of words (as columns) that come from some independent $[1, 2]$-set of $P_m \Box P_n$ and it calculates the minimum among the number of $0's$ of each array. This minimum is equal to $i_{[1,2]}(P_m \Box P_n)$.

It is clear that not every word of length m can belong to such labeling, for instance there can not be consecutive $0's$ in a word, because of independence. The first objective is to identify the words that belong to the labeling associated to some independent $[1, 2]$-set, that we will call suitable words.

We need to have in mind the not every suitable word can be in the first column nor in the last one. For instance a word with the sequence 120 can not be in the first column, because the vertex labeled with 2 does not follow the labeling rules, in the absence of a previous column. Also in the last column can not be vertices with label 3, because they would not be dominated. The second task is to identify those words that can be in the first column and those words that can be in the last one.

It is also clear that not any two suitable words can follow each other in a labeling associated to some independent $[1, 2]$-set, for instance if a word has 3 in the r^{th} position, then the following word must have 0 in the r^{th} position, to preserve domination. The last objective regarding to words is to identify which of them can follow a given one.

The final part of the algorithm calculates the minimum number of zeros in an array, among all possible arrays of labels associated to an independent $[1, 2]$-set. This can be done by the successive addition of columns, beginning with just one and ending with n.

2.1 The suitable words

Let S be an independent $[1, 2]$-set of $P_m \Box P_n$ and consider the associated labeling. It is clear that a column can not contain two consecutive $0's$ because of independence, nor two consecutive $2's$ because in the previous column should be two consecutive $0's$, also it can not contain two consecutive $3's$ because in the following column will be two consecutive $0's$. Moreover a column can not contain any of the sequences $03, 30, 010$ by definition of the labeling. If a column contains the sequence 11 then the previous label or the following label in the column (or both) must be 0, because in other case would be two consecutive $0's$ in the previous column. If a column contains the sequence 32 then the other label next to 2 in the column must be 0, by definition of the labeling.

Similarly sequence 23 must be preceded by 0 and both sequences 21 ,12 must be placed between two $0's$. Words satisfying all those rules are called *suitable*. We denote the cardinal of the set of all suitable words by k, so every suitable word can be identified with $p \in \{1, \dots, k\}$.

2.2 The first column, the last column and the initial vector

Let S be an independent $[1, 2]$-set of $P_m \Box P_n$ and consider the associated labeling. Note that the first column is a suitable word such that every 2 is placed between two $0's$ and every 1 is preceded or followed (but not both) by 0, because in other case the labeling would not follow the definition. A suitable word that satisfies these properties is called *initial*.

Similarly a word can be placed in the last column if it does not contain any 3, because in other case one vertex would be not dominated. We call this word *final*.

Defintion 1 *A suitable word is called* initial *if every vertex labeled as* 2 *is placed between two* $0's$ *and every vertex labeled as* 1 *is preceded or followed (but not both) by* 0 *and it is called* final *if it has no vertex labeled as* 3.

The initial vector X^1 *is a column vector of size k such that for every suitable word* $p \in \{1, 2, \ldots, k\}$, *the* p^{th} *entry of the vector is*

$$X^1(p) = \begin{cases} \textit{number of zeros of word } p & \ldots \quad \textit{if } p \textit{ is an initial word} \\ \infty & \ldots \quad \textit{if } p \textit{ is not an initial word} \end{cases}$$

Finally we define the *initial vector* $X^1 = (X^1(1), X^1(2), \ldots, X^1(k))$ which is a vector of size k such that for every word $p \in \{1, 2, \ldots, k\}$, $X^1(p)$ is number of zeros of p, if it is an initial word, and $X^1(p) = \infty$ in other case.

$$\text{If } q_i = 0 \text{ then} \begin{cases} (\textit{if } i = 1) \begin{cases} p_1 = 1, \; p_2 \neq 0 \textit{ or} \\ p_1 = 2, \; p_2 = 0 \end{cases} \\ (\textit{if } 1 < i < m) \begin{cases} p_i = 1, \; p_{i+1} \neq 0, \; p_{i-1} \neq 0 \textit{ or} \\ p_i = 2, \; p_{i+1} \neq 0, \; p_{i-1} = 0 \\ p_i = 2, \; p_{i+1} = 0, \; p_{i-1} \neq 0 \end{cases} \\ (\textit{if } i = m) \begin{cases} p_m = 1, \; p_{m-1} \neq 0 \textit{ or} \\ p_m = 2, \; p_{m-1} = 0 \end{cases} \end{cases}$$

Conditions can be defined similarly for cases $q_i = 1, q_i = 2$ and $q_i = 3$.

The *transition matrix* is the square matrix A of size k such that, for every pair $p, q \in \{1, 2, \ldots, k\}$, the entry A_{pq} in row p and column q is defined to be the number of zeros of the word p if p can follow q, and $A_{pq} = \infty$ otherwise.

2.3 Adding many columns via the min-plus multiplication

Using the particular matrix multiplication in $(min, +)$-algebra see [?] ,we can successively obtain vectors $X^2 = A \boxtimes X^1, \ldots, X^n = A \boxtimes X^{(n-1)}$. These vectors allows us to characterize grids having an independent $[1, 2]$-set.

For every word $p \in \{1, \ldots k\}$, the entry $X^n(p)$ is the minimum number of zeros in an independent vertex subset of $P_m \Box P_n$, which has word p in the n^{th}-column and $[1, 2]$-dominates the graph, except for may be some vertices in the last column with label 3.

Now we can characterize grids having an independent $[1, 2]$-set, using the vectors obtained by means of the $(\min, +)$ matrix multiplication.

Theorem 2

1. *There exists an independent* $[1, 2]$*-set in* $P_m \Box P_n$ *if an only if there exists a final word* p *such that* $X^n(p) < \infty$.
2. $i_{[1,2]}(P_m \Box P_n) = \min\{X^n(p) \colon p \textit{ is a final word}\}$.

2.4 The recursion rule to calculate $i_{[1,2]}(P_m \Box P_n)$

The above procedure allows us to obtain the value of $i_{[1,2]}(P_m \Box P_n)$ for fixed m and n. However a recurrence argument gives a formula for $i_{[1,2]}(P_m \Box P_n)$ for fixed m and any $n \geq m$. We use the following result which is similar to Theorem 2.2 in [?].

Theorem 3 *Suppose that there exist integers* $n_0, c, d > 0$ *satisfying the equation* $X^{n_0+d}(p) = X^{n_0}(p) + c$, *for every suitable word* p. *If* $P_m \Box P_r$ *has an independent* $[1, 2]$*-set for* $n_0 \leq r \leq n_0 + d - 1$, *then*

1. $P_m \Box P_n$ *has an independent* $[1, 2]$*-set, for all* $n \geq n_0$,
2. $i_{[1,2]}(P_m \Box P_{n+d}) = i_{[1,2]}(P_m \Box P_n) + c$, *for all* $n \geq n_0$.

Now let m be fixed and consider the suitable words of length m and the initial vector X^1. Then apply the operation $\boxtimes$ to calculate vectors X^i, until obtaining a natural number n_0 such that $X^{n_0+d} = c \boxtimes X^{n_0}$. Using the above theorem we get the finite difference equation $i_{[1,2]}(P_m \Box P_{n_0+d}) - i_{[1,2]}(P_m \Box P_{n_0}) = c$. The unique solution obtained from solving this equation for $n \geq n_0$, is the desired formula for $i_{[1,2]}(P_m \Box P_n)$.

2.5 Results

Applying the construction described in the previous section we have obtained that the independent $[1,2]$-number agrees with the independent dominating number in almost every grid of small size $1 \leq m \leq 13, m \leq n$

$$i_{[1,2]}(P_m \Box P_n) = \begin{cases} i(P_m \Box P_n) + 1 & \dots \quad m = 12, n \equiv 10 (mod\ 13) \\ i(P_m \Box P_n) & \dots \quad otherwise \end{cases}$$

We would like to point out that applying the algorithm for large values of m, n is possible but the running time needed is extremely long. However the calculation of $i_{[1,2]}(P_m \Box P_n)$ for $14 \leq m \leq n$ can be solved using a constructive regular pattern, following the ideas in [**?**, **?**]. We have obtained in these cases

$$i_{[1,2]}(P_m \Box P_n) = \begin{cases} i(P_m \Box P_n) & \dots \quad m = 14, 15, \ m \leq n \\ \gamma(P_m \Box P_n) & \dots \quad 16 \leq m \leq n \end{cases}$$

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Degree constrained 2-partitions of tournaments

Jørgen Bang-Jensen and *Tilde My Christiansen*[1]

Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark

Extended Abstract

1 Introduction

A 2-partition (V_1, V_2) of a digraph $D = (V, A)$ is a partition of V into disjoint sets V_1, V_2 such that $V = V_1 \cup V_2$. We will consider problems of deciding whether a given digraph has a 2-partition (V_1, V_2) such that V_i induces a digraph with some specified property. In [1] and [2] Bang-Jensen, Cohen and Havet determined the complexity of 120 such problems. Several of these problems are NP complete for general digraphs and it is natural to ask whether this is still the case for tournaments. This is the topic of this note. More specifically we consider 2-partitions where each partition induces a digraph with minimum out/in/semidegree at least k for some integer k.

We consider directed simple digraphs. Notation follows [3]. Let a_1, a_2 be graph properties, then (V_1, V_2) is an (a_1, a_2)-partition if V_i induces a digraph with property a_i for $i \in [2]$. While $N^+(v)$ denote the set of out-neighbours of a vertex v, given a set $X \subseteq V$, $N^+[X]$ is the closed out-neighbourhood of X containing X and all out-neighbours of the vertices of X.

Proofs left out in this note will appear in the full version of the note together with more detailed descriptions of algorithms that also finds partitions when they exist.

2 $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition

In [8] Thomassen showed that $\delta^+(D) \geq 3$ is sufficient for the existence of a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition of D. Furthermore there exist a polynomial algorithm to check if a general digraph has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition. This is true if and only if D contains two disjoint cycles and deciding if D has two disjoint cycles is polynomial by McCuaig[6].

Theorem 1 *It is NP complete to decide whether a digraph $D = (V, A)$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition, for k_1, k_2 natural numbers with $k_1 + k_2 \geq 3$.*

For tournaments this is another story. In [5] Lichadopol showed that every tournament with out-degree at least $\frac{k_1^2+3k_1+2}{2} + k_2$ has a $(\delta^+ \geq k_1, \delta^+ \geq k_2)$-partition. By modifying the proof of Lichadopol we describe a polynomial algorithm that decides if such a partition exist for any tournament. For simplicity we will assume that $k_1 = k_2 = k$ in the following but the result can easily be generalized.

A vertex $v \in V(T)$ is said to be ***k-out-dangerous*** if $d^+(v) \leq 2k - 2$. Furthermore let $X \subseteq V$, then an ***X-out-critical*** set X' is a set containing X which is critical in the sense that no subset of $X' - X$ can be removed from X' without decreasing the out-degree of $T\langle X' \rangle$.

Lemma 2 *Let k be a fixed integer and let T be a tournament with minimum out-degree at least k. Then the number of k-out-dangerous vertices of T is at most $4k - 3$.*

Proof: Let X be the set of out-dangerous vertices of T. Then the number of arcs in the subtournament $T\langle X \rangle$ is at most $|X|(2k - 2)$. Hence $\frac{|X|(|X|-1)}{2} \leq |X|(2k - 2)$, implying that $|X| \leq 4k - 3$.

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Lemma 3 *Let T be a tournament with minimum out-degree at least k and let $X \subseteq V(T)$. Then T contains an X-out-critical set X' with $|X'| \leq \frac{k^2+3k+2}{2} + |X|$.*

Proof: We will prove the theorem by induction over $|V(T)|$. Let X be a fixed set in $V(T)$. If $|V(T)| \leq \frac{k^2+3k+2}{2} + |X|$ we are done, so assume $|V(T)| > \frac{k^2+3k+2}{2} + |X|$. Let M be the set of vertices that have out-degree k in T and let $m = |M|$.

As each $v \in M$ has $d^+(v) = k$ and $T\langle M \rangle$ is a tournament we have

$$|N^+[M]| \leq m + mk - \frac{m(m-1)}{2} = -\frac{m^2}{2} + \left(\frac{3}{2} + k\right) m =: P(m).$$

Now $P(m)$ has global maximum at $(3/2 + k)$ and maximum for m integer at $k+1$ and $k+2$ with $P(k+1) = P(k+2) = \frac{k^2+3k+2}{2}$. Hence as $|V(D)| > \frac{k^2+3k+2}{2} + |X|$ there exist a vertex $u \in V - (N^+[M] \cup X))$ such that $\delta^+(T\langle V - u \rangle) \geq k$ and the result follows by induction.

We are now ready to prove the existence of a polynomial algorithm for tournaments.

Theorem 4 *Let k be a fixed integer. Then there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k, \delta^+ \geq k)$-partition of a tournament or correctly conclude that none exists.*

Proof: The idea is to consider all possible partitions (O_1, O_2) of the out-dangerous vertices and for each such partition and for all O_1-critical sets X of bounded size, check whether there is a $(\delta^+ \geq k, \delta^+ \geq k)$-partition with $X \subseteq V_1$ and $O_2 \subseteq V_2$. Starting with the partition $(V_1, V_2) = (X, V - X)$ this subalgorithm will repeatedly move a vertex $v \in V_2 - O_2$ with $d^+_{T\langle V_2 \rangle}(v) < k$ to V_1. If at any time there is a vertex $u \in O_2$ such that $d^+_{T\langle V_2 \rangle}(u) < k$ there is no good partition with $X \subseteq V$ and $O_2 \subseteq V_2$, so the subalgorithm terminates. Otherwise it terminates after moving all low degree vertices of $V_2 - O_2$ and if $V_2 \neq \emptyset$ it has found a $(\delta^+ \geq k, \delta^+ \geq k)$-partition.

Correctness follows as we only move vertices from V_2 to V_1 that are not out-dangerous and hence have k out-neighbours in the (current) V_1. Hence $\delta^+(T\langle V_1 \rangle) \geq k$ throughout the algorithm.

Clearly the subalgorithm is polynomial as vertices are only moved from V_2 to V_1 and in at most n steps V_2 is empty. Furthermore, as the number of out-dangerous vertices and the size of O_1-critical sets are bounded by Lemma 2 and Lemma 3, it follows that there are only a polynomial number of calls to the subalgorithm.

We can replace 'tournament' by 'semicomplete digraph' in all of the above, as the presence of 2-cycles will only decrease the number of out-dangerous vertices and the size of an X-out-critical set.

On the other hand if we have a semicomplete digraph and want the partition to also induce tournaments, this problem becomes NP complete even for $k = 1$.

Theorem 5 *It is NP complete to decide if a semicomplete digraph has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition such that each partition also induces a tournament.*

3 $(\delta^+ \geq k, \delta^- \geq k)$- and $(\delta^0 \geq k, \delta^0 \geq k)$-partitions

Both $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition and $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition were proved to be NP complete for general digraphs by Bang-Jensen, Cohen and Havet in [1]. Unfortunately we cannot use the same approach as in Section 2 to find a polynomial algorithm for tournaments. This is because there is no longer a nice way to define dangerous vertices; In a partition (V_1, V_2) we may have a vertex v that has neither k out-neighbours in V_1 nor k in-neighbours in V_2.

Using known results on cycles and complementary cycles in tournaments we can find a polynomial algorithm in the case where $k = 1$. Clearly in both problems the existence of cycles are necessary, for the first it is also sufficient.

Theorem 6 *A tournament T has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition if and only if it has two disjoint cycles.*

Proof: By the above we may assume that T has a pair of disjoint cycles. Let $D_1, \ldots, D_s$ be the strong components of T (if D is strong then $s = 1$). Notice that if we find a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition (V_1', V_2') for the subdigraph induced by $V(D_i) \cup \cdots \cup V(D_j)$ for $i, j \in [s]$ and $i \leq j$, then letting $V_1 = V_1' \cup V(D_1) \cup \cdots \cup V(D_{i-1})$ and $V_2 = V - V_1$, we have a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition for T.

Assume that there are two non-trivial strong components D_i and D_j with $i < j$. Then letting $V_1' = V(D_{i+1}) \cup \cdots \cup V(D_j)$ and $V_2' = V(D_i)$, (V_1', V_2') is a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition for $T\langle V(D_i) \cup \ldots V(D_j)\rangle$ and we are done. Hence there is only one non-trivial component D_i. By the assumption D_i contains two disjoint cycles C_1 and C_2. Now for $j \in [2]$, if $T_j = D_i\langle V(D_i) - V(C_j)\rangle$ has minimum in-degree at least 1, then clearly $(V(C_j), V(D_i) - V(C_j))$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition so suppose x_j is a vertex in $V(D_i) - C_j$ with in-degree 0 in T_j. As x_j must have an in-neighbour on C_j for $j \in [2]$ we have that $x_1 \neq x_2$. But $x_j \notin C_j$ for $j \in [2]$ and hence there is an arc between x_1 and x_2, contradicting that $d^-_{T_j}(x_j) = 0$ for $j \in [2]$.

Theorem 7 *There exists a polynomial algorithm that given a semicomplete digraph $S = (V, A)$ decides if it contains a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition.*

Proof: Assume first that S is not strong and let $D_1, \ldots, D_r$ be the strong components. If D_1 or D_r is a trivial component, then clearly there is no $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. On the other hand if D_1, D_r and D_i for some $i \in [r-1]_2$ are all non-trivial, then $(D_i, V - D_i)$ is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition. Hence we may assume that all but D_1 and D_r are trivial. Now it is easy to see that there is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if and only if D_1 or D_r has such a partition.

Assume now that S is strong. If $n < 8$ all possible partitions is checked. Otherwise it is first checked if S has disjoint cycles using the algorithm of McCuaig [6] and if these exist checked whether it has complementary cycles, using the algorithm of Bang-Jensen and Nielsen [4]. Hence we may now assume that S contains disjoint cycles but not complementary cycles and that is not 2-strong by Reid [7].

Let x be a separator of S and let $D_1, \ldots, D_r$ be the strong components of $S - x$.

The following follows as there are no complementary cycles in S: If $r \geq 3$ then $|D_i| = 1$ (denote d_i from now on) for all $i \in [r-1]_2$. If D_1 is non-trivial, then $d_2x \in A$ (unless $r = 2$), x only dominates vertices y of D_1 that are separators of D_1 and only the first strong component of $D_1 - y$ (denoted D_{11}) can be non-trivial. Similarly if D_r is non-trivial, then $xd_{r-1} \in A$, x is only dominated be vertices z of D_r that are separators of D_r and only the last strong component of $D_r - z$ (denoted D_{rs}) can be non-trivial.

Case 1) $r = 2$ and $|D_1|, |D_2| \geq 3$.
If $D_i \cup x$ is strong for $i \in [2]$ then we have complementary cycles. So x must dominate all vertices of D_1 and all vertices of D_2 must dominate x. Now as $n \geq 8$ then for some $i \in [2]$ there exists a vertex $v \in D_i$ such that $D_i - v$ is strong. Again contradicting that there are no complementary cycles.

Case 2b) $r \geq 2$, $\min\{|D_1|, |D_2|\} = 1$ and $\max\{|D_1|, |D_r|\} \geq 3$.
We may assume $|D_1| = 1$ and $|D_r| \geq 3$. Assume that (V_1, V_2) is a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition of S with $x \in V_1$. Then as x is the only in-neighbour of d_1, d_1 also belongs to V_1. Continuing this way we see that $\{x, d_1, \ldots, d_{r-1}\} = V_1' \subseteq V_1$. As $xd_{r-1} \in A$ any $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition (V_1, V_2) will have vertices of D_r in both V_1 and V_2, and $V_2 \subset D_r$. Indeed it is not hard to see

that the semicomplete digraph S' obtained by deleting $d_1, \ldots, d_{r-1}$ and adding an arc from x to each vertex of D_r will have a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if and only if S does.

Thus we can solve the problem by calling the algorithm on S'.

Case 3) $r > 2$ and $|D_1|, |D_r| \geq 3$.
We have the 3-cycles $C_3 = \{x, y, d_2\}$ and $C_3' = \{x, d_{r-1}, z\}$ in S. If $|D_{11}| \geq 3$ we have the $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition $(V(C_3), V - V(C_3))$. Similarly if $|D_{rs}| \geq 3$ we have a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition $(V(C_3'), V - V(C_3'))$. So assume $|D_{11}| = |D_{rs}| = 1$. Now $\{x, y, z\}$ is a feedback vertex set of S and there is only a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition if x, y and z belongs to disjoint cycles and the x-cycle does not use any vertices of D_1 and D_r. The latter follows from the fact that $\{x, y', z'\}$ is a feedback vertex set for every choice of y', z' as out-neighbours of x in D_1 respectively in-neighbour of x in D_r. Such an x-cycle exits if and only if $xd_i, d_jx \in A$ for some $i, j \in [r-1]_2$ and $i < j$.

For S strong clearly all but Case 2b can be done in polynomial time. Furthermore for Case 2b, notice that the recursive call will be on a semicomplete digraph which has at least 2 vertices less. Hence in at most n calls the algorithm we will terminate and we have a polynomial time algorithm. Now the polynomial algorithm for S not strong follows directly.

The following can be proved in a similar manner as Theorem 5.

Theorem 8 *It is NP complete to decide if a semicomplete digraph omits a partition such that each partition is strong and a tournament.*

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The neighbour-sum-distinguishing edge-colouring game

Olivier Baudon [1], *Jakub Przybyło* [2], *Elżbieta Sidorowicz* [3], *Éric Sopena* [1], *Mariusz Woźniak* [2], and Mohammed Senhaji [1]

[1] Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France
[2] AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland
[3] Faculty of Mathematics, Computer Science and Econometrics,
University of Zielona Góra, Zielona Góra, Poland

Extended Abstract

Let $\gamma : E(G) \longrightarrow \mathbb{N}^*$ be an edge colouring of a graph G and $\sigma_\gamma : V(G) \longrightarrow \mathbb{N}^*$ the vertex colouring given by $\sigma_\gamma(v) = \sum_{e \ni v} \gamma(e)$ for every $v \in V(G)$. A *neighbour-sum-distinguishing edge-colouring* of G is an edge colouring γ such that for every edge uv in G, $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. The study of neighbour-sum-distinguishing edge-colouring of graphs was initiated by Karoński, Łuczak and Thomason [8]. They conjectured that every graph with no isolated edge admits a neighbour-sum-distinguishing edge-colouring with three colours.

We consider a game version of neighbour-sum-distinguishing edge-colouring. The *neighbour-sum-distinguishing edge-colouring game* on a graph G is a 2-player game where the two players, called Alice and Bob, alternately colour an uncoloured edge of G. Alice wins the game if, when all edges are coloured, the so-obtained edge colouring is a neighbour-sum-distinguishing edge-colouring of G. Therefore, Bob's goal is to produce an edge colouring such that two neighbouring vertices get the same sum, while Alice's goal is to prevent him from doing so. The neighbour-sum-distinguishing edge-colouring game on G with Alice having the first move will be referred to as the *A-game on G*. The neighbour-sum-distinguishing edge-colouring game on G with Bob having the first move will be referred to as the *B-game on G*.

We study the neighbour-sum-distinguishing edge-colouring game on various classes of graphs. In particular, we prove that Bob wins the game on the complete graph K_n, $n \geq 3$, whoever starts the game, except when $n = 4$. In that case, Bob wins the game on K_4 if and only if he starts the game.

General results

A *balanced edge* in a graph G is an edge $uv \in E(G)$ with $\deg_G(u) = \deg_G(v)$.

Lemma 1 *Let G be a graph containing a balanced edge.*

1. *If $|E(G)|$ is even then Bob wins the A-game on G.*
2. *If $|E(G)|$ is odd then Bob wins the B-game on G.*

For a graph G, for every vertex $v \in V(G)$, we denote by $\deg^p_G(v)$ the number of pendant neighbours of v. An *internal vertex* in G is a vertex with $\deg_G(v) > 1$.

Theorem 2 *Let G be a graph such that $\deg^p_G(v) \geq \frac{1}{2}\deg_G(v) + 1$ for every internal vertex $v \in V(G)$.*

(1) *If $|E(G)|$ is odd then Alice wins the A-game on G.*

(2) *If $|E(G)|$ is even then Alice wins the B-game on G.*

Theorem 2 allows us to prove that Alice wins the A-game or the B-game on some special trees. A *caterpillar* is a tree T whose set of internal vertices induces a path, called the *central path* of T. We then have:

Corollary 3 (Special caterpillars) *Let T be a caterpillar, with central path $v_1v_2 \ldots v_k$, such that $\deg_T(v_1) \geq 4$, $\deg_T(v_k) \geq 4$ and $\deg_T(v_i) \geq 6$ for every i, $2 \leq i \leq k-1$. We then have:*

(1) *If $|E(G)|$ is odd then Alice wins the A-game on G.*

(2) *If $|E(G)|$ is even then Alice wins the B-game on G.*

Graphs families

The *double-star* $DS_{m,n}$, $m \geq n \geq 1$, is obtained from the two stars $K_{1,m}$ and $K_{1,n}$ by adding an edge joining their two centers. We prove the following:

Theorem 4 (Double-stars)

1. *For every integer $n \geq 1$, Bob wins the B-game on $DS_{n,n}$.*
2. *For every integer $n \geq 1$, Alice wins the A-game on $DS_{n,n}$.*
3. *For every integer $m > n \geq 1$, Alice wins the A-game on $DS_{m,n}$.*
4. *For every integer $m > n \geq 1$, Alice wins the B-game on $DS_{m,n}$.*

Theorem 5 (Complete graphs)

1. *For every integer $n \geq 3$, Bob wins the A-game on K_n.*
2. *For every integer $n \geq 3$, Bob wins the B-game on K_n if and only if $n \neq 4$.*

Theorem 6 (Complete bipartite graphs)

1. *For every integer $n \geq 2$, Bob wins the A-game on $K_{2,n}$.*
2. *For every integer $n \geq 2$, Alice wins the B-game on $K_{2,n}$.*

We leave as an open problem the question of determining who wins the A-game or the B-game on general complete bipartite graphs $K_{m,n}$, $3 \leq m \leq n$.

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On the list incidence chromatic number of graphs

Brahim BENMEDJDOUB [1], *Éric SOPENA* [2,3] and *Isma BOUCHEMAKH* [1]

[1] L'IFORCE, University of Sciences and Technology Houari Boumediene (USTHB)
[2] Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France
[3] CNRS, LaBRI, UMR5800, F-33400 Talence, France

Extended Abstract

In this paper, we introduce and study the list version of incidence colourings. All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively, and by $\Delta(G)$ the maximum degree of G.

Recall that a graph G is *k-list colourable* (or *k-choosable*) if, whenever each vertex v of G is given a list $L(v)$ of k colours, G admits a proper colouring in which each vertex receives a colour from its own list. The *list chromatic number* of G is then defined as the smallest integer k such that G is k-choosable. This type of colouring was independently introduced by Vizing [9] and Erdös, Rubin and Taylor [3] (see the surveys by Alon [1] and Tuza [8]).

An *incidence* of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. We denote by $Inc(G)$ the set of incidences of G. Two incidences (v, e) and (w, f) of $Inc(G)$ are *adjacent* whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f. An *incidence k-colouring* of G is a mapping from $Inc(G)$ to the set of colours $\{1, \dots, k\}$ such that every two adjacent incidences receive distinct colours. The smallest k for which G admits an incidence k-colouring is the *incidence chromatic number* of G, denoted $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Massey [2]. This problem has attracted much interest in recent years, see for instance [6, 5, 10, 11].

The list version of incidence colouring is defined in a way similar to the case of ordinary proper vertex colourings. We thus say that a graph G is *k-list incidence colourable* if, whenever each incidence (v, e) of G is given a list $L(v, e)$ of k colours, G admits an incidence colouring in which each incidence receives a colour from its own list. The *list incidence chromatic number* of G, denoted $\chi_i^\ell(G)$, is then the smallest integer k such that G is k-list incidence colourable.

We clearly have $\chi_i^\ell(G) \geq \chi_i(G)$ and $\chi_i(G) \geq \Delta(G) + 1$ for every graph G. Moreover, since every incidence (v, e) in a graph G has at most $3\Delta(G) - 2$ adjacent incidences, we get:

Proposition 1 *For any graph G, $\Delta(G) + 1 \leq \chi_i(G) \leq \chi_i^\ell(G) \leq 3\Delta(G) - 1$.*

Observe also that an incidence k-colouring of a graph G can be viewed as a proper colouring of the square of its line graph. Since the line graph of an n-cycle is an n-cycle, by a result of Erdös, Rubin and Taylor [3], we get:

Proposition 2 *For every cycle G, $\chi_i^\ell(G) = \chi_i(G)$.*

Recall that a graph G is *d-degenerated* if every subgraph of G contains a vertex of degree at most d. In [5], Hosseini Dolama and Sopena proved the following:

Theorem 3 (Hosseini Dolama and Sopena [5]) *For every d-degenerated graph G, $\chi_i(G) \leq \Delta(G) + 2d - 1$.*

We prove that the same result holds for the list incidence chromatic number:

Theorem 4 *For every d-degenerated graph G, $\chi_i^\ell(G) \leq \Delta(G) + 2d - 1$.*

Since every tree is 1-degenerated, every K_4-minor free graph (and thus every outerplanar graph) is 2-degenerated, and every planar graph is 5-degenerated, Theorem 4 gives the following:

Corollary 5 *For every graph G,*

1. *if G is a tree then $\chi_i^\ell(G) = \Delta(G) + 1$,*
2. *if G is a K_4-minor free graphs (and thus, if G is an outerplanar graph) then $\chi_i^\ell(G) \leq \Delta(G) + 3$,*
3. *if G is a planar graph then $\chi_i^\ell(G) \leq \Delta(G) + 9$.*

The *square grid* $G(m,n)$ is the graph defined as the Cartesian product of two paths, that is, $G(m,n) = P_m \,\square\, P_n$, where P_n denotes the path on n vertices. Since every square grid is 2-degenerated, we get $\chi_i^\ell(G(m,n)) \leq \Delta(G(m,n)) + 3 \leq 7$ for every square grid $G(m,n)$ by Theorem 4. We can decrease this bound to 6:

Theorem 6 *For every integers $m, n \geq 1$, $\chi_i^\ell(G(m,n)) \leq 6$.*

A *Halin graph* is a planar graph obtained from a tree with no vertex of degree 2 by adding a cycle connecting all its leaves. Wang, Chen and Pang proved that $\chi_i(G) = \Delta(G) + 1$ for every Halin graph G with $\Delta(G) \geq 5$ [10] and Meng, Guo and Su that $\chi_i(G) \leq \Delta(G) + 2$ for every Halin graph G with $\Delta(G) = 4$ [7] (it is known, by a result of Maydanskiy [6], that if G is subcubic, then $\chi_i(G) \leq \Delta(G) + 2$).

For this class of graph, we prove the following results:

Lemma 7 *If G is a Halin graph then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 1, 7\}$.*

Lemma 8 *If G is a Halin graph then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 2, 6\}$.*

By Lemmas 7 and 8, we thus get:

Theorem 9 *If G is a Halin graph then*

$$\begin{cases} \chi_i^\ell(G) \leq 6, & \textit{if } \Delta(G) \in \{3,4\}, \\ \chi_i^\ell(G) \leq 7, & \textit{if } \Delta(G) = 5, \\ \chi_i^\ell(G) = \Delta(G) + 1, & \textit{otherwise.} \end{cases}$$

A *cactus* is a (planar) graph such that every vertex belongs to at most one cycle. We prove the following:

Theorem 10 *If G is a cactus then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 1, 8\}$. Moreover, if $\Delta(G) \in \{3,4\}$ then $\chi_i^\ell(G) \leq \Delta(G) + 2$.*

Apart from improving, if possible, some of the above given bounds, we propose the following open questions:

1. What is the best upper bound on the list incidence chromatic number of subcubic graphs? (By Theorem 4, we know that this bound is at most 8.)
2. What is the value of $\chi_i^\ell(K_n)$? (By Proposition 1, we know that this value is at most $3n - 4$.)
3. Which classes of graphs satisfy the incidence version of the list colouring conjecture, that is, for which graphs G do we have $\chi_i^\ell(G) = \chi_i(G)$? (By Proposition 1 and Theorem 4, we know that this equality holds for every tree.)

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On a conjecture of Barát and Thomassen

Julien Bensmail [1], *Ararat Harutyunyan* [2], *Tien Nam Le* [3], *Martin Merker* [1] and *Stéphan Thomassé* [3]

[1] Technical University of Denmark
[2] Mathematical Institute, University of Toulouse (Paul Sabatier)
[3] LIP, ENS de Lyon

Extended Abstract

Given a graph H, we say that G has an H-*decomposition* if there exists a partition of $E(G)$ into parts isomorphic to H. One necessary condition for the existence of an H-decomposition is of course that $|E(H)|$ divides $|E(G)|$. Since this condition is obviously not sufficient in general, it is a natural question to ask what additional conditions guarantee the existence of an H-decomposition. In 2006, Barát and Thomassen [2] considered decompositions of graphs into trees and conjectured that sufficiently large edge-connectivity may be one such additional sufficient condition. More precisely, they conjectured the following.

Conjecture 1 [2] *For any tree T on m edges, there exists an integer k_T such that every k_T-edge-connected graph with size divisible by m has a T-decomposition.*

Conjecture 1 has attracted growing attention since its introduction, and was mainly verified for T being among particular families of trees. These families include stars [9], some bistars [1, 11], and paths of particular length [5, 7, 8, 10]. Very recently, some breakthrough results were obtained by Merker [6], who proved the conjecture for T being any tree of diameter at most 4 (hence covering some of the results above), and by Botler, Mota, Oshiro, and Wakabayashi [4], who proved the conjecture for T being any path. The latter result was recently improved by Bensmail, Harutyunyan, Le, and Thomassé [3], who showed that, for path-decompositions, large minimum degree is a sufficient condition provided the graph is 24-edge-connected.

We verify Conjecture 1 for every tree T, hence settling the conjecture in the affirmative.

Theorem 2 (Bensmail, Harutyunyan, Merker, Le, Thomassé '16+) *The Barát-Thomassen conjecture is true.*

Our proof uses a mixture of structural and probabilistic techniques.

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Minimally LS_+-imperfect claw-free graphs

Silvia Bianchi [1], *Mariana Escalante* [1] *Graciela Nasini* [1] and *Annegret Wagler* [2]

[1] CONICET and FCEIA, Universidad Nacional de Rosario, Argentina
[2] LIMOS (UMR 6158 CNRS), University Blaise Pascal, Clermont-Ferrand, France

Extended Abstract

Context The context of this work is the study of the stable set polytope, some of its linear and semi-definite relaxations, and graph classes for which certain relaxations are tight. The *stable set polytope* STAB(G) of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all stable sets of G. Two canonical relaxations of STAB(G) are the *edge constraint stable set polytope*

$$\text{ESTAB}(G) = \{\mathbf{x} \in [0,1]^V : x_i + x_j \leq 1, ij \in E\},$$

and the *clique constraint stable set polytope*

$$\text{QSTAB}(G) = \{\mathbf{x} \in [0,1]^V : x(Q) = \sum_{i \in Q} x_i \leq 1, \; Q \subseteq V \text{ clique}\}.$$

We have STAB(G) $\subseteq$ QSTAB(G) $\subseteq$ ESTAB(G) for any graph, where STAB(G) equals ESTAB(G) for bipartite graphs, and QSTAB(G) for perfect graphs only [5].

According to a famous characterization achieved by Chudnovsky et al. [4], perfect graphs are precisely the graphs without chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$. This shows that odd holes and odd antiholes are the only minimally imperfect graphs.

As common generalization of cliques and odd (anti)holes, an *antiweb* A_n^k is a graph with n nodes $0, \ldots, n-1$ and edges ij if and only if $k \leq |i-j| \leq n-k$ and $i \neq j$. Inequalities associated with the join of some antiwebs $A_1, \ldots, A_k$ and a clique Q

$$\sum_{i \leq k} \frac{1}{\alpha(A_i)} x(A_i) + x(Q) \leq 1,$$

are called *joined antiweb constraints* (where $\alpha(G)$ denotes the stability number of G and the inequality is scaled to have right hand side 1). We denote the linear relaxation of STAB(G) obtained by all joined antiweb constraints by ASTAB*(G). By construction, we see that

$$\text{STAB}(G) \subseteq \text{ASTAB}^*(G) \subseteq \text{QSTAB}(G) \subseteq \text{ESTAB}(G).$$

Lovász and Schrijver introduced in [11] the PSD-operator LS_+ which, applied to ESTAB(G), generates a positive semi-definite relaxation $LS_+(G)$ of STAB(G) satisfying

$$\text{STAB}(G) \subseteq LS_+(G) \subseteq \text{ASTAB}^*(G).$$

Graphs G with STAB(G) $= LS_+(G)$ are called *LS_+-perfect*, and all other graphs *LS_+-imperfect*. A conjecture has been recently proposed in [2], which can be equivalently reformulated as follows [9]:

Conjecture 1 (LS_+-Perfect Graph Conjecture) *A graph G is LS_+-perfect if and only if $LS_+(G) = \text{ASTAB}^*(G)$.*

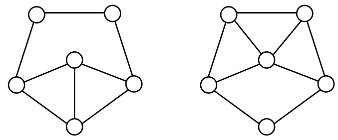

Figure 1: The graphs G_{LT} (on the left) and G_{EMN} (on the right).

Minimally LS_+-imperfect graphs One way to verify Conjecture 1 for a class $\mathcal{G}$ of graphs is to show that all facet-inducing graphs G in $\mathcal{G}$ different from cliques, antiwebs, and their complete joins are LS_+-imperfect, i.e., contain a minimally LS_+-imperfect subgraph. The two smallest such graphs, found by [7] and [10], are depicted in Figure 1.

Further LS_+-imperfect graphs can be obtained by applying operations preserving LS_+-imperfection.

In [10], the *stretching* of a node v is introduced as follows: Partition its neighborhood $N(v)$ into two nonempty, disjoint sets A_1 and A_2 (so $A_1 \cup A_2 = N(v)$, and $A_1 \cap A_2 = \emptyset$). A stretching of v is obtained by replacing v by two adjacent nodes v_1 and v_2, joining v_i with every node in A_i for $i \in \{1, 2\}$, and either subdividing the edge v_1v_2 by one node w (S_0-stretching) or subdividing every edge between v_2 and A_2 with one node (δ-stretching).

In [10] it is shown that both node stretchings preserve LS_+-imperfection. Hence, all stretchings of G_{LT} and G_{EMN} are LS_+-imperfect, see Figure 2 for some examples.

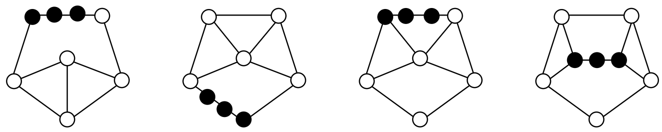

Figure 2: Some S_0-stretchings (v_1, w, v_2 in black) of G_{LT} and G_{EMN}.

In order to verify Conjecture 1 for line graphs, a characterization of minimally LS_+-imperfect line graphs is given in [9]. The proof relies on a result of Edmonds & Pulleyblank [6] who showed that a line graph $L(H)$ is facet-defining if and only if H is a 2-connected hypomatchable graph (that is, for all nodes v of H, $H - v$ admits a perfect matching). Such graphs H have an ear decomposition $H_0, H_1, \ldots, H_k = H$ where H_0 is an odd hole and H_i is obtained from H_{i-1} by adding an odd path (ear) between distinct nodes of H_{i-1}. In [9], it is shown that the line graph $L(H_1)$ of any ear decomposition equals G_{LT}, G_{EMN} or one of its S_0-stretchings and that these graphs are the only minimally LS_+-imperfect line graphs.

In order to verify Conjecture 1 for webs $W_n^k = \overline{A}_n^k$, it is shown in [8] that all imperfect not minimally imperfect webs with stability number 2 contain G_{EMN} and all webs W_n^2 different from W_7^2, W_{10}^2, some stretching of G_{LT}. Furthermore, all other webs contain some LS_+-imperfect $W_{n'}^2$. Hence, the only minimally LS_+-imperfect web is W_{10}^2.

Moreover, G_{LT} and G_{EMN} are generalized in a different way by [1] who studied graphs G with $\alpha(G) = 2$ such that $G - v$ is an odd antihole for some node v and showed that G is minimally LS_+-imperfect if and only if v is not completely joined to $G - v$.

Minimally LS_+-imperfect claw-free graphs Note that all 3 previously mentioned graph classes, i.e. line graphs, webs and graphs G with $\alpha(G) = 2$, are subclasses of claw-free graphs. In [3], Conjecture 1 has been verified for claw-free graphs. Here, we exhibit that the proofs presented in [3] imply also a characterization of the minimally LS_+-imperfect claw-free graphs.

For that, we firstly consider quasi-line graphs (i.e., graphs where the neighborhood of any node splits into two cliques). Quasi-line graphs clearly constitute an intermediate class between line graphs and claw-free graphs. In addition, they contain all webs, but no complete joins of odd (anti)holes with a further node. With the help of the results in [3], one can prove:

Theorem 2 *Every LS_+-imperfect quasi-line graph contains the web W_{10}^2 or an LS_+-imperfect line graph.*

For claw-free but not quasi-line graphs, it turns out that webs do not play a role, but instead the minimally LS_+-imperfect graphs G with $\alpha(G) = 2$. Here, one can prove with the help of the results in [3]:

Theorem 3 *Any LS_+-imperfect claw-free but not quasi-line graph contains an LS_+-imperfect graph G with $\alpha(G) = 2$ or an LS_+-imperfect line graph.*

Therefore, taking all the previous results into account, we obtain the following characterization of minimally LS_+-imperfect claw-free graphs:

Theorem 4 *A claw-free graph G is minimally LS_+-imperfect if and only if*

- *G has $\alpha(G) = 2$ such that $G - v$ is an odd antihole for some node v, not completely joined to $G - v$,*
- *G is the web W_{10}^2,*
- *G is an LS_+-imperfect line graph (i.e., G_{LT}, G_{EMN} or one of its S_0-stretchings).*

Conclusions and future lines of research The above results imply particularly that all other minimally LS_+-imperfect graphs contain a claw. The smallest such graph G_{BENW} is depicted in Figure 3(a). In fact, we have:

Theorem 5 *G_{BENW} is the only minimally LS_+-imperfect graph with 7 nodes.*

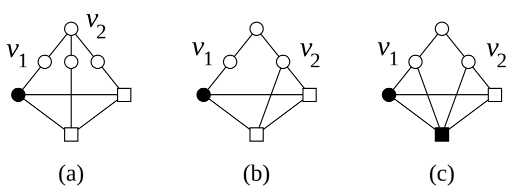

Figure 3: Some stretchings of K_4: (a) a δ-stretching, (b) an S_0-stretching, (c) an S_1-stretching (in all three cases, black-filled nodes belong to A_1, squared nodes belong to A_2).

Note that G_{BENW} can be obtained as δ-stretching of K_4 (Figure 3(a)). Hence, G_{BENW} is an example of a graph where this operation creates a minimally LS_+-imperfect graph, starting from an LS_+-perfect graph. Similarly, G_{LT} can be obtained as S_0-stretching of K_4 (Figure 3(b)), and G_{EMN} by applying a more general operation[1], the S_1-stretching, to K_4 (Figure 3(c)).

Hence, we shall investigate under which conditions minimally LS_+-imperfect graphs can be generated by one of several possible operations preserving LS_+-imperfection, but also when such operations create new minimally LS_+-imperfect graphs starting from a facet-defining LS_+-perfect graph. The study of minimally LS_+-imperfect graphs can, in turn, open new perspectives towards verifying or falsifying Conjecture 1.

[1] In [2], the S_i-stretching of a node was introduced as generalization of the S_0-stretching where the two node subsets A_1 and A_2 intersect in a clique of size i.

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Homomorphisms from graphs to integers

Jan Bok and *Jaroslav Nešetřil*

Computer Science Institute of Charles University, Prague

Extended Abstract

Our work deals with homomorphisms from graphs (simple, connected, rooted and finite) to integers and its generalizations. More precisely, we use the following definition:

A *Lipschitz mapping* of graph $G = (V, E)$ with root $v_0 \in V$ is a mapping of V to $\mathbb{Z}$, such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ holds that $|f(u) - f(v)| \leq 1$.

This definition was motivated by papers [3] and [4] and was introduced in mentioned form by Loebl, Nešetřil and Reed [1].

One can view such mapping as a homomorphism from graph to an infinite path with a loop added to each vertex. This definition generalizes the concept of random walk on path into graph-indexed random walk and its slightly different version was originally motivated by a problem in statistical physics [3].

We will be interested in two parameters regarding the Lipschitz mappings. A *range* of Lipschitz mapping f of the graph G is the size of set $\{x|x = f(v) \text{ for some } v \in V\}$. An *average range* of the graph G is an arithmetic average of ranges of all Lipschitz mappings of G.

We would like to point out that neither a range nor an average range depend on the choice of the root. Also, the reason for using the rooted graphs is purely technical since we would otherwise get an infinitely many mappings. The mappings f with $f(v_0) \neq 0$ would then be just shifted mappings.

Main conjecture on an average range states that paths are extremal with regard to this parameter. Formally:

Conjecture 1 (Loebl, Nešetřil, Reed 2003 [1]) *The maximum value of an average range over all n-vertex graphs G is achieved by P_n.*

A new result was published recently in this direction with the theorem partially solving the conjecture:

Theorem 2 (Wu, Yaokun and Xu, Zeying and Zhu, Yinfeng [2]) *The maximum value of $a(T)$ over all trees T on n vertices is achieved by P_n.*

Our main result is the generalization of this theorem to all pseudotrees – graphs with at most one cycle.

Theorem 3 *The maximum value of an average range over all pseudotrees on n vertices is achieved by P_n.*

We prove this theorem by introducing the operation which transforms a unicyclic graph into *dragon graph* in finitely many steps. A dragon graph is a graph obtained by joining a cycle graph to a path graph by an edge. We show that this operation does not decrease an average range. We then prove that an average range for the dragon graph D on n vertices is lesser or equal to an average range of P_n.

We prove an explicit formula for an average range of cycles as a by-product:

Theorem 4 *An average range of the cycle graph C_n is $\frac{3^n+(-1)^n}{2\cdot\binom{n}{0}_2}$, where $\binom{n}{0}_2$ stands for the central trinomial coefficient.*

We also show some technical lemmas that can be useful in strengthening our result to cacti – graphs with each edge contained in at most one cycle.

The second part of our talk is aimed on issues concerning the M-Lipschitz mappings which are the natural generalization of our main definition. A *M-Lipschitz mapping* of graph $G = (V, E)$ with root $v_0 \in V$ is a mapping of V to $\mathbb{Z}$, such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ holds that $|f(u) - f(v)| \leq M$.

The first problem deals with the *maximum range* parameter. That is the maximum of ranges over all M-Lipschitz mappings. We show that it is tightly connected to a diameter.

Theorem 5 *For every graph G with diameter d, maximum range over all M-Lipschitz mappings of G is equal to $M \cdot (d + 1)$.*

The last topic of our talk is the problem of *partial M-Lipschitz mapping extension* – M-LipExt. M-LipExt gets as an input a graph G and some subset H of the vertices of G together with the function $f_H : H \to \mathbb{Z}$. Our task is to determine if there is an M-Lipschitz mapping f of G such that $f_H \subseteq f$.

We show the quadratic-time algorithm for M-LipExt assuming that the input graph is a tree.

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Packing coloring of circular ladders and generalised $H-$graphs

Daouya LAÏCHE [1], Éric SOPENA [2,3] and Isma BOUCHEMAKH [1]

[1] L'IFORCE, University of Sciences and Technology Houari Boumediene (USTHB)
[2] Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France
[3] CNRS, LaBRI, UMR5800, F-33400 Talence, France

Extended Abstract

All the graphs we consider are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. The *distance* $d_G(u, v)$ between vertices u and v in G is the length (number of edges) of a shortest path joining u and v. We denote by P_n, $n \geq 1$, the path of order n and by C_n, $n \geq 3$, the cycle of order n.

A *packing k-coloring* of G is a mapping $\pi : V(G) \to \{1, \ldots, k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d_G(u, v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k-coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$. This type of coloring has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [6, 7] under the name *broadcast coloring* and has been studied by several authors in recent years (see for instance [1, 2, 3, 4, 5, 9, 10, 11]).

In this work, we determine some exact values, or upper and lower bounds, of the packing chromatic number of some families of graphs, namely circular ladder graphs, H-graphs and generalized H-graphs.

We first recall a few known results. The following proposition, which states that having packing chromatic number at most k is a hereditary property, will be useful in the sequel:

Proposition 1 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [7]) *If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.*

The packing chromatic numbers of paths and cycles have been determined by Goddard *et al.* [7]:

Theorem 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [7])

- $\chi_\rho(P_n) = 2$ *if* $n \in \{2, 3\}$,
- $\chi_\rho(P_n) = 3$ *if* $n \geq 4$,
- $\chi_\rho(C_n) = 3$ *if* $n = 3$ *or* $n \equiv 0 \pmod 4$,
- $\chi_\rho(C_n) = 4$ *if* $n \geq 5$ *and* $n \equiv 1, 2, 3 \pmod 4$.

The *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbor to every vertex of G. We call such a degree-one neighbor a *pendant vertex* or a *pendant neighbor*. In [8], we have determined in particular the packing chromatic number of coronae of cycles:

Theorem 3 (Laïche, Bouchemakh, Sopena [8]) *The packing chromatic number of the corona graph $C_n \odot K_1$ is given by:*

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

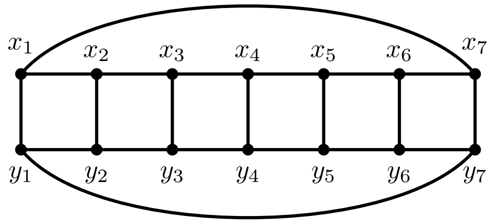

Figure 1: The circular ladder CL_7

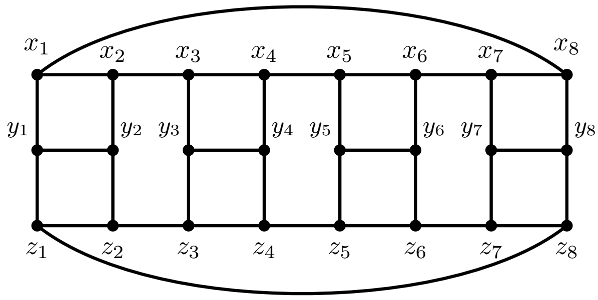

Figure 2: The H-graph $H(4)$

Recall that the Cartesian product $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, two vertices (u, u') and (v, v') being adjacent if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

The *circular ladder* CL_n of length $n \geq 3$ is the Cartesian product $CL_n = C_n \,\square\, K_2$. Figure 1 depicts the circular ladder CL_7. Note that the circular ladder CL_n contains the corona graph $C_n \odot K_1$. Therefore, by Proposition 1, Theorem 3 provides a lower bound for the packing chromatic number of circular ladders.

William and Roy [12] proved that the packing chromatic number of a circular ladder of length $n = 6q$, $q \geq 1$, is at most 5. Our following result determines the packing chromatic number of any circular ladder:

Theorem 4 *For every integer $n \geq 3$,*

$$\chi_\rho(CL_n) = \begin{cases} 5 & \text{if } n \text{ is even and } n \geq 16, \text{ or } n \in \{6, 10, 12\}, \\ 6 & \text{otherwise.} \end{cases}$$

The *H-graph* $H(r)$, is the 3-regular graph of order $3n$, $n = 2r$, with vertex set

$$V(H(r)) = \{x_i : 1 \leq i \leq n\} \,\cup\, \{y_i : 1 \leq i \leq n\} \,\cup\, \{z_i : 1 \leq i \leq n\},$$

and edge set

$$\begin{aligned} E(H(r)) \;=\; & \{(x_i, x_{i+1}),\ (z_i, z_{i+1}) : 1 \leq i \leq n-1\} \,\cup\, \{(x_n, x_1),\ (z_n, z_1)\} \\ & \cup\, \{(y_{2i-1}, y_{2i}) : 1 \leq i \leq r\} \,\cup\, \{((x_i, y_i), (y_i, z_i) : 1 \leq i \leq n\}. \end{aligned}$$

Figure 2 depicts the H-graph $H(4)$.

William and Roy [12] proved that the packing chromatic number of any H-graph $H(r)$ with $r \geq 4$, r even, is at most 5. Again, we complete their result as follows:

Theorem 5 *For every integer $r \geq 2$, $\chi_\rho(H(r)) = 5$ if r is even and $6 \leq \chi_\rho(H(r)) \leq 7$ if n is odd.*

We know consider a natural extension of H-graphs. The *generalized H-graph* $H^\ell(r)$ with ℓ levels, $\ell \geq 1$, is the 3-regular graph of order $n(\ell+2)$, $n = 2r$, with vertex set

$$V(H^\ell(r)) = \{x_j^i : 0 \leq i \leq \ell+1, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(H(r)) &= \{(x_j^0, x_{j+1}^0),\ (x_j^{\ell+1}, x_{j+1}^{\ell+1}) : 1 \leq j \leq n-1\} \cup \{(x_n^0, x_1^0),\ (x_n^{\ell+1}, x_1^{\ell+1})\} \\ &\cup \{(x_{2j-1}^i, x_{2j}^i) : 1 \leq i \leq \ell,\ 1 \leq j \leq r\} \\ &\cup \{(x_j^i, x_j^{i+1}) : 0 \leq i \leq \ell,\ 1 \leq j \leq n\}. \end{aligned}$$

Figure 3 depicts the generalized H-graph with three levels $H^3(4)$.

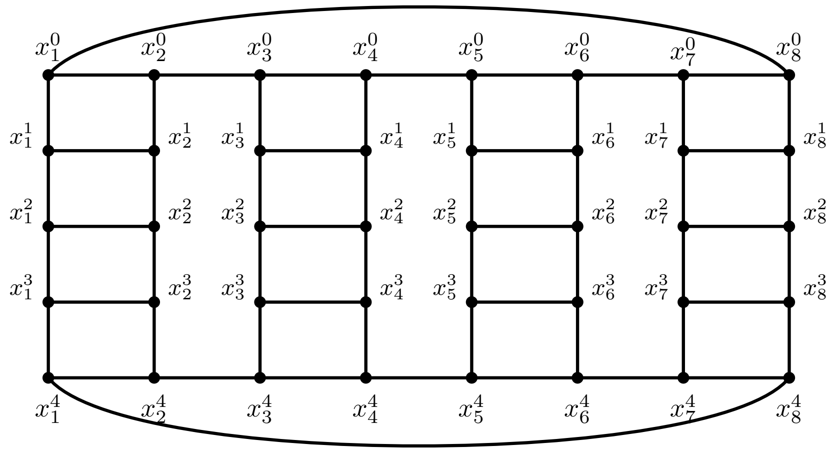

Figure 3: The generalized H-graph $H^3(4)$

Observe that any H-graph is a generalized H-graph with one level. For generalized H-graphs with two levels, we have the following:

Theorem 6 *For every integer $r \geq 2$,*

$$\chi_\rho(H^2(r)) = \begin{cases} 7 & \text{if } r \in \{2,4,7,8,11\}, \\ 6 & \text{otherwise.} \end{cases}$$

Finally, for generalized H-graphs with more than two levels, we have the following:

Theorem 7 *For every integers $r \geq 2$ and $\ell \geq 3$,*

$$\chi_\rho(H^\ell(r)) = \begin{cases} 5 & \text{if } r \text{ is even,} \\ 6 & \text{otherwise.} \end{cases}$$

Finally, we propose the following directions for future research:

1. In Theorem 5, we proved that $6 \leq \chi_\rho(H(r)) \leq 7$, for every odd r. What is the exact value of $\chi_\rho(H(r))$ in this case?
2. It would be interesting to study the packing coloring of some other classes of graphs such as, for instance, cactuses, triangulated graphs, outerplanar or planar graphs...

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A Vizing-like theorem for union vertex-distinguishing edge coloring

Nicolas Bousquet, *Antoine Dailly*, *Éric Duchêne*, *Hamamache Kheddouci* and *Aline Parreau*

LIRIS, University of Lyon, France

Extended Abstract

A vertex-distinguishing coloring of a graph G consists in an edge or a vertex coloring (not necessarily proper) of G leading to a labeling of the vertices of G, where all the vertices are distinguished by their labels.

There are several possible rules for both the coloring and the labeling. For instance, in a *set irregular edge coloring* [5], the label of a vertex is the union of the colors of its adjacent edges. Other rules for the labeling of a vertex from an edge coloring have been studied: the multiset of its adjacent colors [1], their sum [3], product or difference [6] (for those three rules, the colors must be integers)... The variant where the edge coloring is proper has also been studied [2]. If the vertices are colored, we can define the *identifying coloring* [4], in which each vertex is assigned a label corresponding to the union of its closed neighbourhood colors.

Motivated by a generalization of the set irregular edge coloring problem, we introduce a variant of the problem: to each edge is associated a nonempty set of colors. Given a simple graph G, a *k-coloring* of G is a function $f : E(G) \to 2^{\{1\dots,k\}}$ where every edge is labeled using a non-empty subset of $\{1, \dots, k\}$. For any k-coloring f of G, we define, for every vertex u, the set $id_f(u)$ as follows:

$$id_f(u) = \bigcup_{v \text{ s.t. } uv \in E} f(uv).$$

If the context is clear, we will simply write $id(u)$ for $id_f(u)$. A k-coloring f is *union vertex-distinguishing* if, for all distinct u, v in $V(G)$, $id(u) \neq id(v)$. Figure 1 shows an example of such a 4-coloring. For a given graph G, we denote by $\chi_\cup(G)$ the smallest integer such that there exists a union vertex-distinguishing coloring of G. The union vertex-distinguishing edge coloring being defined only on graphs without any connected component of size 1 or 2, we will only consider such graphs. It is easy to notice that such graphs admit a valid union vertex-distinguishing edge coloring.

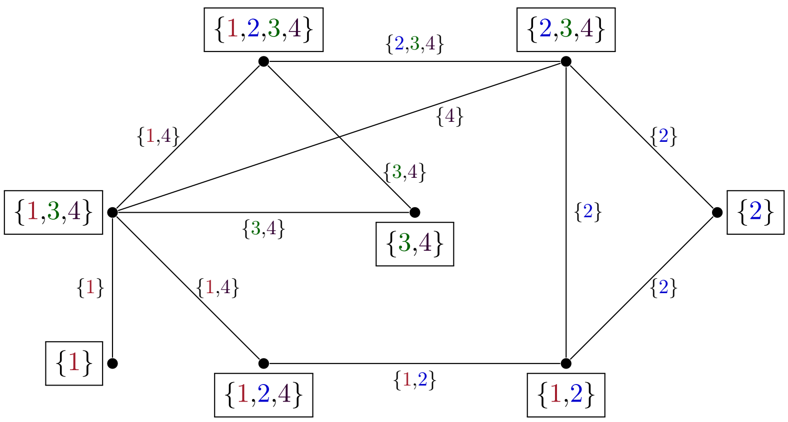

Figure 1: An example of a vertex-distinguishing edge coloring.

We have both lower and upper bounds for $\chi_\cup$:

Proposition 1. *For any graph G with no connected component of size 1 or 2, we have the following inequalities:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \min(\chi_S(G), \chi_{id}(G))$$

where $\chi_S(G)$ is the set irregular edge coloring number of G, and $\chi_{id}(G)$ is the identifying number of G.

The lower bound comes from the fact that, from k colors, one can have at most $2^k - 1$ labels, since labels are nonempty subsets of $\{1, \ldots, k\}$. The upper bound comes from the relationship between our parameter $\chi_\cup$ and other vertex-distinguishing parameters: a set irregular edge coloring is a union vertex-distinguishing edge coloring where only singletons are allowed, and any identifying coloring induces a valid union vertex-distinguishing edge coloring.

We say that a graph G is *optimally colored* if $\chi_\cup(G) = \lceil \log_2(|V| + 1) \rceil$. For example, the graph shown on Figure 1 is optimally colored, since it has 8 vertices and the coloring uses 4 colors. We will see that some classes of graphs can be optimally colored, and that any graph with no connected component of size 1 or 2 admits a union vertex-distinguighing edge coloring with at most 2 more colors than the optimal bound. This is the main result of our paper.

Theorem 2. *For any graph G, we have the following property:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \lceil \log_2(|V(G)| + 1) \rceil + 2$$

Sketch of proof. For any graph G, if H is a graph such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$, then we call H an $edge-subgraph$ of G.

Our proof follows the following schema:

1. We prove that for any edge-subgraph H of a graph G, we have $\chi_\cup(G) \leq \chi_\cup(H) + 1$.

2. We then prove that any graph G admits an edge-subgraph isomorphic to a disjoint union of stars subdivided at most once.

3. We now prove that stars subdivided at most once can be optimally colored.

4. Finally, we prove that a disjoint union of graphs that can be separately optimally colored can be colored together using at most the optimal number of colors plus one.

Thus, Theorem 2 is proved.

We will present sketches of proofs for each item:

The first point is easily proved: if we assign to each edge of $E(G) \backslash E(H)$ a color that has not been used in the coloring of H, then the resulting coloring will be union vertex-distinguishing.

The second point is proved by contradiction. We study the smallest graph G which does not admit a disjoint union of stars subdivided at most once as an edge-subgraph. By minimality, if we take u a vertex of maximum degree, then for every neighbour v of u, the component of v in $G \setminus (u, v)$ is reduced to a single vertex or an edge. This implies that the component of v in $G \setminus (u, v)$, as well as in G, is a star subdivided at most once, a contradiction.

The third point is proved by finding a valid union vertex-distinguishing edge coloring of any star subdivided at most once. There are two cases according to there are $2^k - 1$ vertices of degree 2 in the neighbourhood of the central vertex or not. In both cases, such a coloring can be constructed.

The fourth point is a generalization of a smaller result: let H_1 and H_2 be two graphs such that $2^k \leq |V(H_1)|, |V(H_2)| \leq 2^{k+1} - 1$. If both H_1 and H_2 can be optimally colored, then their disjoint union $H_1 \cup H_2$ can be optimally colored. Indeed, we only need to add the color $k + 1$ to each edge of H_2. The resulting coloring will be union vertex-distinguishing and optimal for $H_1 \cup H_2$.

Thus, if we have a family of graphs that can be separately optimally colored, we begin by grouping the graphs H_i, H_j verifying $2^k \leq |V(H_i)|, |V(H_j)| \leq 2^{k+1} - 1$ for a certain k until no graph satisfies this anymore. We then use induction to join the remaining graphs, again by using a color that was not used in any of the graphs that we join. The plus one comes from the fact that if we have two graphs H_i and H_j such that $2^{k_i} \leq |V(H_i)| \leq 2^{k_i+1} - 1 <$

$2^{k_j} \leq |V(H_j)| \leq 2^{k_j+1} - 1$ and verify $|V(H_i)| + |V(H_j)| < 2^{k_j+1}$, then we use $k_j + 1$ colors and thus the coloring is not optimal. □

As seen in the proof of Theorem 2, we actually "lose" a value on the bound by proving the second point. We thus conjecture that the upper bound can be improved:

Conjecture 3. *For any graph G with no connected component of size 1 or 2, we have:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq \chi_\cup(G) \leq \lceil \log_2(|V(G)| + 1) \rceil + 1$$

There are several possibilities to try and prove or disprove this conjecture: a first idea would be to study the exact value of $\chi_\cup$ for trees or stars subdivided at most once. If graphs from one of these two classes are optimally colorable, then the conjecture is true.

In addition, we proved that paths, cycles and complete binary trees can be optimally colored, which validates the above conjecture for these classes and all the graphs that contain a graph of one of these classes as an edge subgraph (e.g. Hamiltonian graphs).

Theorem 4. *We have* $\chi_\cup(G) = \lceil \log_2(|V(G)|+1) \rceil$ *if* G *belongs to one of the following classes of graphs:*

1. *Paths of length greater than 2;*

2. *Cycles of length greater than 3 and different from 7;*

3. *Complete binary trees.*

We have $\chi_\cup(G) = \lceil \log_2(|V(G)| + 1) \rceil + 1$ *for the following classes of graphs:*

4. *Cycles of length 3 and 7;*

5. *The complete graphs of order* $2^k - 1$.

Sketch of proof. For the first point, we actually prove a slightly larger statement: for $n \geq 3$, there exists an optimal union vertex-distinguishing m-coloring for a path $P_n = (u_1, \dots, u_n)$ such that $id(u_1) = \{1\}$, $id(u_n) = \{m\}$ and the only vertex that satisfies $id(u_j) = \{1, m\}$ is u_{n-1}. This is proved by induction on n. If $n = 2^k + \ell$ where $0 \leq \ell \leq 2^k - 1$, we use the optimal colorings of P_{2^k-1} and P_ℓ to create an optimal coloring of P_n.

For the second point, we have two cases: either $n = 2^k - 1$ or not. In the latter case, we connect the first and last vertices of the path P_n to create an optimal coloring of C_n. The former case is proved by induction on k.

For the third point, we use induction on the height of the complete binary tree.

For the fourth point, it is easily seen that both C_3 and C_7 can not be optimally colored, but using respectively 3 and 4 colors a valid vertex-distinguishing edge coloring can be obtained.

The fifth point is proven by contradiction: if complete graphs of order $2^k - 1$ could be optimally colored, then two vertices u and v would be identified each by a different singleton $\{i\}$ and $\{j\}$. This is contradictory, since the edge uv would have to be colored with both the singleton $\{i\}$ and the singleton $\{j\}$. □

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On the proper connection number and the 2-proper connection number of graphs

Christoph Brause [1], *Trung Duy Doan* [1,2] and *Ingo Schiermeyer* [1]

[1] Institute of Discrete Mathematics and Algebra, TU Bergakademie Freiberg, Germany
[2] School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Vietnam

Extended Abstract

1 Introduction

We use [4] for terminology and notation not defined here and consider simple and undirected graphs only.

The concept of proper connections in graphs is an extension of proper colourings and is motivated by rainbow connections of graphs. Andrews et al. [1] and, independently, Borozan et al. [3] introduced the concept as follows:

An edge-coloured graph G is called *k-properly connected* if every two vertices $u, v \in V(G)$ are connected by k internally vertex disjoint paths whose edges are properly coloured. The proper connection number $pc_k(G)$ is the smallest number of colours needed to colour a k-connected graph G k-properly connected. Simplifying notion, we say a 1-properly connected graph is properly connected and we write $pc(G)$ instead of $pc_1(G)$.

Further we say, an edge-colouring c has the *strong property* if for every two vertices $u, v \in V(G)$ there exists two properly coloured paths $P_1 : u = w_1w_2 \dots w_k = v$ and $P_2 : u = z_1z_2 \dots z_l = v$ such that $c(w_1w_2) \neq c(z_1z_2)$ and $c(w_{k-1}w_k) \neq c(z_{l-1}z_l)$. We note that $pc(G) = 1$ if and only if G is complete [3].

For example, colouring the edges of a cycle C alternatingly by two colours makes C properly connected. If $m(C)$ is even, then this colouring has the strong property as well as it makes C 2-properly connected, while if $m(C)$ is odd, then there exists two vertices in C such that there exists exactly one properly coloured path between them. On the other hand, if $m(C)$ is odd uv is any edge, then colouring the edges of $C - uv$ alternatingly by two colours and uv by a third one, makes G properly 2-connected and the colouring has the strong property. Therefore, $pc(C) = 2$, $pc_2(C) = 2 + (m(C) \mod 2)$.

For simplifying notation, let $[k]$ be the set $\{1, 2, \dots, k\}$ for some positive integer k. Following common notation, we say G contains an *induced subgraph* F if there is a vertex subset $U \subseteq V(G)$ such that $G[U] \cong F$. Therefore, G is *F-free* ($\mathcal{F}$-free) if and only if G contains F (all graphs of $\mathcal{F}$) not as an induced subgraph. Let $S_{i,j,k}$ be the graph consisting of three induced paths of lengths i,j, and k with a common initial vertex, and $\mathcal{S}$ be the set of graphs whose every component is of the form $S_{i,j,k}$ for some $0 \leq i \leq j \leq k$.

In many fields of graph theory, forbidden subgraphs and the connectivity of a graph play an important role. In [2], Bedrossian characterized pairs of forbidden subgraphs for 2-connected graphs implying hamiltonicity. Thus, since every noncomplete, hamiltonian graph has proper connection number 2 [3], his characterization is the starting point for our work to find sufficient conditions in terms of connectivity and forbidden subgraphs such that $pc(G) = 2$ holds for a graph G. We note that all pairs in Bedrossian's characterization contain the claw. Our first result improves the above observation on the proper connection number of a graph by forbidding the claw only.

Theorem 1 *Let G be a connected, claw-free, and noncomplete graph. Then $pc(G) = 2$.*

Next, we focus on necessary conditions on forbidden subgraphs, implying a 2-connected graph to have proper connection number 2.

Proposition 2

1. *Let $\mathcal{F}$ be a finite set of graphs. If $\mathcal{F} \cap \mathcal{S} = \emptyset$, then there exists a 2-connected, $\mathcal{F}$-free graph G such that $pc(G) = 3$.*

2. *Let $0 \leq i \leq j \leq k$. If $i \geq 3$ or $j + k \geq 15$, then there exists a 2-connected, $S_{i,j,k}$-free graph G such that $pc(G) = 3$.*

Using this characterization, it is natural to forbid $S_{i,j,k}$ with small i,j, and k, for example $P_5, S_{1,1,2}$, or $S_{1,1,6}$.

Theorem 3 *If G is a noncomplete, 2-connected, P_5-free graph, then $pc(G) = 2$.*

Theorem 4 *If G is a noncomplete, 2-connected, $S_{1,1,2}$-free graph, then $pc(G) = 2$.*

Theorem 5 *If G is a noncomplete, 2-connected, $S_{1,1,6}$-free graph of minimum degree at least 3, then $pc(G) = 2$.*

Some basic results, which are important for our proofs, make only use of the connectivity of a graph.

Theorem 6 (Borozan et al. [3]) *If G is a 2-connected graph, then there exists an edge-colouring $c : E(G) \to [3]$ having the strong property.*

Theorem 7 (Borozan et al. [3]) *If G is a 2-connected bipartite graph, then there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

The authors claim that their results still hold if one replaces 2-connectivity by 2-edge-connectivity. As a further consequence, by a result of Paulraja in [5], every 3-connected graph G has a 2-connected bipartite spanning graph. Therefore, Borozan et al. deduced the following result.

Theorem 8 (Borozan et al. [3]) *If G is a 3-connected graph, then there exists an edge-colouring $c : E(G) \to [2]$ having the strong property.*

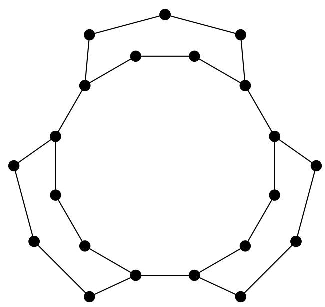

Figure 1: Graph B

There are 2-connected graphs having proper connection number 3, for example graph B in Figure 1 [3]. Since all known graphs have a 3-cut, we study the proper connection number of 3-edge-connected graphs.

Theorem 9 *If G is a 3-edge-connected and noncomplete graph, then $pc(G) = 2$.*

We note that Theorem 9 closes the gap in transforming Theorems 6, 7, and 8 to their edge-connected version.

By the results of Borozan et al. [3], one can bound $pc(G)$ for any 2-connected graph by 3 from above. Further it is known that $pc(G) = 1$ if and only if G is complete [3]. Therefore, our results contribute to a full characterization in terms of forbidden subgraphs of 2-connected graphs of proper connecting number 2.

Clearly, it makes only sense to consider the 2-proper connection number $pc_2(G)$ in graphs of connectivity 2 or larger. Therefore, one can readily observe that any proper edge-colouring in a 2-connected graph G makes G 2-properly connected. Thus, $pc_2(G) \leq \chi'(G)$ and, by Vizing's Theorem [6], $pc_2(G) \leq \Delta(G) + 1$. Using our next result, we can characterize graphs achiving equality $pc_2(G) = \Delta(G) + 1$. Further, our result shows that, differently to the proper connection number, there does not exist a constant C such that $pc_2(G) \leq C$ for all 2-connected graphs G.

Theorem 10 *If G is a 2-connected graph, different from an odd cycle, then $pc_2(G) \leq \Delta(G)$ and, for every integer $\Delta \geq 2$, there exists an infinite class of graphs of maximum degree $\Delta(G) = \Delta$ with 2-proper connection number $\Delta(G)$.*

As shown above, the 2-proper connection number of an odd cycle C is 3 but $\Delta(C) = 2$. Therefore, the next corollary follows from Theorem 10.

Lemma 11 *If G is a 2-connected graph of 2-proper connection number $\Delta(G) + 1$, then it is an odd cycle.*

Although, there exist graphs having equal 2-proper connection number and maximum degree, the difference $\Delta(G) - pc_2(G)$ can be arbitrarily large, as one can observe from our next result. Let $G_1 \Box G_2$ be the cartesian product of G_1 and G_2.

Theorem 12 *If G_1 and G_2 are two traceble graphs of order at least 2, then $pc_2(G_1 \Box G_2) = 2$.*

Therefore, $\Delta(K_n \Box K_n) = 2(n-1)$, $pc_2(K_n \Box K_n) = 2$, and $\Delta(K_n \Box K_n) - pc_2(K_n \Box K_n) = 2(n-2)$.

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On upper bounds for the independent transversal domination number

Christoph Brause [1], Michael A. Henning [2], Kenta Ozeki [3], Ingo Schiermeyer [1] and Elkin Vumar [4]

[1] Institute of Discrete Mathematics and Algebra, TU Bergakademie Freiberg, Germany
[2] Department of Pure and Applied Mathematics, University of Johannesburg, South Africa
[3] National Institute of Informatics and JST, ERATO, Kawarabayashi Large Graph Project, Japan
[4] College of Mathematics and System Science, Xinjiang University, China

Extended Abstract

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

A set $D \subseteq V(G)$ is *dominating* if $D \cap N[v] \neq \emptyset$ for all vertices $v \in V(G)$. The cardinality of a smallest dominating set is the *domination number*, $\gamma(G)$, and such a set is called a *minimum dominating set* of G, which we refer to as a $\gamma(G)$-*set* (for example, the set of black vertices in Figure 1 is a $\gamma(C_5)$-set).

A set $I \subseteq V(G)$ is *independent* if every two vertices of I are non-adjacent. The set I is *maximal* if there exists no independent set I' such that $I \subset I'$. Further, the cardinality of a largest independent set in G is the *independence number* $\alpha(G)$ of G and an independent set having cardinality $\alpha(G)$ in G is called a *maximum independent set.* We refer to a maximum independent set in G as an $\alpha(G)$-*set* (for example, the set of black vertices in Figure 2 is an $\alpha(C_5)$-set). We say that a graph G is *well-covered* if every maximal independent set in G is maximum. Clearly, a set $C \subseteq V(G)$ is independent in $\overline{G}$ if and only if it is a clique in G. Therefore, maximal and maximum cliques in G are maximal and maximum independent sets in $\overline{G}$, respectively.

Problems on dominating sets as well as on independent sets have a long history in graph theory, see [1, 6, 7]. Combining the properties of domination and independence, we have the concept of an *independent dominating set* in G, which is a set that is both dominating and independent in G. The cardinality of a minimum independent dominating set in G is the *independent domination number* of G, denoted $i(G)$ (for example, the set of black vertices in Figure 2 is a $i(C_5)$-set). We refer to [5] for a recent survey on independent dominating sets in graphs.

As a new concept combining independent and dominating sets, Hamid introduced independent transversal dominating sets in [4]. An *independent transversal dominating set* in G, abbreviated by ITD-*set*, is a dominating set that intersects every maximum independent set in G. Thus, if D is an ITD-set of G, then D is a dominating set of G and $\alpha(G) > \alpha(G - D)$. The *independent transversal domination number*, denoted by $\gamma_{\text{it}}(G)$, of G is the smallest cardinality of an ITD-set in G. An ITD-set of cardinality $\gamma_{\text{it}}(G)$ is called a *minimum independent transversal dominating set*, and is referred to as an $\gamma_{\text{it}}(G)$-*set* (for example, the set of black vertices in Figure 3 is a $\gamma_{it}(C_5)$-set).

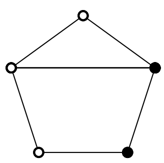

Figure 1: a $\gamma(C_5)$-set

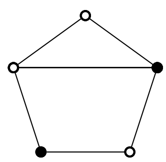

Figure 2: an $i(C_5)$-set and $\alpha(C_5)$-set

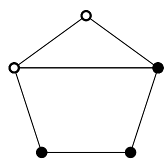

Figure 3: a $\gamma_{it}(C_5)$-set

We will see, that sets intersecting all cliques will play an important role for our results. Therefore, let us introduce the concept of clique transversals. More precisely, a *clique transver-*

sal is a set $T \subseteq V(G)$ which intersects every maximal clique in at least one vertex. The cardinality of a smallest clique transversal in a graph G, denoted by $\tau_C(G)$, is called the *clique transversal number* and any clique transversal of cardinality $\tau_C(G)$ is called a $\tau_C(G)$*-set*. Since every maximal independent set in G is a maximal clique in $\overline{G}$, every $\tau_C(\overline{G})$-set intersects every maximal independent set in G. As an immediate consequence of this observation, we have the following result.

Lemma 1 *If G is a well-covered graph, then $\gamma_{\rm it}(G) \geq \tau_C(\overline{G})$.*

The *eccentricity* of a vertex v in G, denoted by $\mathrm{ecc}_G(v)$ or simply $\mathrm{ecc}(v)$ if G is clear from context, is the maximum distance of a vertex from v in G.

The following classic result due to Ore [10] states that every graph without isolated vertices has a dominating set of cardinality at most one-half its order.

Theorem 1 (Ore [10]) *If G is a graph without isolated vertices, then $\gamma(G) \leq \frac{n(G)}{2}$.*

The *corona* $\mathrm{cor}(G)$ of a graph G, also denoted by $G \circ K_1$ in the literature, is the graph obtained from G by adding a pendant edge to each vertex of G. Payan and Xuong [11] characterized those graphs with no isolated vertex and with domination number exactly half their order.

Theorem 2 (Payan, Xuong [11]) *If G is a graph with no isolated vertex, then $\gamma(G) = \frac{n(G)}{2}$ if and only if the components of G are C_4 or cor(H) for some connected graph H.*

We note that the graphs G given in Theorem 2 that achieve equality in Ore's Theorem 1, all have the property that there exists a $\gamma(G)$-set which is also an ITD-set. A natural question is to establish an upper bound on the independent transversal domination number of a graph without isolated vertices. As observed by Hamid [4], every vertex in a complete graph K_n forms a maximum independent set in the graph, implying that $\gamma_{\rm it}(K_n) = n$ for all $n \geq 1$. Hamid [4] posed the following conjecture.

Conjecture 1 (Hamid [4]) *If G is a connected non-complete graph, then $\gamma_{\rm it}(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil$.*

Since every ITD-set in a graph G is by definition a dominating set in G, we observe that $\gamma(G) \leq \gamma_{\rm it}(G)$ always holds. Hamid observed in [4] the following upper bound on $\gamma_{\rm it}(G)$ in terms of $\gamma(G)$ and the minimum degree $\delta(G)$ of G.

Theorem 3 (Hamid [4]) *If G is a graph, then $\gamma_{\rm it}(G) \leq \gamma(G) + \delta(G)$.*

By Theorem 3, one can bound $\gamma_{\rm it}(G)$ from above by $\gamma(G) + 1$ for all graphs G having a vertex of degree 1. In particular, if G is a tree, then $\gamma_{\rm it}(G) \leq \gamma(G) + 1$. However, Hamid believed that the same upper bound holds for a larger class of graphs.

Conjecture 2 (Hamid [4]) *If G is a connected bipartite graph, then $\gamma_{\rm it}(G) \leq \gamma(G) + 1$.*

Motivated by Conjectures 1 and 2, our two immediate aims are first to disprove Conjecture 1 and second to prove Conjecture 2. Our third aim is to strengthen the upper bound of Theorem 3, and establish improved upper bounds on $\gamma_{\rm it}(G)$ for a connected graph G.

In this talk we present four infinite classes of graphs that disprove Conjecture 1. In particular, two of our classes are defined as follows:

Let $k \geq 3$ be an integer and let $H(k)$ be the graph obtained from the disjoint union of two cliques, K_k, on k vertices by adding an edge joining the two cliques.

Example 1 *For $k \geq 3$, the graph $G \cong H(k)$ of order $n(G) = 2k$ satisfies $\gamma_{\rm it}(G) = k + 1 = \frac{n(G)}{2} + 1$.*

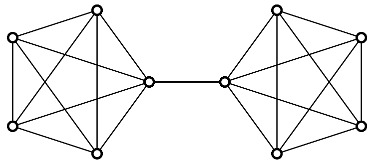

Figure 4: $H(5)$

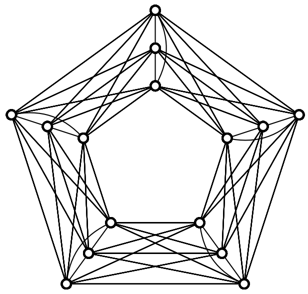

Figure 5: $C_5[K_3]$

Let H be a graph vertex disjoint from G. Therefore, the graph $G[H]$ is obtained by replacing vertices $v \in V(G)$ by disjoint sets $S(v)$ such that the graph induced by $S(v)$ is isomorphic to H and all vertices of $S(w)$ are adjacent to all vertices of $S(z)$ if and only if $wz \in E(G)$.

Example 2 *For $k \geq 1$, the graph $G \cong C_5[K_k]$ of order $n(G) = 5k$ satisfies $\gamma_{\rm it}(G) = 3k = \frac{3}{5}n(G)$.*

Furthermore, by using Kim's result on the Ramsey number $R(3,t)$ [8] and some results on clique transversals of Erdős, Gallai, and Tuza [2], we can prove the following result.

Theorem 4 *There do not exist positive constants $C < 1$ and $k \geq 2$ that guarantee that $\gamma_{\rm it}(G) \leq C \cdot n(G)$ whenever $\alpha(G) \geq k$.*

Our examples inspired us to study upper bounds on the independent transversal domination number of a general graph. The following result strengthens the result of Theorem 3 and establishes an improved upper bound on $\gamma_{\rm it}(G)$ for connected graphs G. A proof of Theorem 5 is presented in this talk.

Theorem 5 *If G is a connected graph and u is a vertex of minimum degree in G, then*

$$\gamma_{\rm it}(G) \leq \begin{cases} \delta(G)+1 & \text{if } {\rm ecc}_G(u) \leq 2, \\ \frac{n(G)}{2}+1 & \text{if } {\rm ecc}_G(u) \geq 3, \end{cases}$$

and these bounds are tight.

Theorem 5 implies that for graphs of large diameter the independent transversal domination number is bounded from above by nearly one-half the order of the graph. However, when the minimum degree is large, the diameter is small and, as our examples show, the independent transversal domination number increases.

In this talk we prove that if a graph has independence number at least one-half its order, then the independent transversal domination number is bounded from above by the domination number plus one.

Theorem 6 *If G is a graph with $\alpha(G) \geq \frac{n(G)}{2}$, then $\gamma_{\rm it}(G) \leq \gamma(G)+1$, and this bound is tight.*

As a special case of Theorem 6, we remark that Conjecture 2 is proven, since every bipartite graph has independence number at least one-half its order.

In order to prove Theorem 6, we use the concept of crown decomposition. In particular, we say that G admits a *crown decomposition* (C,H,R) if (C,H,R) is a partition of $V(G)$ such that

1. C is an independent set,
2. there is a maximum matching in $G[C,H]$ covering all vertices of H, and
3. there is no edge in G between a vertex of C and a vertex of R.

Using Hall's marriage Theorem [3], we prove that every graph with independence number at least $\frac{n(G)}{2}$ admits a crown decomposition (C,H,R). Furthermore, we show that a set intersecting all $\alpha(G[C,H])$-sets intersects all $\alpha(G)$-sets. By Kőnig's Theorem [9], we prove that if $|C|=|H|$, then there exists a $\gamma(G)$-set D and a vertex $v \in (C \cup H) \setminus D$ such that $D \cup \{v\}$ is an ITD-set of G, and, if $|C|>|H|$, then there exists a vertex v in C which is contained in every $\alpha(G[C,H])$-set. Therefore, $\gamma_{\rm it}(G) \leq \gamma(G)+1$ follows.

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On the chromatic number of $2K_2$-free graphs

Christoph Brause [1], *Bert Randerath* [2], Ingo Schiermeyer [1], and *Elkin Vumar* [3]

[1] Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, Germany
[2] Institut für Nachrichtentechnik, Technische Hochschule Köln, Germany
[3] College of Mathematics and System Science, Xinjiang University, Urumqi, China

Extended Abstract

1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Let G be a graph. An *induced subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, and $uv \in E(H)$ if and only if $uv \in E(G)$ for all $u, v \in V(H)$. Given graphs G and F we say that G *contains* F if F is isomorphic to an induced subgraph of G. We say that a graph G is F-free, if it does not contain F. For two graphs G, H we denote by $G + H$ the disjoint union and by $G \vee H$ the join of G and H, respectively.

A graph G is called *k-colourable*, if its vertices can be coloured with k colours so that adjacent vertices obtain distinct colours. The smallest k such that a given graph G is k-colourable is called its *chromatic number*, denoted by $\chi(G)$. It is well-known that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ for any graph G, where $\omega(G)$ denotes its clique number and $\Delta(G)$ its maximum degree. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A *hole* in a graph is an induced cycle of length at least four, and an *antihole* is the complement of a hole.

A family $\mathcal{G}$ of graphs is called χ-bound with binding function f if $\chi(G') \leq f(\omega(G'))$ holds whenever $G \in \mathcal{G}$ and G' is an induced subgraph of G. For a fixed graph H let $\mathcal{G}(H)$ denote the family of graphs which are H-free.

The following theorems are well known in chromatic graph theory.

Theorem 1 *(Erdős) [9]*
For any positive integers $k, l \geq 3$ there exists a graph G with girth $g(G) \geq l$ and chromatic number $\chi(G) \geq k$.

Theorem 2 *The Strong Perfect Graph Theorem [5]*
A graph is perfect if and only if it contains neither an odd hole of length at least five nor its complement.

In this paper we study the chromatic number of $2K_2$-free graphs. Our work was motivated by the following problem posed by Gyárfás.

Problem 3 *(Gyárfás [11])*
What is the order of magnitude of the smallest χ-binding function for $\mathcal{G}(2K_2)$?

One of the earliest results is due to Wagon, who has considered graphs without induced matchings.

Theorem 4 *[16] Let G be a $2K_2$-free graph with clique number $\omega(G)$. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.*

Theorem 5 *[16] The family $\mathcal{G}(pK_2)$ has an $O(\omega^{2p-2})$ χ-binding function for all $p \geq 1$.*

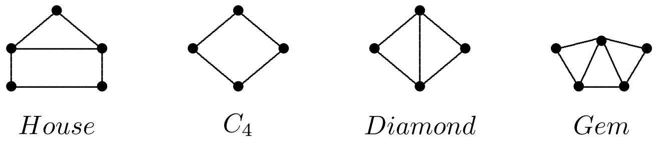

Figure 1: The graphs *House*, C_4, *Diamond*, and *Gem*.

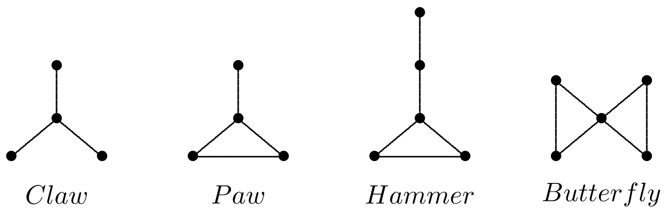

Figure 2: The graphs *Claw*, *Paw*, *Hammer*, and *Butterfly*.

2 Results for $(2K_2, H)$-free graphs

For $(2K_2, C_4)$-free graphs and $(2K_2, Diamond)$-free graphs the following results are known.

Theorem 6 *[3] Let G be a $(2K_2, C_4)$-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq \omega(G) + 1$. Here equality holds if and only if G is not a split graph.*

Theorem 7 *[2] Let G be a $(2K_2, Diamond)$-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq \omega(G) + 1$.*

Our first two results are for $(2K_2, House)$-free graphs and $(2K_2, Gem)$-free graphs.

Theorem 8 *Let G be a $(2K_2, House)$-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

The graph $K_k[C_5]$ shows that the bound $\frac{3}{2}\omega(G)$ is sharp.

Theorem 9 *Let G be a $(2K_2, Gem)$-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq 2\omega(G)$.*

Next we consider claw-free graphs, which have been studied by Chudnovsky and Seymour.

Theorem 10 *[6] Let G be a Claw-free graph with independence number $\alpha(G) \geq 3$. Then $\chi(G) \leq 2\omega(G)$.*

If G is a *claw*-free graph with independence number $\alpha(G) = 2$, then $\chi(G) \geq \frac{|V(G)|}{2}$. But $\omega(G)$ maybe of order $\sqrt{|V(G)| \cdot log|V(G)|}$ due to Kim [12].

For the class of $(2K_2, Claw)$-free graphs we have shown the following results.

Theorem 11 *Let G be a $(2K_2, Claw)$-free graph with independence number $\alpha(G) \geq 3$. Then G is perfect.*

For the proof we will make use of Ben Rebea's Lemma.

Ben Rebea's Lemma [7]. *Let G be a connected claw-free graph with $\alpha(G) \geq 3$. If G contains an odd antihole then it contains a hole of length five.*

Now we turn to $(2K_2, Claw)$-free graphs with independence number $\alpha(G) = 2$. Surprisingly we can show that there is no linear binding function.

For this purpose we consider the complement $\overline{G}$ of a graph G, which is $(K_1 + K_3)$-free and C_4-free. Suppose, $\overline{G}$ is K_3-free. Hence $g(\overline{G}) \geq 5$. Now we apply a result due to Bollobás [1] about the independence ratio of graphs with given girth. Using this, for every natural number Δ, there is a graph G with $\omega(G)/n < 2(log\Delta)/\Delta$. However, $\chi(G) \geq \frac{n}{2}$ implying $\chi(G) > \frac{\Delta}{4log\Delta} \cdot \omega(G)$.

Theorem 12 *Let $\mathcal{G}$ be the class of $(2K_2, Claw, 3K_1)$-free graphs. Then $\mathcal{G}$ contains no linear binding function.*

The proof given above admits an immediate generalization as follows.

Theorem 13 *Let H be a graph with independence number $\alpha(H) \geq 3$, and let $\mathcal{G}$ be the class of $(2K_2, H, 3K_1)$-free graphs. Then $\mathcal{G}$ contains no linear binding function.*

If we add an edge to a claw, we obtain the Paw. Paw-free graphs have been characterized.

Theorem 14 *[13] Let G be a connected Paw-free graph. Then G is K_3-free or multipartite.*

Using this characterization we can show the following result.

Theorem 15 *[13] Let G be a connected $(2K_2, Paw)$-free graph. Then G is perfect or $G \cong C_5[n_1K_1, n_2K_1, n_3K_1, n_4K_4, n_5K_1]$. In the later case, $\omega(G) = 2$ and $\chi(G) = 3$.*

Here $C_5[n_1K_1, n_2K_1, n_3K_1, n_4K_4, n_5K_1]$ denotes the composition of a $C_5 = v_1v_2v_3v_4v_5$, where each vertex v_i is replaced by n_iK_1 for $1 \leq i \leq 5$.

3 Superclasses of $2K_2$-free graphs

A structural characterization of $2K_2$-free graphs has been shown recently in [8].

Theorem 16 *A connected graph is $2K_2$-free if and only if it forbids $P_5, Butterfly$ and $Hammer$ as induced subgraphs.*

This characterization evokes the two graph classes of $(P_5, Butterfly)$-free graphs and of $(P_5, Hammer)$-free graphs, which are both superclasses of $2K_2$-free graphs and subclasses of P_5-free graphs. The class of $(P_5, Butterfly)$-free graphs has been considered in [15].

Theorem 17 *Let H be a graph such that $\mathcal{G}(H)$ has an $O(\omega^t)$ χ-binding function for some $t \geq 1$, and let G be a connected $(P_5, K_1 \vee H)$-free graph with clique number $\omega(G)$. Then G has an $O(\omega^{t+1})$ χ-binding function.*

So we can apply Theorem 5 and Theorem 17 to obtain the following result for $(P_5, Butterfly)$-free graphs.

Theorem 18 *Let G be a $(P_5, Butterfly)$-free graph, then $\chi(G) \leq c \cdot \omega^3$ for a constant c.*

Next we consider the class of $(P_5, Hammer)$-free graphs. Here we can show the following binding function.

Theorem 19 *Let G be a $(P_5, Hammer)$-free graph with clique number $\omega = \omega(G)$. Then $\chi(G) \leq \omega^2$.*

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The complexity of signed graph and edge-coloured graph homomorphisms

Richard Brewster [1], Florent Foucaud [2], Pavol Hell [3] and Reza Naserasr [4]

[1] Dept. of Mathematics and Statistics, Thompson Rivers University, Kamloops, Canada
[2] LIMOS - CNRS UMR 6158, Université Blaise Pascal, Clermont-Ferrand, France
[3] School of Computing Science, Simon Fraser University, Burnaby, Canada
[4] IRIF - CNRS UMR 8243, Université Paris Diderot - Paris 7, France

Extended Abstract

Graph homomorphisms and their variants play a fundamental role in the study of computational complexity. For example, the celebrated CSP Dichotomy Conjecture of Feder and Vardi [6] can be reformulated in terms of digraph homomorphisms or graph retractions. On the other hand, the dichotomy theorem of Hell and Nešetřil [10] shows that there are no NP-intermediate graph homomorphism problems. We study two natural types of homomorphism problems on signed graphs, one with switching and one without. For one, we prove a dichotomy theorem for a large class of signed graphs, while for the other we prove that a dichotomy theorem is equivalent to the CSP Dichotomy Conjecture.

A *signed graph* is a graph G together with a signing function $\sigma : E(G) \to \{+, -\}$. By setting $\Sigma = \sigma^{-1}(-)$, we use the notation (G, Σ) to denote this signed graph. The set Σ of negative edges is referred to as the *signature* of (G, Σ). Signed graphs were introduced by Harary in [8], and studied in depth by Zaslavsky (see for example [16, 17, 18, 19, 20]). The notion of distinguishing a set Σ of edges can also be found in the work of König [12]. On the one hand, signed graphs may be viewed as 2-edge-coloured graphs. On the other hand, the choice of $\{+, -\}$ to distinguish the edges leads to two natural concepts for (and which are fundamental to the development of) signed graphs: the sign of a cycle and the operation of switching.

A cycle or closed walk of (G, Σ) is said to be *negative* if the product of signs of all edges is the negative sign, and *positive* otherwise. A (signed) subgraph of (G, Σ) is called *balanced* if it contains no negative cycle. A cycle of length 2 is a *digon*. We only consider negative digons.

To *switch* at a vertex v means to multiply the signs of all edges incident to v by $-$, that is, to switch the sign of each of these edges. (In the case of a loop at v, its sign is multiplied twice and hence it is invariant under switching.) Given signatures Σ and Σ' on a graph G, the signature Σ' is said to be *switching equivalent* to Σ, denoted $\Sigma \equiv \Sigma'$, if it can be obtained from Σ by a sequence of switches. Testing if $\Sigma \equiv \Sigma'$ can be done in polynomial time. Zaslavsky proved that two signatures are equivalent if and only if they induce the same cycle signs [17].

Given two graphs G and H, a *homomorphism* φ of G to H is a mapping of the vertices $\varphi : V(G) \to V(H)$ such that $xy \in E(G)$ implies $\varphi(x)\varphi(y) \in E(H)$. We write $G \to H$ to denote the existence of a homomorphism. One natural extension of this idea to signed graphs is to additionally require that homomorphisms preserve the sign of edges.

Definition 1 *Let (G, Σ) and (H, Π) be two signed graphs. An* edge-coloured homomorphism *of (G, Σ) to (H, Π) is a (graph) homomorphism $\varphi : G \to H$ such that for any two adjacent vertices x and y of G, the signs of the edges xy in (G, Σ) and $\varphi(x)\varphi(y)$ in (H, Π) are the same. When there is such a homomorphism, we write $(G, \Sigma) \xrightarrow{ec} (H, \Pi)$.*

For classical undirected graphs, the complexity of homomorphism testing is completely determined by the dichotomy theorem of Hell and Nešetřil. Let H be a fixed graph. The decision problem $\textsc{Hom}(H)$, also known as H-Colouring, has as an instance a graph G, and asks the question does $G \to H$?

Theorem 2 (Hell and Nešetřil [10]) *If a graph H is bipartite or contains a loop, then $\textsc{Hom}(H)$ is polynomial-time solvable; otherwise, it is NP-complete.*

Our main focus is a possible extension of Theorem 2 for homomorphism problems for signed graphs. To this end, let (H, Π) be a fixed signed graph. We define the following decision problem.

EC-HOM(H, Π)
INSTANCE: A signed graph (G, Σ).
QUESTION: Does $(G, \Sigma) \xrightarrow{ec} (H, \Pi)$?

The notion of edge-coloured homomorphisms of signed graphs (and graphs with any number of edge-colours) was studied in [1, 9, 13] from a non-computational point of view. See also [2, 3, 5] for studies of the computational complexity of edge-coloured homomorphism problems.

As discussed in [14], it is natural to study signed graph homomorphisms incorporating switching.

Definition 3 *Let (G, Σ) and (H, Π) be signed graphs. A* signed homomorphism *of (G, Σ) to (H, Π) is a mapping $\varphi : V(G) \to V(H)$ such that there exist a switching (G, Σ') of (G, Σ) and a switching (H, Π') of (H, Π), such that $\varphi : (G, \Sigma') \xrightarrow{ec} (H, \Pi')$ is an edge-coloured homomorphism. When there is such a homomorphism, we write $(G, \Sigma) \xrightarrow{s} (H, \Pi)$.*

In this work we have adopted a "one-object, two-homomormophisms" point of view. Work exclusively on edge-coloured graphs (resp. signed graphs) would simply use the term *homomorphism of edge-coloured graphs* (resp. *homomorphism of signed graphs*). In particular, our term signed homomorphism refers to the latter. It does not indicate a signing of the mapping.

Given signed graphs (G, Σ) and (H, Π) together with a graph homomorphism $\varphi : G \to H$, an *associated edge-mapping* is a function $\varphi^{\#} : E(G) \to E(H)$ such that for each edge e in G with edge-points $\{u, v\}$ the image $\varphi^{\#}(e)$ is an edge in H with edge-points $\{\varphi(u), \varphi(v)\}$. If e and $\varphi^{\#}(e)$ have the same sign for all $e \in E(G)$, then φ is an edge-coloured homomorphism. By definition, if there exists a resiging Σ' of G, so that φ preserves edge signs, then φ is a signed homomorphism. Alternatively the following theorem shows signed homomorphisms may be viewed as mappings that preserve the important structures in signed graphs: the adjacency of vertices and the signs of cycles.

Theorem 4 *Let (G, Σ) and (H, Π) be signed graphs. Then $\varphi : V(G) \to V(H)$, with a given associated mapping $\varphi^{\#} : E(G) \to E(H)$, is a signed homomorphism $(G, \Sigma) \xrightarrow{s} (H, \Pi)$ if and only if $\varphi : G \to H$ is a homomorphism of the underlying graphs and, for every cycle C of G, the sign of C in (G, Σ) is the same as the sign of the closed walk $\varphi^{\#}(C)$ in (H, Π).*

Let (H, Π) be a fixed signed graph. We define the signed homomorphism decision problem for (H, Π) analogously as in the edge-coloured case.

S-HOM(H, Π)
INSTANCE: A signed graph (G, Σ).
QUESTION: Does $(G, \Sigma) \xrightarrow{s} (H, \Pi)$?

A primary concern in this work is the computational complexity of the S-HOM(H, Π) problem. Note that when Π is switching equivalent to $E(H)$ or to $\emptyset$, then S-HOM(H, Π) has the same complexity as HOM(H). Then, the computational complexity of the problem is decided by Theorem 2. Prior to this work, the only other known case of the problem's complexity is the study of [7] about signed cycles, where it is proved that S-HOM(C_{2k}, Π) is NP-complete if Π has an odd number of elements, and polynomial-time solvable otherwise. (Note that S-HOM(C_{2k+1}, Π) for $k \geq 1$ is always NP-complete.) Here, we give a full dichotomy characterization in the case where (H, Π) is a simple signed graph. Indeed we prove something even stronger. We require the following definition: a signed graph (H, Π) is an *s-core* if for every endomorphism $\varphi : (H, \Pi) \xrightarrow{s} (H, \Pi)$, φ is an automorphism. Every signed graph (H, Π) contains (as a subgraph) an s-core that is unique up to isomorphism and switching.

Theorem 5 *Let* (H, Π) *be a connected signed graph that does not contain all three of a negative digon, a negative loop, and a positive loop. Then,* s-Hom(H, Π) *is polynomial-time solvable if the s-core of* (H, Π) *has at most two edges; it is NP-complete otherwise.*

We believe that a full dichotomy holds for signed graph homomorphisms. In fact, we conjecture that all cases not covered in Theorem 5 are NP-complete.

Conjecture 6 *Let* (H, Π) *be a connected signed graph. Then,* s-Hom(H, Π) *is polynomial-time solvable if the s-core of* (H, Π) *has at most two edges; it is NP-complete otherwise.*

Note that the polynomial half of Conjecture 6 is proved in the proof of Theorem 5, that is, the polynomial case holds for all signed graphs.

The fundamental tool in our work is to reduce signed homomorphism problems to edge-colour homomophism problems.

Definition 7 *Let* (G, Σ) *be a signed graph. The* switching graph *of* (G, Σ) *is a signed graph, denoted* $P(G, \Sigma)$*, constructed as follows.*

(i) For each vertex u *in* $V(G)$ *we have two vertices* u_0 *and* u_1 *in* $P(G, \Sigma)$*.*

(ii) For each edge uv *in* G *we have four edges* u_iv_j *for* $i, j \in \{0, 1\}$ *in* $P(G, \Sigma)$*, with* u_iv_i *having the same sign as* uv *(for* $i \in \{0, 1\}$*) and* u_iv_{1-i} *having the opposite sign as* uv*. (In particular, loops do not change sign.)*

Edge-coloured switching graphs were defined by Brewster and Graves [4] in a more general setting related to permutations. This construction is also used by Ochem, Pinlou, and Sen in [15]. See Klostermeyer and MacGillivray [11] for a similar definition in the context of digraphs. In the context of signed graphs, Zaslavsky used a construction similar (but different) to that of switching graphs [18]. The following proposition allows us to transform questions about signed homomorphisms to the setting of edge-coloured homomorphisms.

Proposition 8 *Let* (G, Σ) *and* (H, Π) *be two signed graphs. The following are equivalent.*

(a) $(G, \Sigma) \xrightarrow{s} (H, \Pi)$,

(b) $(G, \Sigma) \xrightarrow{ec} P(H, \Pi)$,

(c) $P(G, \Sigma) \xrightarrow{ec} P(H, \Pi)$.

Hence Theorem 5 may be viewed as a dichotomy result for edge-coloured homomorphism for targets of the form $P(H, \Pi)$. This class of targets is rich enough to express vertex-colourings of signed graphs (as studied [18, 19, 20]) as homomorphisms. By contrast we prove that a dichotomy theorem for all edge-coloured homomoprhisms would settle the CSP Dichotomy Conjecture.

Theorem 9 *For each CSP template* T*, there is a signed graph* (H, Π) *such that* ec-Hom(H, Π) *and* Csp(T) *are polynomially equivalent. Moreover,* (H, Π) *can be chosen to be bipartite and homomorphic to a path whose signs alternate* $+$ *and* $-$*.*

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Strengthening a Theorem of Tutte on the Hamiltonicity of Polyhedra

Gunnar Brinkmann and *Carol T. Zamfirescu*

Ghent University, Belgium

Extended Abstract

A classic result in graph theory is Whitney's theorem from 1931 that 4-connected triangulations of the plane are hamiltonian [9]. This result was generalised by Tutte in 1956. He showed that all 4-connected planar graphs are hamiltonian [8]. In the past decades, stronger versions of Whitney's theorem have appeared, with one of the most far-reaching results being a theorem of Jackson and Yu that a hamiltonian cycle exists even if there are up to three 3-cuts in a plane triangulation [4]. (Henceforth, all triangulations considered here are plane.)

Although Tutte's seminal theorem has been generalised in several ways as well—see for instance Sanders' result that in a 4-connected plane graph a hamiltonian cycle through any two edges exists [5]—the aforementioned theorem by Jackson and Yu was not generalised to all 3-connected plane graphs with at most three 3-cuts. We have done so in [1], and this will be the focal point of our talk. We shall provide context and present an essential stepping stone leading to the result. Let us emphasise that the theorem of Jackson and Yu does not only concern the number of 3-cuts, but also their relative position, encoded by a *decomposition tree*. Such decomposition trees, which are unique for triangulations, are not defined for general plane graphs, so only the part about the number of 3-cuts can be generalised.

In this talk, with Steinitz' theorem in mind [6], the word *polyhedron* will be used for 3-connected plane graphs. All cuts here are vertex cuts, and a cut of cardinality k shall be called a *k-cut*. Let G be a polyhedron and $X = \{u, v, w\}$ a 3-cut in G. If (V', E') is a component of $G - X$, then $G[V' \cup X]$ is called a *closed component* of $G - X$. If (V'', E'') is a closed component of $G - X$, then $(V'', E'' \cup \{uv, vw, wu\})$ is called an *edge closed component* of $G - X$. In the talk, we shall outline the main idea behind these closed components, and emphasise the differences between triangulations and polyhedra with respect to hamiltonian properties and the way we can use or not use certain techniques to achieve our goals.

But there are also significant structural similarities between triangulations and polyhedra; an example of such a similarity presents itself as follows. While in triangulations 3-cuts are separating triangles that lie properly inside each other, the relative position in polyhedra can be more complicated: in a polyhedron, vertices of a 3-cut X may end up in different edge closed components of a 3-cut X'. (This makes translating the decomposition tree concept so difficult.) Thus, it is worthwhile to explicitly state the following result, which is trivial for triangulations.

Proposition 1 *Let G be a polyhedron with $k \geq 1$ 3-cuts. Then G contains a 3-cut X such that at least one edge closed component of $G - X$ has no 3-cuts, i.e. the edge closed component is 4-connected or isomorphic to K_4.*

A graph G shall be called *k-hamiltonian* if for each set S of k vertices, the graph $G - S$ is hamiltonian. In 1994, Thomas and Yu proved the following result which was originally conjectured by Plummer.

Theorem 2 (Thomas and Yu [7]) *4-connected polyhedra are 2-hamiltonian.*

We present the following useful lemma.

Lemma 3 *A polyhedron G with k 3-cuts contains a spanning subgraph that can be obtained from a 4-connected polyhedron by deleting at most k vertices.*

Together with this lemma, Theorem 2 implies:

- Polyhedra with at most two 3-cuts are hamiltonian.
- Polyhedra with at most one 3-cut are 1-hamiltonian.

Theorem 2 can obviously not be strengthened to imply 3-hamiltonicity, so in order to prove that polyhedra with at most three 3-cuts are hamiltonian, we need a different strategy. In the talk we will present an important ingredient of the proof of our main theorem from [1], which now follows.

Theorem 4 *A polyhedron with at most three* 3*-cuts is hamiltonian.*

Concerning the connection between the hamiltonicity and the traceability of a polyhedron, we have the following.

Theorem 5 *If all polyhedra with at most k* 3*-cuts are hamiltonian, then all polyhedra with at most $k+1$* 3*-cuts are traceable. In particular, all polyhedra with at most four* 3*-cuts are traceable.*

A natural question is how low we can go with the number of 3-cuts and still find non-traceable or non-hamiltonian examples.

Proposition 6 (Brinkmann, Souffriau, and Van Cleemput [3]) *For all $d \geq 6$ there exist non-hamiltonian triangulations with exactly d* 3*-cuts.*

Proposition 7 *For all $d \geq 8$ there exist non-traceable triangulations with exactly d* 3*-cuts.*

Combining the above conclusions, the obvious open questions are:

- Are polyhedra with four or five 3-cuts hamiltonian?
- Are polyhedra with five, six or seven 3-cuts traceable?

Unfortunately, the hope for an easy (i.e. structurally simple) answer to either question is extinguished by the following result, which revolves around toughness and scattering numbers—in the talk, a brief outline of the connection between these concepts and hamiltonicity will be given.

Theorem 8 *Polyhedra with at most five* 3*-cuts are* 1*-tough, and polyhedra with at most seven* 3*-cuts have scattering number at most* 1.

At the end of the presentation, we give an overview of results on hamiltonian properties of polyhedra with few 3-cuts [2].

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Toughness and hamiltonicity for special graph classes

Hao Qi [1], *Hajo Broersma* [2], Adam Kabela [3], and *Elkin Vumar* [1]

[1] College of Mathematics and System Sciences, Xinjiang University, P.R. China
[2] Faculty EEMCS, University of Twente, The Netherlands
[3] Department of Mathematics and European Centre of Excellence NTIS, University of West Bohemia, Czech Republic

Extended Abstract

1 Introduction

Much of the background for this work and references to related work can be found in [1] and [5]. We consider only undirected graphs with no loops or multiple edges.

Let $\omega(G)$ denote the number of components of a graph G. A *vertex cut* of a connected graph $G = (V, E)$ is a set $S \subseteq V$ with $\omega(G - S) > 1$. A graph G is said to be *t-tough* if $|S| \geq t \cdot \omega(G - S)$ for every vertex cut S of G. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for the complete graph K_n on $n \geq 1$ vertices). Hence if G is not a complete graph, $\tau(G) = \min\{|S|/\omega(G - S)\}$, where the minimum is taken over all vertex cuts S of G. Following Plummer [14], a vertex cut S of G is called a *tough set* if $\tau(G) = |S|/\omega(G - S)$, i.e. a tough set is a vertex cut S of G for which this minimum is achieved. A graph G is *hamiltonian* if G contains a *Hamilton cycle*, i.e. a cycle containing every vertex of G.

Historically, most of the research on toughness has been based on a number of conjectures in [8]. The most challenging of these conjectures, which is still open, states that there is a constant t such that every t-tough graph is hamiltonian. This conjecture is called Chvátal's Conjecture and has been shown to be true when restricted to a number of graph classes, including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs. In order to keep this exposition short, we refer the interested reader to [1] or [5] for more information and a survey of results, and more specifically to [7, 10, 2] for results on chordal graphs.

For split graphs, a subclass of chordal graphs, it was shown in [13] that every 3/2-tough split graph on at least three vertices is hamiltonian. Moreover, it was shown there that this result is best possible, in the sense that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to 3/2$. Similar toughness results on subclasses of chordal graphs were obtained in [11] (for spider graphs) and in [12] (for interval graphs). Recently, the result on split graphs was extended in [6], where it was shown that 25-tough $2K_2$-free graphs are hamiltonian.

2 New results

In this presentation, we first consider an extension of the class of split graphs to the class of *multisplit* graphs, motivated by *bisplit* graphs that were introduced in [4]. A graph $G = (V, E)$ is called a *k-multisplit* graph ($k \geq 2$) if V can be partitioned into two disjoint sets S and T such that S is an independent set in G and T induces a complete k-partite subgraph in G. For $k = 2$, the class coincides with the class of bisplit graphs, and in the degenerate case that each class in the k-partition consists of only one vertex, the k-multisplit graph is just a split graph.

As a new contribution to the field, using matching and alternating path techniques, we show that Chvátal's Conjecture holds for k-multisplit graphs by proving the following theorem.

Theorem 1 *Every 2-tough k-multisplit graph on at least three vertices is hamiltonian.*

While this establishes Chvátal's Conjecture for a new graph class, we do not know whether our bound on the toughness is best possible.

In contrast, as a second main result, we prove a best possible toughness result for a class of graphs that extends the class of C_5^*-graphs. The latter class of graphs was recently introduced and studied in [6] as a special subclass of $2K_2$-free graphs, in fact it is precisely the class of triangle-free non-bipartite $2K_2$-free graphs.

For an integer $p \geq 3$, a graph $G = (V, E)$ is called a C_p^*-graph if V consists of p non-empty disjoint independent sets $A_1, A_2, \ldots, A_p$ such that, for all $i = 1, 2, \ldots, p-1$, the sets A_i and A_{i+1}, as well as the sets A_p and A_1 induce complete bipartite graphs in C_p^*, so the edges of these bipartite graphs are the only edges of E. Note that, in particular, C_p^*-graphs are P_p-free and triangle-free if $p \geq 4$, so they have to be dealt with as special cases if one aims to prove Chvátal's Conjecture for P_p-free or triangle-free graphs. In [6], it is proved that a C_5^*-graph is hamiltonian if and only if it is 1-tough. Here, we extend this result by using the Max-Flow Min-Cut Theorem due to Ford and Fulkerson [9] to prove the following result.

Theorem 2 *A C_p^*-graph ($p \geq 3$) is hamiltonian if and only if it is 1-tough.*

Our other results deal with the computational complexity of determining the toughness of graphs in the above two classes. It was shown in [3] that the problem of recognizing t-tough graphs is coNP-complete for every fixed positive rational t. This implies that it is NP-hard to compute the toughness of a general input graph. On the other hand, toughness can be computed efficiently when the input graph is restricted to certain graph classes; see [1] and [5] for more details. In particular, recognizing t-tough graphs is polynomially solvable within the classes of claw-free graphs and split graphs [15]. This complexity question is still open for many well-studied graph classes, e.g. for (maximal) planar graphs and for chordal graphs.

In contrast to the result of [15] on split graphs, we use a result of [13] to show that determining the toughness of k-multisplit graphs is NP-hard.

Theorem 3 *For any fixed integer $k \geq 2$, determining the toughness of k-multisplit graphs is an NP-hard problem.*

Next to this, we prove that the toughness of C_p^*-graphs can be determined in polynomial time, using a direct approach to characterize and restrict the candidates for possible tough sets.

Theorem 4 *The toughness of a C_p^*-graph ($p \geq 3$) can be determined in polynomial time.*

Theorems 2 and 4 together imply that it can be checked in polynomial time whether a C_p^*-graph is hamiltonian.

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Limited broadcast domination

J. Cáceres [1], *C. Hernando* [2], *M. Mora* [2], *I.M. Pelayo* [2] and *M.L. Puertas* [1]

[1] Department of Mathematics, Universidad de Almería, Spain

[2] Department of Mathematics, Universitat Politècnica de Catalunya, Spain

Extended Abstract

Domination in graphs has shown as a extremely fruitful concept, since it was originally defined in the late fifties [1] and named in the early sixties [7]. A dominating set of a graph G is a vertex set S such that any vertex not in S has at least one neighbor in S. Multiple variants of domination have been defined over the past fifty years, putting the focus on different aspects. One of them is the idea of broadcasting, firstly introduced in [6] and taken up more recently in [3]. This model reflects the idea of several broadcast stations, with associated transmission powers, that can broadcast messages to places at distance greater than one. We recall the formal definition from [3].

Definition 1 (Erwing'04 [3]) *For a graph G any function $f\colon V(G) \to \{0, 1, \ldots, diam(G)\}$ is called a broadcast on G. A vertex $v \in V(G)$ with $f(v) > 0$ is a f-dominating vertex and the set $V_f^+ = \{v \in V(G)\colon f(v) > 0\}$ is the f-dominating set. An f-dominating vertex v is said to f-dominate every vertex u with $d(u, v) \leq f(v)$. A dominating broadcast on G is a broadcast f such that every vertex in G is f-dominated. The cost of a f-dominating broadcast is $\sigma(f) = \sum_{v \in V_f^+} f(v)$ and the dominating broadcast number $\gamma_B(G)$ is the minimum value of $\sigma(f)$ over all dominating broadcast of G.*

In this work we follow the suggestion posed in [4] as an open problem of considering the broadcast dominating problem with limited broadcast power.

Definition 2 (Dunbar et all.'06 [4]) *Let G be a graph, let $f\colon V(G) \to \{0, 1, 2\}$ be a function and let $V_f^+ = \{v \in V(G)\colon f(v) \geq 1\}$. We say that f is a dominating 2-broadcast if for every $u \in V(G)$ there exists $v \in V_f^+$ such that $d(u, v) \leq f(v)$. The cost of a 2-dominating broadcast f is $\omega(f) = \sum_{v \in V_f^+} f(v)$ and the 2-broadcast dominating number of G is*

$$\gamma_{B_2}(G) = \min\{\omega(f)\colon f \text{ is a 2-dominating broadcast of } G\}.$$

We study this parameter in [2]. Among other results, we have obtained a general upper bound in terms of the order of the graph, which is reached for graphs of every order.

Theorem 3 *For every graph G of order n,*

$$\gamma_{B_2}(G) \leq \lceil 4n/9 \rceil.$$

Moreover there are graphs of order as great as desired that attain this bound.

This upper bound is smaller in the particular case of the caterpillars. Recall that a caterpillar C is a tree such that the graph obtained by removing all its leaves is a path. The broadcast number has been studied in this graph class in [8]. In this case we have obtained the following upper bound of the 2-broadcast domination number.

Theorem 4 *Let C be a caterpillar of order n, then*

$$\gamma_{B_2}(C) \leq \lceil 2n/5 \rceil.$$

Moreover there are caterpillars of order as great as desired that attain this bound.

There is a natural upper bound for the broadcast number in every graph [3]

$$\gamma_B(G) \leq \min\{\gamma(G), \mathrm{rad}(G)\}$$

that provides the following classification of graphs [4]

- Type I graphs: $\gamma_B(G) = \gamma(G)$
- Type II graphs, also called radial graphs: $\gamma_B(G) = \mathrm{rad}(G)$
- Type III: $\gamma_B(G) < \min\{\gamma(G), \mathrm{rad}(G)\}$

In [8] author characterizes Type I caterpillars and radial trees are studies in [5]. The 2-broadcast domination number is independent from the radius of the graph, however it keeps the domination number as an upper bound. So the question of characterize graphs with $\gamma_{B_2}(G) = \gamma(G)$ arises. Following the ideas in [8] we have obtained a characterization of caterpillars satisfying this condition.

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Packing graphs of bounded codegree

Wouter Cames van Batenburg[1] and Ross J. Kang

Department of Mathematics, Radboud University Nijmegen, Netherlands

Extended Abstract

Let G_1 and G_2 be two graphs (with neither loops nor multiple edges) on n vertices. They are said to *pack* if there exist injective mappings of their vertex sets into $[n] = \{1, \dots, n\}$ so that their edge sets have disjoint images. Equivalently, they pack if G_1 is a subgraph of the complement of G_2. The *maximum codegree* $\Delta^\wedge(G)$ of a graph G is the maximum over all vertex pairs of their common degree. The *maximum adjacent codegree* $\Delta^\vartriangle(G)$ of G is the maximum over all pairs of *adjacent* vertices of their common degree. Clearly, $\Delta^\vartriangle(G) \leq \Delta^\wedge(G)$ always. Let Δ_1 and Δ_2 denote the maximum degrees of G_1 and G_2, respectively, and $\Delta_1^\wedge$ and $\Delta_2^\vartriangle$ the corresponding maximum (adjacent) codegrees. Our aim is to obtain good sufficient conditions for G_1 and G_2 to pack in terms of Δ_1, Δ_2, $\Delta_1^\wedge$, $\Delta_2^\vartriangle$ and n.

The study of graph packing was initiated in the 1970s [2, 6, 7, 16] and has attracted considerable attention over the years. The central question is the following.

Conjecture 1 (Bollobás and Eldridge [2] and Catlin [7]) *If G_1, G_2 are as above, then G_1 and G_2 pack provided $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$.*

If true, the statement would be sharp and would significantly generalise a celebrated result of Hajnal and Szemerédi [11] on equitable colourings. Sauer and Spencer [16] showed that $2\Delta_1\Delta_2 < n$ is a sufficient condition for G_1 and G_2 to pack.

The Bollobás–Eldridge–Catlin (BEC) conjecture is a notorious problem in combinatorics. Despite concerted efforts, for instance with probabilistic and regularity methodology, the BEC conjecture has been confirmed in only the following special cases: $\Delta_1 = 2$ [1]; $\Delta_1 = 3$ and n sufficiently large [10]; G_1 bipartite and n sufficiently large [9]; and G_1 d-degenerate, $\Delta_1 \geq 40d$ and $\Delta_2 \geq 215$ [4]. It has also been shown that an approximate BEC condition, $(\Delta_1 + 1)(\Delta_2 + 1) \leq 3n/5 + 1$, is sufficient for G_1 and G_2 to pack, provided that $\Delta_1, \Delta_2 \geq 300$ [14].

It is intuitive to expect that additional sparsity conditions on G_1 or G_2 makes packing with respect to the BEC conjecture easier. This has already borne out of two of the results mentioned above [4, 9]. We continue in this vein by investigating when the maximum codegree $\Delta_1^\wedge$ of G_1 is bounded. Note that $\Delta_1^\wedge < 2$ is equivalent to having no 4-cycle as a subgraph in G_1. Our main result is the following.

Theorem 2 *Let G_1, G_2 be as above. Let $t \geq 2$ be an integer. Then G_1 and G_2 pack provided $\Delta_1(\Delta_2 + 1) \leq n + 1$ and $\Delta_1^\wedge < t$ and $\Delta_1 > 17t \cdot \Delta_2$.*

This immediately implies that, for $t \geq 2$, the BEC conjecture holds under the additional conditions that $\Delta_1^\wedge < t$ and $\Delta_1 > 17t \cdot \Delta_2$. By taking G_2 to be a collection of (nearly) equal-sized cliques, Theorem 2 also implies that, if G is a $K_{2,t}$-free graph of maximum degree Δ with $\Delta \geq \sqrt{17t \cdot n}$, then its equitable chromatic number is at most $\Delta(G)$. (This is not a direct corollary to the result of Hajnal and Szemerédi.)

We were unable to avoid the linear dependence on Δ_2 in the lower bound condition on Δ_1 in Theorem 2. Although we have not seriously attempted to optimise the factor $17t$, we can improve on it if we also assume that $\Delta_2^\vartriangle$ is bounded, as exemplified by the following.

Theorem 3 *Let G_1, G_2 be as above. Let $t \geq 2$ be an integer. The BEC conjecture holds under the additional conditions that $\Delta_1^\wedge < t$, G_2 is triangle-free, and $\Delta_1 > (4 + \sqrt{5})t \cdot \Delta_2$.*

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We actually derive our main results from the following more general albeit more technical statements. For integers $t \geq 2$ and $\Delta_2 \geq 1$, we define

$$\alpha^*(t, \Delta_2) := \frac{1}{2}(2 + \gamma + \sqrt{4\gamma + \gamma^2}), \quad \text{where} \quad \gamma = \frac{\Delta_2}{\Delta_2 + 1} \cdot \frac{t-1}{t}.$$

Note that α^* satisfies $(\alpha^* - 1)^2 - \gamma\alpha^* = 0$ and $\frac{1}{8}(9 + \sqrt{17}) \leq \alpha^* \leq \frac{1}{2}(3 + \sqrt{5})$.

Theorem 4 *Let G_1, G_2 be as above. Let $t \geq 2$ be an integer and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ and $0 < \epsilon < 1/2$ be reals. Then G_1 and G_2 pack if $\Delta_1^\wedge < t$ and n is larger than each of the following quantities:*

$$\left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \tag{1}$$

$$(2\alpha t + 2) \cdot \Delta_2 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2 + (1 - \epsilon) \cdot \Delta_1\Delta_2, \tag{2}$$

$$1 + \left(2 + \frac{\epsilon}{1 - 2\epsilon}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \quad \textit{and} \tag{3}$$

$$\left(t + \frac{3-\epsilon}{2}\right) \cdot \Delta_2 + \frac{3-\epsilon}{2}(t-1) \cdot \Delta_2^2 + \frac{1+\epsilon}{2} \cdot \Delta_1\Delta_2. \tag{4}$$

Theorem 5 *Let G_1, G_2 be as above. Let $s \geq 1$ and $t \geq 2$ be integers and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ be real. Then G_1 and G_2 pack if $\Delta_1^\wedge < t$, $\Delta_2^\triangle < s$, and n is larger than both of the following quantities:*

$$\left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2 \quad \textit{and} \tag{5}$$

$$(2 + 2\alpha t) \cdot \Delta_2 + (s - 1) \cdot \Delta_1 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2. \tag{6}$$

The proofs of Theorems 4 and 5 proceed by considering a hypothetical edge-minimal counterexample (G_1, G_2), meaning that there exist injective mappings of the vertex sets of G_1, G_2 into $[n]$ such that the images of their edge sets have exactly one common element. Such a hypothetical counterexample has a rich structure. First, by definition it is not possible to obtain a packing by relabelling the vertices in G_1. In particular, this fact implies that there is a fixed vertex u such that every other vertex is connected to u by an edge of G_1 followed by an edge of G_2, or an edge of G_2 followed by an edge of G_1. This in itself already implies the relatively crude bound $2\Delta_1\Delta_2$. Second, the fact that G_1 is $K_{2,t}$-free implies that any two vertices of G_1 cannot have neighbourhoods that overlap too much. So, if one knows that there is a vertex set A and at least q other vertices for each of which the intersection of their neighbourhood with A is rather large, then one knows that either q or the size of A must be rather small. This intuition is made precise with a lemma of Corrádi [8]. Last, bounds on the maximum degrees immediately yield bounds on several neighbourhood sets. These are the three main ingredients we use in a case analysis to show that the entire vertex set is covered by several *small* (intersections of) (neighbourhoods of) neighbourhoods, ultimately leading to a contradictory upper bound on n, thus proving the theorem. The precise details can be found in the full version of the paper [5].

The BEC conjecture notwithstanding, naturally one might wonder whether Theorem 4, or rather Theorem 2, could be improved according to a weaker form of the BEC condition, as was the case for d-degenerate G_1 [4]. In other words, it would be interesting to improve upon the $\Omega(\Delta_1\Delta_2)$ terms appearing in each of (1)–(4). We leave this to further study, but point out the following constructions where G_1 has low maximum codegree, which mark boundaries for this problem.

- When n is even, there are non-packable pairs (G_1, G_2) of graphs where G_1 is a perfect matching (so $\Delta_1^\wedge = 0$) and $2\Delta_1\Delta_2 = n$, cf. [13].

- Bollobás, Kostochka and Nakprasit [3] exhibited a family of non-packable pairs (G_1, G_2) of graphs where G_1 is a forest (so $\Delta_1^\wedge = 1$) and $\Delta_1 \ln \Delta_2 \geq cn$ for some $c > 0$.
- If $\Delta^\wedge(G) = 1$, then the chromatic number of G satisfies $\chi(G) = O(\Delta(G)/\ln \Delta(G))$ as $\Delta(G) \to \infty$, and there are standard examples having arbitrarily large girth that show this bound to be sharp up to a constant factor, cf. [15, Ex. 12.7]. Since the equitable chromatic number is at least the chromatic number, these examples moreover yield non-packable pairs (G_1, G_2) of graphs having $\frac{\Delta_1}{\ln \Delta_1}(\Delta_2 + 1) \geq cn$ for some $c > 0$ and $\Delta_1^\wedge = 1$.

Since the examples can also have the maximum adjacent codegree $\Delta_1^\triangle$ being zero, this last remark hints at another natural line to pursue, which could significantly extend both the result of Csaba [9] and a result of Johansson [12]. If Δ_1 is large enough and G_1 is triangle-free, is some condition of the form $\frac{\Delta_1}{\ln \Delta_1}(\Delta_2 + 1) = cn$ for some constant $c > 0$ sufficient for G_1 and G_2 to pack?

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α-labellings of lobsters with maximum degree three

Atílio G. Luiz [1], C. N. Campos [1] and R. Bruce Richter [2]

[1] Institute of Computing, University of Campinas, Brazil
[2] Combinatorics and Optmization, University of Waterloo, Canada

Extended Abstract

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *graceful labelling* of G is an injection $f\colon V(G) \to \{0, \ldots, |E(G)|\}$ such that $\{|f(u) - f(v)|\colon uv \in E(G)\} = \{1, 2, \ldots, |E(G)|\}$. A graceful labelling f of G is an *α-labelling* if, additionally, there exists an integer $k \in \{1, 2, \ldots, |E(G)|\}$ such that, for each edge $uv \in E(G)$, either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$. If a graph G has a graceful labelling (α-labelling), we say that G is *graceful* (*α-labellable*). A result that follows directly from the definition of α-labelling is that if a graph G has an α-labelling, then G is bipartite. Therefore, the only graphs which are expected to have such a labelling are bipartite graphs, such as trees.

In 1967, Rosa introduced four types of labellings of graphs, including graceful labellings and α-labellings, and posed the *Graceful Tree Conjecture* which states that all trees are graceful [6]. Rosa proved that the Graceful Tree Conjecture is a strenghtened version of the well-known *Ringel-Kotzig Conjecture* which states that the complete graph K_{2m+1} has a cyclic decomposition into subgraphs isomorphic to a given tree T with m edges. In his seminal article, Rosa also proved that, for any positive integer p, if a graph G with m edges has an α-labelling, then there exists a cyclic decomposition of the complete graph K_{2pm+1} into subgraphs isomorphic to G. Thus, as in the case of graceful labellings, this second result stresses the importance of α-labellings in the study of cyclic decompositions of complete graphs.

The Graceful Tree Conjecture is a very important open problem in Graph Theory [3]. On the other hand, unlike graceful labellings, there are several examples of trees that do not have an α-labelling [4]. Nevertheless, α-labellings of trees have been investigated by many authors [1, 2, 4, 5] due to their connection with cyclic decompositions of complete graphs and also due to the fact that α-labellings also have suitable properties that allow one to obtain larger α-labellable trees or graceful trees from smaller α-labellable trees [4, 5].

A family of trees for which α-labellings are still not characterized is the family of lobsters, defined as follows. Let T be a tree and P be one of its longest paths. We say that T is *k-distant* if all of its vertices are at distance at most k from P. Path P is called the *spine* of T. The k-distant trees with $k \in \{0, 1, 2\}$ have specific names in the literature: 0-distant trees are *paths*, 1-distant trees are *caterpillars*, and 2-distant trees are *lobsters*. It is well-known that every caterpillar has an α-labelling but the same is not true of lobsters [6]. For example, Huang et al. [4] proved that no lobster of diameter four that is not a caterpillar has an α-labelling.

A characterization of trees with maximum degree three and having α-labellings is still not known, despite some attempts towards this end [1, 2]. In particular, Brinkmann et al. [2] determined all trees with maximum degree three and at most 36 vertices that do not have an α-labelling. They obtained a necessary condition for a tree to admit an α-labelling and they found a family $\mathcal{F}$ of trees that do not admit α-labellings, defined as follows: a tree T belongs to $\mathcal{F}$ if it can be obtained from a tree T' with $4k$ vertices, $k \geq 1$, all of odd degree, by subdividing each one of its edges. Furthermore, the authors observe that all trees T with $15 \leq |V(T)| \leq 36$ and maximum degree three either have an α-labelling or are in $\mathcal{F}$, leading them to ask the following.

Question 1 (Brinkmann et al. [2])**.** *Do all trees without an α-labelling, with maximum degree three and at least 15 vertices belong to family $\mathcal{F}$?*

Motivated by Question 1, in this work, we investigate α-labellings of trees with maximum degree three restricted to lobsters. We approach this family by using an important tool, called

π-representation, introduced by Kotzig in order to deal with α-labellings and we present an infinite family of lobsters with maximum degree three that have an α-labelling. The result obtained in this work, points towards an affirmative answer to Question 1 when restricted to the family of lobsters.

In order to state our result, some additional definitions are needed. Let G be a lobster with $\Delta(G) = 3$. The *legs* of G are the non-trivial connected components obtained by removing the edges of its spine. Note that, since $\Delta(G) = 3$, the legs of G are isomorphic to P_2, P_3, or $K_{1,3}$, and are called 1-*leg*, 2-*leg* and Y-*leg*, respectively. Let P be the spine $(v_1, v_2, \dots, v_t)$ of G. For integers i, j with $1 \le i \le j \le t$, let $P^{i,j}$ denote the subpath $(v_i, v_{i+1}, \dots, v_j)$ of P. The *subgraph inherited from* $P^{i,j}$ is the subgraph consisting of $P^{i,j}$ and all legs having a vertex in $P^{i,j}$. For $1 \le i \le t$, the i-prefix is the subgraph inherited from $P^{1,i}$, while the i-suffix is the subgraph inherited from $P^{t-i+1,t}$. An i-ending is either the i-prefix or the i-suffix. When i is irrelevant, we will speak of 'a/an prefix/suffix/ending'. A *prohibited suffix* is one of the three suffixes shown in Figure 1. Symmetrically, there are three *prohibited prefixes*. A *prohibited ending* is either a prohibited suffix or a prohibited prefix.

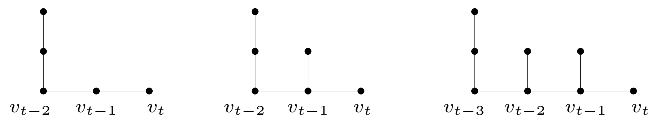

Figure 1: The three prohibited suffixes.

In this work, we prove the following theorem:

Theorem 2. *Let G be a lobster with $\Delta(G) = 3$ and without Y-legs. If G has at most one prohibited ending, then G has an α-labelling.*

1 Preliminaries

In 1973, Kotzig [5] studied α-labellings of arbitrary bipartite graphs and he introduced a type of graphical representation for a bipartite graph whose existence is equivalent to the existence of an α-labelling. This graphical representation is called π-representation and we define it in the context of trees as follows.

Let L_0, L_1 and L_2 be three distinct parallel lines lying on a plane such that L_1 and L_2 are both equidistant from L_0. Also, let T be a tree with bipartition $\{V_1, V_2\}$. Consider a drawing of T such that: (i) the vertices of V_1 and V_2 lie on the lines L_1 and L_2, respectively, and there is a real number d such that any two consecutive vertices on either L_1 or L_2 are distance d; (ii) the edges of T are drawn as straight lines; and (iii) if two edges intersect, then their intersection does not lie on line L_0. Such a drawing of T is called a π-*representation* of T and we say that T is π-*representable* if it can be drawn this way. As an example, Figure 2 exhibits two π-representations of the path P_8.

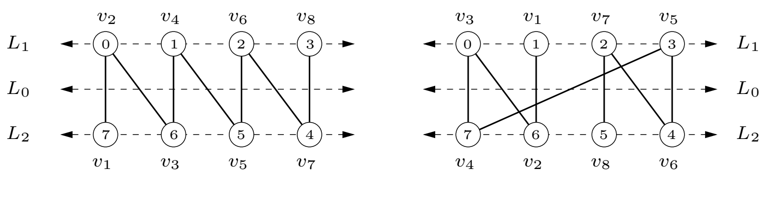

Figure 2: Two π-representations of P_8 and their respective α-labellings.

When dealing with π-representations, it is necessary to be able to talk about the distance of a vertex from the end of a π-representation. Following Kotzig, consider a π-representation of a tree T with vertices of V_1 on L_1 and vertices of V_2 on L_2. For $i = 1, 2$, label the vertices of

V_i in order on L_i as $p_i(0), p_i(1), \ldots, p_i(|V_i|-1)$. (We use the same direction for both L_1 and L_2.) For $j = 0, 1, \ldots, |V_i| - 1$, we set $depth^+(p_i(j)) = j$ and $depth^-(p_i(j)) = |V_i| - 1 - j$. For example, in the first π-representation of Figure 2, the vertex $v_1 = p_2(0)$ has $depth^+(v_1) = 0$ and $depth^-(v_1) = 3$, and the vertex $v_6 = p_1(2)$ has $depth^+(v_6) = 2$ and $depth^-(v_6) = 1$.

Kotzig turns a π-representation of a tree into an α-labelling by labelling $p_1(i)$ with i and $p_2(i)$ with $|V(T)| - 1 - i$. He shows this is an α-labelling. Conversely, the inverse function converts an α-labelling of a tree into a π-representation.

Theorem 3 (Kotzig [5])**.** *A tree T has an α-labelling if and only if it has a π-representation.*

The importance of π-representations as a tool to study α-labellings is in the fact that two π-representations can be linked in order to obtain a larger π-representation, which, equivalently, provides us a way to obtain a larger α-labellable tree from two smaller ones. This construction is widely used in our proofs:

Lemma 4 (Kotzig [5])**.** *Let T' and T'' be two π-representable trees and $x \in V(T')$, $y \in V(T'')$ such that T' and T'' have π-representations in which $depth^-(x) = depth^+(y)$. Then, the tree T obtained from $T' \cup T''$ by the addition of the new edge xy is π-representable.* □

As previously stated, every caterpillar is α-labellable and, therefore, has a π-representation. However, in our proof, we are interested in a specific π-representation of caterpillars which is presented in Lemma 5. We say that a caterpillar is a *$\mathcal{C}$-block* when its π-representation under consideration is the π-representation stated at Lemma 5.

Lemma 5 (Kotzig [5])**.** *Let T be a caterpillar with spine $(v_1, \ldots, v_n)$, $n \geq 1$. Then, T has a π-representation in which $depth^+(v_1) = depth^-(v_n) = 0$.* □

2 Results

In this section, we present our main results. In the proof of Theorem 2, we decompose lobster G into vertex-disjoint subgraphs isomorphic to specific caterpillars called $\mathcal{L}$-blocks, which are defined as follows. Let T be a caterpillar with maximum degree at most three and spine $(v_1, \ldots, v_n)$, $n \geq 6$, which satisfies the following condition: if $\Delta(T) = 3$, then its vertices of degree three are $v_{i_1}, \ldots, v_{i_k}$ such that $3 < i_1 < i_2 < \cdots < i_k < n - b$, for $b \in \{0, 1, 2\}$. Caterpillar T is denoted by the ordered tuple $(\{i_1, \ldots, i_k\}, b)$. We say that $(\{i_1, \ldots, i_k\}, b)$ is an *$\mathcal{L}$-block* if it has a π-representation such that $depth^+(v_3) = depth^-(v_{n-b}) = 0$. In Lemma 6, we present the $\mathcal{L}$-blocks that are used in the proof of Theorem 2. Lemmas 7 and 8 present additional results on π-representations of caterpillars that are considered in our proof.

Lemma 6. *The following caterpillars are $\mathcal{L}$-blocks: $(\emptyset, 2)$; $(\{4\}, 2)$; $(\{4, 5\}, 2)$; $(\{7\}, 2)$; $(\{7, 8\}, 2)$; $(\{4, 5, 9\}, 2)$; $(\{4, 9\}, 2)$; $(\emptyset, 1)$ not isomorphic to P_8; $(\{4, 5\}, 1)$ with $|V(T)| \geq 9$ and $|V(T)| \neq 12$; $(\{4\}, 1)$ with $|V(T)| \geq 8$ and $|V(T)| \neq 11$; $(\{7\}, 1)$ with $|V(T)| \geq 11$; $(\{7, 8\}, 1)$ with $|V(T)| \geq 12$; $(\{4, 5, 9\}, 1)$ with $|V(T)| \geq 14$; and $(\{4, 9\}, 1)$ with $|V(T)| \geq 13$.* □

Lemma 7. *The following caterpillars have a π-representation such that $depth^+(v_3) = 0$: $(\{7\}, 0)$ with $|V(T)| \in \{9, 10\}$; $(\{7, 8\}, 0)$ with $|V(T)| = 11$; $(\{4, 5, 9\}, 0)$ with $|V(T)| = 13$; $(\{4, 9\}, 0)$ with $|V(T)| = 12$; and $(\{4\}, 1)$ with $|V(T)| = 7$.* □

Lemma 8. *Let G be a lobster with $\Delta(G) = 3$ and without Y-legs. For $i \in \{3, 4\}$, let H be an i-suffix of G. If H is isomorphic to a prohibited suffix, then H does not have a π-representation in which the vertex v_{t-i+1} has depth 0.* □

2.1 Sketch of the proof of Theorem 2

Let G be a lobster with $\Delta(G) = 3$ and without Y-legs. Let the path $(v_1, v_2, \ldots, v_t)$ be the spine of G and adjust notation so that no suffix of G is one of the three prohibited suffixes. This is possible, since such an ending exists by hypothesis.

Given the spine $(v_1, v_2, \ldots, v_t)$ and proceeding in order from v_1, we partition G into $\mathcal{C}$- and $\mathcal{L}$-blocks as follows. If there is no 2-leg, then G is a caterpillar; G is a $\mathcal{C}$-block and is the only block. Otherwise, let v_{j+1} be the first vertex that is in a 2-leg; the first block B_1 is the $\mathcal{C}$-block inherited by $P^{1,j}$.

We suppose we have $i \geq 1$ such that the i^{th} block B_i has been determined with last spine vertex v_j; the leg, if there is one, containing v_j is in B_i. Suppose first that there is a $k > j$ such that there is a 2-leg containing v_k. Choose the minimal such k. If $k > j + 1$, then the next block B_{i+1} is the $\mathcal{C}$-block that is the subgraph inherited by $P^{j+1,k-1}$. In the case $k = j + 1$, if there is an $l > k$ such that the subgraph inherited by $P^{k,l}$ is an $\mathcal{L}$-block, then, choosing the minimal such l yields one of the graphs in Lemma 6 as B_{i+1}. If no such l exists, then subgraph inherited from $P^{k,t}$ is the suffix that makes the last block B_{i+1}. Finally, if there is no $k > j$ such that v_k is in a 2-leg, the suffix that is the subgraph inherited from $P^{j+1,t}$ is the last block B_{i+1}. Lemmas 6 and 7 combine with Lemma 4 to show that we can glue together the blocks (with depths always equal to 0).

3 Acknowledgments

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A Proof for a Conjecture of Gorgol

Victor Campos [1] and *Raul Lopes* [2]

Departamento de Computação, Federal University of Ceará, Brazil
ParGO, UFC
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[2] Partially supported by CAPES

Extended Abstract

1 Introduction

The first known result in extremal graph theory was published in 1907 when Mantel [4] proved that the maximum number of edges in a simple graph without a triangle as a subgraph is $\lfloor n^2/4 \rfloor$. Turán [6] generalized this result to find the maximum number of edges without K_r as a subgraph, where K_r denotes the complete graph on r vertices. It's important to note that these results do not hold if the graphs considered are not simple, therefore, throughout this text, we consider only simple graphs.

The *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in a graph on n vertices which does not contain H as a subgraph. Using this notation, Mantel's Theorem states that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. Let $H_{\mathrm{ex}}(n, G)$ represent an *extremal graph* on n vertices without G as a subgraph with $\mathrm{ex}(n, G)$ edges. Turán's Theorem states that $H_{\mathrm{ex}}(n, K_{r+1}) = T_r(n)$, where $T_r(n)$ is the complete r-partite graph on n vertices in which all parts have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

For a general graph H, the value of $\mathrm{ex}(n, H)$ is asymptotically well known. Erdős, Stone and Simonovits' Theorem [2] states that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{n^2} = \frac{\chi(H) - 2}{2\chi(H) - 2}.$$

Although this theorem gives a lot of information on the asymptotic growth of $\mathrm{ex}(n, H)$, it should be noted that it is only of interest for nonbipartite graphs. If H is bipartite, it asserts merely that $\mathrm{ex}(n, H) \ll n^2$. In this paper, we focus on the Turán number for bipartite graphs. In particular, we consider when $H = kP_3$, where P_r is a path on r vertices and kG consists of k vertex-disjoint copies of the graph G.

For graphs G and H, let $G + H$ and $G \vee H$ be the disjoint union and join of G and H, respectively. The join of G and H is the graph obtained from $G+H$ by adding edges from all vertices of G to all vertices of H. Simonovits [5] showed that the graph $K_{k-1} \vee T_r(n-k+1)$ is the unique extremal graph forbidding kK_{r+1} for sufficiently large n. Gorgol [3] gave upper and lower bounds for the Turán numbers forbidding kH, for a connected graph H on r vertices. The lower bound was obtained by noting that neither $G_1(n, kH) = K_{kr-1} + H_{\mathrm{ex}}(n-kr+1, H)$ nor $G_2(n, kH) = K_{k-1} \vee H_{\mathrm{ex}}(n - k + 1, H)$ contain k disjoint copies of H. Indeed, for $G_1(n, kH)$, K_{kr-1} does not have enough vertices to contain kH and $H_{\mathrm{ex}}(n - kr + 1, H)$ contains no copies of H. In the second case, since $H_{\mathrm{ex}}(n - k + 1, H)$ contains no copies of H, any copy of H in $G_2(n, kH)$ must contain at least one vertex in K_{k-1} which implies that there can be at most $k - 1$ copies of H in $G_2(n, kH)$. Gorgol's lower bound is, therefore,

$$\mathrm{ex}(n, kH) \geq \max\{\mathrm{e}(G_1(n, kH)), \mathrm{e}(G_2(n, kH))\}$$

where $\mathrm{e}(G)$ denotes the number of edges in G.

We consider the case when $H = P_3$. Let $\mathrm{g}_1(n, k) = \mathrm{e}(G_1(n, kP_3))$, $\mathrm{g}_2(n, k) = \mathrm{e}(G_2(n, kP_3))$ and $\mathrm{Gorgol}(n, k) = \max\{\mathrm{g}_1(n, k), \mathrm{g}_2(n, k)\}$. Note that $H_{\mathrm{ex}}(n, P_3) = M_n$, where M_n is a *near perfect matching* on n vertices. If n is even, then M_n consists of a matching on n vertices and, if n is odd, then $M_n = M_{n-1} + K_1$. Thus, $\mathrm{e}(M_n) = \lfloor n/2 \rfloor$ and we have

$$\begin{aligned}
\mathrm{g}_1(n,k) &= \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor; \\
\mathrm{g}_2(n,k) &= \binom{k-1}{2} + (n-k+1)(k-1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor; \text{ and} \\
\mathrm{Gorgol}(n,k) &= \begin{cases} \mathrm{g}_1(n,k), & \text{for } 3k \le n < 5k, \\ \mathrm{g}_2(n,k), & \text{for } n \ge 5k. \end{cases}
\end{aligned}$$

When $n < 3k$, a graph on n vertices cannot contain kP_3 as a subgraph. Gorgol's conjecture states that $\mathrm{ex}(n, kP_3) = \mathrm{Gorgol}(n,k)$ when $n \ge 3k$ and, in the same paper, Gorgol proved this is true when $k \in \{2,3\}$. Bushaw and Kettle [1] also proved the conjecture to be true for $n \ge 7k$.

In this paper, we give a constructive proof for Gorgol's Conjecture for all values of n and k. In particular, we give an algorithm that finds a set of disjoint copies of P_3 given a graph G as input. We prove Gorgol's Conjecture by showing that, if G has n vertices and more than $\mathrm{Gorgol}(n,k)$ edges, then this algorithm returns at least k disjoint copies of P_3 in G.

2 Algorithm overview

A P_3 *configuration on* G is a family of disjoint sets of vertices $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ such that, for $1 \le i \le s$, $|Q_i| = 3$ and $G[Q_i]$, the subgraph of G induced by Q_i, contains a copy of P_3 as a subgraph. The *size* of the P_3 configuration $\mathcal{Q} = \{Q_1, \ldots, Q_s\}$ is s and is denoted by $\mathrm{s}(\mathcal{Q})$. The *number of triangles in* $\mathcal{Q}$ is the number of Q_i's that induce a triangle in G. We say that a P_3 configuration $\mathcal{Q}'$ is an *improvement* of $\mathcal{Q}$ if either $\mathrm{s}(\mathcal{Q}') > \mathrm{s}(\mathcal{Q})$ or if $\mathrm{s}(\mathcal{Q}') = \mathrm{s}(\mathcal{Q})$ and $\mathcal{Q}'$ has more triangles than $\mathcal{Q}$.

We start with a P_3 configuration $\mathcal{Q}_0$ of size 0 and proceed through a sequence of iterations. For $i \ge 1$, in the i-th iteration, the algorithm follows a sequence of steps looking for an improvement $\mathcal{Q}_i$ from the current known P_3 configuration $\mathcal{Q}_{i-1}$. The steps will be performed in order and, whenever a step is successful in finding an improvement, we begin a new iteration starting from the first step. This guarantees that if a step is unsuccessful, then all previous steps are also unsuccessful. If no improvement is found on any step of an iteration, then the algorithm stops.

3 Algorithm Iteration

In this section, we describe the behaviour of the algorithm on each iteration. Each iteration follows a sequence of steps in a given order.

On the i-th iteration, each step checks for a condition on $\mathcal{Q}_{i-1}$ to obtain an improvement $\mathcal{Q}_i$. In what follows, we describe the sequence of steps in the algorithm indexed by the order they are checked. In these steps, consider $\mathcal{Q} = \mathcal{Q}_{i-1} = \{Q_1, \ldots, Q_s\}$, $U = Q_1 \cup \cdots \cup Q_s$ and $F = \mathrm{V}(G) - U$.

To help the reader understand the relationship between the algorithm steps and Gorgol's conjecture we prove a few results which are valid whenever a step cannot find an improvement on a given iteration. When proving these results, we always assume G has n vertices, $\mathrm{e}(G) > \mathrm{Gorgol}(n,k)$ and $\mathrm{s}(\mathcal{Q}) < k$. We state these conditions here and not on each result to avoid the repetition.

Step 1

If F has any vertex u that has two distinct neighbours also in F, then choose Q containing u and two of its neighbours in F and define $\mathcal{Q}_i = \mathcal{Q} \cup \{Q\}$.

For $A \subseteq \mathrm{V}(G)$, let $\mathrm{e}_G(A)$ be the number of edges of $G[A]$. For $A, B \subseteq \mathrm{V}(G)$, let $\mathrm{e}_G(A,B)$ be the number of edges in G with one endpoint in A and another in B. Let $\mathrm{d}_G(v)$ and

$d_G(v, A)$ be the *degree* of v in G and the number of neighbours of v in A, respectively. Let $\mathrm{ne}_G(A, B) = |A||B| - \mathrm{e}_G(A, B)$ be the number of non-edges with one endpoint in A and another in B. When G is clear from the context, we drop the subscript on this notation.

When Step 1 does not apply, it is expected that there are some sets Q_j with many edges to vertices in F. Therefore, we rely on finding improvements of our current configuration by rebuilding those sets Q_j with many edges to F. Let $\mathrm{EE}_j(G) = \mathrm{e}_G(Q_j) + \mathrm{e}_G(Q_j, F) - 9$. We say that a set Q_j has *excess edges* if $\mathrm{EE}_j(G) \geq 1$. Such sets will be key to find a desired improvement for $\mathcal{Q}$, therefore we show that at least one set Q_j has excess edges. We do so as a corollary of the following inequality which is important in our proof of Gorgol's Conjecture.

Lemma 1 *Let $\mathcal{Q}^+ = \{Q_1, \cdots, Q_{k-1}\}$ where the sets $Q_{s(\mathcal{Q})+1}, \ldots, Q_{k-1}$ are disjoint sets of three vertices in F and let $U^* = U \cup Q_{s+1} \cup \cdots \cup Q_{k-1}$. If $\mathrm{EE}^*_j(G) = \mathrm{e}_G(Q_j) + \mathrm{e}_G(Q_j, V(G) \setminus U^*) - 9$, then*

$$\sum_{1 \leq i < k} \mathrm{EE}^*_i(G) > \sum_{1 \leq i < j < k} \mathrm{ne}(Q_i, Q_j).$$

Corollary 2 *There is at least one set in $\mathcal{Q}$ with excess edges.*

Let $\mathcal{L} = \{Q_j \in \mathcal{Q} : \mathrm{EE}_j(G) \geq 1\}$. For every $Q_j \in \mathcal{L}$, let x_j be a vertex of Q_j with the largest number of neighbours in F. In the remaining steps, we use the following auxiliary lemmas to find improvements in $\mathcal{Q}$.

Lemma 3 *If $Q_j \in \mathcal{L}$, then $\mathrm{e}(Q_j, F) \geq 7$ and $\mathrm{d}(x_j, F) \geq 3$.*

Lemma 4 *Let C be a 4-cycle contained in a graph G. If C has at least two neighbours w_1 and w_2 not in C, then $G[\mathrm{V}(C) \cup \{w_1, w_2\}]$ contains two copies of a P_3.*

Step 2

For all $Q \in \mathcal{Q}$, do one of the following if the conditions are satisfied.

1. If $G[Q \cup F]$ contains two disjoint copies of P_3 with vertex sets Q' and Q'', then define $\mathcal{Q}_i = \{Q', Q''\} \cup \mathcal{Q} \setminus \{Q\}$.
2. If Q does not induce a triangle but $G[Q \cup F]$ contains a triangle with vertex set Q', then define $\mathcal{Q}_i = \{Q'\} \cup \mathcal{Q} \setminus \{Q\}$.

We get the following results if Step 2 does not apply.

Lemma 5 *No vertex in F has two neighbours in any $Q_a \in \mathcal{L}$.*

Lemma 6 *If $Q_a \in \mathcal{L}$, then $\mathrm{d}(x_a, F) \geq 6$.*

When we look for an improvement to $\mathcal{Q}$, Lemma 6 tells us that if $Q_a, Q_b \in \mathcal{L}$, then we can choose disjoint neighbours of x_a and x_b in F to replace Q_a and Q_b as Q'_a and Q'_b using only vertices in $\{x_a, x_b\} \cup F$. This observation will be recurrent in the following steps to find an improvement for $\mathcal{Q}$ by using the vertices in $(Q_a \cup Q_b) \setminus \{x_a, x_b\}$ to find another copy of a P_3.

Step 3

For all distinct $Q_a, Q_b \in \mathcal{Q}$, do one of the following if the conditions are satisfied.

1. If $G[Q_a \cup Q_b \cup F]$ contains three disjoint copies of P_3 with vertex sets Q, Q' and Q'' as a subgraph, then define $\mathcal{Q}_i = \{Q, Q', Q''\} \cup \mathcal{Q} \setminus \{Q_a, Q_b\}$.
2. If neither Q_a nor Q_b induce triangles in G but $G[Q_b \cup Q_a \cup F]$ contains a triangle with vertex set Q and a disjoint copy of P_3 with vertex set Q', then define $\mathcal{Q}_i = \{Q, Q'\} \cup \mathcal{Q} \setminus \{Q_a, Q_b\}$.

We get the following results if Step 3 does not apply.

Lemma 7 *If $Q_a, Q_b \in \mathcal{L}$, then* $\mathrm{ne}(Q_a, Q_b) \geq 4$.

Lemma 8 *If* $\mathrm{ne}(Q_a, Q_b) \leq 1$ *for $Q_a \in \mathcal{L}$ and $Q_b \in \mathcal{Q} \setminus \mathcal{L}$, then* $\mathrm{e}(Q_b, F) = 0$.

Step 4

For all distinct $Q_a \in \mathcal{Q} \setminus \mathcal{L}$ and $Q_b, Q_c \in \mathcal{L}$ do as follows. If $G[(Q_a \cup Q_b \cup Q_c) \setminus \{x_b, x_c\}]$ contains two disjoint copies of P_3 with vertex sets Q and Q', then let Q'_a and Q'_b be disjoint containing x_a and x_b and contained in $F \cup \{x_a, x_b\}$ and that contain copies of P_3 in G. Define $\mathcal{Q}_i = \{Q, Q', Q'_a, Q'_b\} \cup \mathcal{Q} \setminus \{Q_a, Q_b, Q_c\}$.

We get the following result if Step 4 does not apply.

Lemma 9 *If* $\mathrm{ne}(Q_a, Q_b) \leq 1$ *for $Q_a \in \mathcal{Q} \setminus \mathcal{L}$ and $Q_b \in \mathcal{L}$, then* $\mathrm{ne}(Q_a, Q_c) \geq 6$ *for every $Q_c \in \mathcal{L} \setminus \{Q_b\}$.*

4 Proof of Gorgol's Conjecture

Let G be a graph with n vertices and $\mathrm{e}(G) > \mathrm{Gorgol}(n, k)$. Suppose we apply the provided algorithm to G and it returns a P_3 configuration $\mathcal{Q}$. Assume to the contrary that $\mathrm{s}(\mathcal{Q}) < k$.

Lemma 10 *If $n < 5k$ and $Q_j \in \mathcal{L}$, then* $\mathrm{EE}^*_j(G) \leq 2k - 4$.

When $n \geq 5k$, we can get the following strengthening of Lemma 1.

Lemma 11 *Let $\mathcal{Q}^+$ and* EE^* *be defined as in Lemma 1. If $n \geq 5k$, then*

$$\sum_{1 \leq i < k} (\mathrm{EE}^*_i(G) - n + 5k - 7) > \sum_{1 \leq i < j < k} \mathrm{ne}(Q_i, Q_j).$$

Lemma 12 *If $n \geq 5k$ and $Q_j \in \mathcal{L}$, then* $\mathrm{EE}^*_j(G) - n + 5k - 7 \leq 2k - 10$.

Theorem 13 *If a graph G has n vertices with $\mathrm{e}(G) \geq$ Gorgol(n, k), then the provided algorithm will find a P_3 configuration of size at least k.*

Most results in this paper are used to find upper bounds on EE^* or lower bounds on the number of non-edges between sets in a P_3 configuration of size smaller than k. The main ideia behind the proof of Theorem 13 is to use these bounds to find a contradiction for the inequalities in Lemma 1 when $n < 5k$ and Lemma 11 when $n \geq 5k$.

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Edge clique covers in graphs with independence number 2

Pierre Charbit [1], *Geňa Hahn* [2], *Marcin Kamiński* [3], *Manuel Lafond* [2], *Nicolas Lichiardopol* [4], *Reza Naserasr* [5], Ben Seamone [2,6] and *Rezvan Sherkati* [7]

[1] IRIF, Université Paris Diderot 7, France
[2] Département d'informatique et de recherche opérationnelle, Université de Montréal, Canada
[3] Institute of Computer Science, University of Warsaw, Poland
[4] Lycée A. de Craponne, Salon, France
[5] LRI, Université Paris-Sud 11, France
[6] Department of Mathematics, Dawson College, Montréal, Canada
[7] Department of Computer Engineering, Sharif University of Technology, Tehran, Iran

Extended Abstract

Throughout, $G = (V, E)$ refers to a simple loopless graph and n denotes the number of vertices in G when the graph in question is clear. Standard graph theory notation is used throughout, but we make the following definitions explicit. We denote by $G[X]$ the subgraph of G *induced* by $X \subseteq V(G)$. A *clique* is a set $Q \subseteq V(G)$ such that $G[Q]$ is a complete graph. The *edge clique cover number* of a graph G, denoted $\text{ecc}(G)$, is the minimum number of complete subgraphs of G whose union contains every edge of G. This parameter, also known as the *intersection number* of a graph, was introduced by Erdős, Goodman, and Posa [3]. In addition to being a interesting parameter in its own right, it is also related to a parameter known as the *competition number* of a graph, introduced by Cohen [2] in the study of food webs.

We begin with the following problem:

Problem 1 (Chen, Jacobson, Kézdy, Lehel, Scheinerman, Wan [1]) *If G is a claw free graph, is $\text{ecc}(G) \leq n$?*

It was shown in [1] that the answer is "yes" in the special case of quasi-line graphs (graphs where the neighborhood of any vertex can be partitioned into two cliques), a question which was of interest from the point of view of competition numbers of graphs. More recently, Javadi and Hajebi [4] showed that $\text{ecc}(G) \leq n$ for any claw-free graph with $\alpha(G) \geq 3$, a result which relies on the claw-free graph structure theorem due to Chudnovsky and Seymour (in particular, see [6]). Our work focuses on the last remaining case:

Conjecture 2 *If G is a graph with $\alpha(G) = 2$, then $\text{ecc}(G) \leq n$.*

Tightness of the conjectured upper bound

In [4], it is shown that the bound given in Problem 1 is tight, and the graphs having $\alpha \geq 3$ which attain the bound are characterized. We show that upper bound is tight even in the case when $\alpha = 2$. A set of edges $E' \subseteq E(G)$ is called a *packing* if for any two distinct edges $e_1, e_2 \in E'$, no clique of G contains both e_1 and e_2. The *packing number* of G, denoted $\text{pack}(G)$, is the maximum size of a packing of G. Since no clique of G can cover two edges of a packing, $\text{ecc}(G) \geq \text{pack}(G)$. We use the following lemmata regarding $\text{pack}(G)$ when $\alpha(G) = 2$.

Lemma 3 *If $\alpha(G) = 2$, then $\text{pack}(G) \leq n$.*

Lemma 4 *If $\alpha(G) = 2$ and pack$(G) = n$, then $\text{ecc}(G) = n$.*

We then construct an infinite family of graphs with independence number 2 and packing number n. Let $n = 3k+1$ for some integer $k \geq 1$, and let $H_k = (V, E)$ with $V = \{0, \ldots, n-1\}$ and $E = \{ij : i-j \equiv 0, 1 \pmod 3\}$. In other words, i and j are adjacent if and only if $i-j+1$ is not a multiple of 3.

Lemma 5 *For any $k \geq 1$, $\alpha(H_k) = 2$ and* $\mathrm{pack}(H_k) \geq n$.

It follows that the bound proposed in Conjecture 2 is tight.

Upper bounds on $\mathrm{ecc}(G)$

We begin the following simple lemma.

Lemma 6 *If G with $\alpha(G) \leq 2$ has no dominating edge, then* $\mathrm{ecc}(G) \leq n$.

Our main theorems below provide upper bounds on $\mathrm{ecc}(G)$ graphs with independence number 2. These are obtained by showing that one may remove dominating edges without increasing the independence number of the graph, then bounding the number of cliques needed to cover the removed dominating edges in two different ways. This method, combined with Lemma 6, yields the following two theorems as a result.

Theorem 7 *If G is a graph with $\alpha(G) = 2$, then* $\mathrm{ecc}(G) \leq n + \delta(G) + 1$.

Theorem 8 *There exists a positive real number c such that if G is a graph with $\alpha(G) = 2$, then* $\mathrm{ecc}(G) \leq 2n - c\sqrt{n \log n}$.

The results in these theorems above are improvements on results of Javadi, Maleki, and Omoomi [5], who show that there is a constant c such that $\mathrm{ecc}(G) \leq cn^{4/3} \log^{1/3} n$ if $\alpha(G) = 2$.

We also consider the fractional version of the problem. A graph G has a *fractional edge clique cover* of size k if there exists a set of cliques $\mathcal{Q}$ and a real-valued weight function w such that

1. $w(Q) \geq 0$ for all $Q \in \mathcal{Q}$,
2. $\displaystyle\sum_{Q\,:\,e \in E(G[Q])} w(Q) \geq 1$ for each $e \in E(G)$, and
3. $\displaystyle\sum_{Q \in \mathcal{Q}} w(Q) \geq k$.

The *fractional edge clique cover number* of a graph G, denoted $\mathrm{ecc}_f(G)$ and first studied in [7], is the smallest k such that G has a fractional edge clique cover of size k. The proofs of the theorems above are easily modified to give upper bounds on $\mathrm{ecc}_f(G)$ which are improvements on their integer counterparts (though, all of which are still greater than n).

Further results

The proof idea behind Theorems 7 and 8 can be used to show that Conjecture 2 holds in some special cases. Two examples are as follows:

Theorem 9 *If G is a graph with $\alpha(G) = 2$ and* $\mathrm{diam}(G) \leq 3$, *then* $\mathrm{ecc}(G) \leq \lceil \frac{n}{2} \rceil + 1$.

Theorem 10 *If G is a C_4-free graph with $\alpha(G) = 2$, then* $\mathrm{ecc}(G) \leq n$.

We also derive a number of properties which a vertex-minimal counterexample to Conjecture 2 must have, if one exists, in terms of connectivity, clique number, and minimum degree.

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Partial cubes without Q_3^- minors

Victor Chepoi [1], *Kolja Knauer* [1] and Tilen Marc [2]

[1] Laboratoire d'Informatique Fondamentale, Aix-Marseille Université and CNRS, Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France

[2] Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

Extended Abstract

Partial cubes are the graphs which admit an isometric embedding into a hypercube. Djoković [4] characterized partial cubes in the following simple but pretty way: a graph $G = (V, E)$ can be isometrically embedded in a hypercube if and only if G is bipartite and for any edge uv, the sets $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$ and $W(v, u) = \{x \in V : d(x, v) < d(x, u)\}$ are convex. In this case, $W(u, v) \cup W(v, u) = V$, whence $W(u, v)$ and $W(v, u)$ are complementary convex subsets of G, called *halfspaces*. The edges between $W(u, v)$ and $W(v, u)$ correspond to a coordinate in a hypercube embedding of G.

Moreover, partial cubes have the separation property S_3: any convex subgraph G' of a partial cube G can be represented as an intersection of halfspaces of G [2]. We call such a representation (or simply the convex subgraph G') a *restriction* of G. A *contraction* of G is the partial cube G' obtained from G by contracting all edges corresponding to a given coordinate in a hypercube embedding. Now, a partial cube H is called a *partial cube-minor* (abbreviated, *pc-minor*) of G if H can be obtained by a sequence of contractions from a convex subgraph of G. If $T_1, \ldots, T_m$ are finite partial cubes, then $\mathcal{F}(T_1, \ldots, T_m)$ is the set of all partial cubes G such that *no* $T_i, i = 1, \ldots, m$, can be obtained as a pc-minor of G. We say that a class of partial cubes $\mathcal{C}$ is *pc-minor-closed* if we have that $G \in \mathcal{C}$ and G' is a minor of G imply that $G' \in \mathcal{C}$. For any set of partial cubes $T_1, \ldots, T_m$, the class $\mathcal{F}(T_1, \ldots, T_m)$ is pc-minor-closed. Many important subfamilies of partial cubes are pc-minor-closed as it can be shown from their basic properties: median graphs, bipartite cellular graphs, graphs of lopsided sets, tope graphs of COMs, the class $\mathcal{S}_4$ also known as Pasch graphs,...

In the talk we present some structural results of graphs in $\mathcal{F}(Q_3^-)$, where Q_3^- denotes the graph of the three dimensional cube minus a vertex. In some sense this is the simplest, non-trivial pc-minor-closed family. It turns out that this class presents a natural generalization to many known families; for the reasons that all finite convex subgraphs of graphs in $\mathcal{F}(Q_3^-)$ can be obtained from the Cartesian products of edges and even cycles (*cells*) by gluing them together, we call them *hypercellular graphs.*

Our results mainly concern their structure. It is well-known [1] that median graphs are exactly the graphs in which the convex hulls of isometric cycles are hypercubes; these hypercubes are gated subgraphs. Moreover, any finite median graph can be obtained by gated amalgams from cubes [5, 7]. Analogously, it was shown in [3] that any isometric cycle of a bipartite cellular graph is a convex and gated subgraph; moreover, the bipartite cellular graphs are exactly the bipartite graphs which can be obtained by gated amalgams from even cycles. We extend these results in the following way:

Theorem 1 *The convex closure of any isometric cycle of a graph $G \in \mathcal{F}(Q_3^-)$ is a gated subgraph isomorphic to a Cartesian product of edges and even cycles.*

We say that a partial cube G satisfies the *3-convex cycles condition* (abbreviated, *3CC-condition*) if for any three convex cycles C_1, C_2, C_3 that intersect in a vertex and pairwise intersect in three different edges the convex hull of $C_1 \cup C_2 \cup C_3$ is a cell. Defining the dimension of a cell X as the number of edge-factors plus two times the number cyclic factors (which corresponds to the topological dimension of $[X]$) one can give a natural generalization of the 3CC-condition. We say that a partial cube G (or its cell complex $\mathbf{X}(G)$) satisfies the *3-cell condition* (abbreviated, *3C-condition*) if for any three cells X_1, X_2, X_3 of dimension $k + 2$ that intersect in a cell of dimension k and pairwise intersect in three different cells of dimension $k + 1$ the convex hull of $X_1 \cup X_2 \cup X_3$ is a cell.

Theorem 2 *For a partial cube $G = (V, E)$, the following conditions are equivalent:*

(i) $G \in \mathcal{F}(Q_3^-)$, *i.e., G is hypercellular;*

(ii) any cell of G is gated and G satisfies the 3CC-condition;

(iii) any cell of G is gated and G satisfies the 3C-condition;

(iv) each finite convex subgraph of G can be obtained by gated amalgams from cells.

A further characterization of hypercellular graphs is analogous to median and cellular graphs. We show that hypercellular graphs satisfy the so-called *median-cell property*, which is essentially defined as follows: for any three vertices u, v, w of G there exists a unique gated cell X of G such that if u', v', w' are the gates of u, v, w in X, respectively, then u', v' lie on a common (u, v)-geodesic, v', w' lie on a common (v, w)-geodesic, and w', u' lie on a common (w, u)-geodesic. Namely, we have:

Theorem 3 *A partial cube G satisfies the median-cell property if and only if G is hypercellular.*

Theorem 2 immediately implies that median graphs and bipartite cellular graphs are hypercellular. Furthermore, a subclass of netlike partial cubes, namely partial cubes which are gated amalgams of even cycles and cubes [6], are hypercellular. In particular, we obtain that these three classes coincide with $\mathcal{F}(Q_3^-, C_6)$, $\mathcal{F}(Q_3^-, Q_3)$, and $\mathcal{F}(Q_3^-, C_6 \times K_2)$, respectively. Other direct consequences of Theorem 2 concern convexity invariants (Helly, Caratheodory, Radon, and partition numbers) of hypercellular graphs which are shown to be either a constant or bounded above.

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Steinberg's Conjecture is false

Vincent Cohen-Addad* Michael Hebdige† Daniel Král'‡

Zhentao Li* Esteban Salgado§

Steinberg's Conjecture is one of the most well-known open problems in planar graph colourings. The conjecture has been open for 40 years and reads as follows:

Conjecture 1 (Steinberg's Conjecture)**.** *Every planar graph that does not have a cycle of length of four and five can be coloured using at most three colours.*

As a possible approach to proving the conjecture, Erdős [**?**] suggested finding the smallest c such that every planar graph that does not contain cycles of length $\{4,...,c\}$ is 3-colourable. The best known bound was proven in 2005 by Borodin et al. [**?**] when they showed that every planar graph that does not have cycles of length four, five, six and seven is 3-colourable i.e. $c \leq 7$.

We resolve Steinberg's conjecture by constructing a planar graph that has no cycles of lengths four or five that is not 3-colourable. This counterexample also acts as a counterexample to the following strengthenings of Steinberg's Conjecture.

Conjecture 2 (Strong Bordeaux Conjecture [**?**])**.** *Every planar graph that does not have any pair of adjacent triangles or a cycle of length five is 3-colourable.*

Conjecture 3 (Novosibirsk 3-Color Conjecture [**?**])**.** *Every planar graph that does not have a cycle of length three sharing an edge with a cycle of length three or five is 3-colourable.*

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*Département d'Informatique, UMR CNRS 8548, École Normale Supérieure, Paris, France. E-mail: vcohen,zhentao@di.ens.fr.

†Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK. E-mail: m.hebdige@warwick.ac.uk. The work of this author was supported by the Leverhulme Trust 2014 Philip Leverhulme Prize of the forth author.

‡Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK. E-mail: d.kral@warwick.ac.uk. The work of this author was supported by the Leverhulme Trust 2014 Philip Leverhulme Prize.

§École Polytechnique, Paris, France, and Pontificia Universidad Católica de Chile, Santiago, Chile. E-mail: easalgadovalenzuela@gmail.com

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Subdivisions of oriented cycles in digraphs with large chromatic number

Nathann Cohen [1], *Frédéric Havet* [2] William Lochet [2] and *Nicolas Nisse* [3]

[1] CNRS, LRI, Univ. Paris Sud, Orsay, France [2] Univ. Côte d'Azur, CNRS, Inria, I3S, France [3] Univ. Côte d'Azur, Inria, CNRS, I3S, France

Extended Abstract

The *chromatic number* $\chi(D)$ of a digraph D is the chromatic number of its underlying graph. The *chromatic number* of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest k such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such k exists. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ *has bounded chromatic number.*

We are interested in the following question : which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by $\mathrm{Forb}(H)$ (resp. $\mathrm{Forb}(\mathcal{H})$) the class of digraphs that do not contain H (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows :

Problem 1. *Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\mathrm{Forb}(\mathcal{D})) < +\infty$?*

An *oriented graph* is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said *directed* if all vertices have in-degree and out-degree at most 1.

Observe that if D is an orientation of a graph G and $\mathrm{Forb}(D)$ has bounded chromatic number, then $\mathrm{Forb}(G)$ has also bounded chromatic number. A classical result by Erdős implies that G must be a tree. Burr proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k and conjectured Burr [3] that it could be further improved to $(2k-2)$-chromatic digraphs.

For special oriented trees T, better bounds on the chromatic number of $\mathrm{Forb}(T)$ are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that $\chi(\mathrm{Forb}(P^+(k))) = k$, where $P^+(k)$ is the directed path of length k (a *directed path* is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path P on n vertices with two blocks (*blocks* are maximal directed subpaths) has been determined by Addario-Berry et al. [1] :

Theorem 2 (Addario-Berry et al. [1]). *Let P be an oriented path with two blocks on $n \geq 4$ vertices, then $\chi(\mathrm{Forb}(P)) = n - 1$.*

In this paper, we are interested in the chromatic number of $\mathrm{Forb}(\mathcal{H})$ when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by S-Forb(D) (resp. S-Forb($\mathcal{D}$)) the class of digraphs that contain no subdivision of D (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of S-Forb($\mathcal{C}$), where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $\vec{C}_k$ the directed cycle of length k. For all k, $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k,\ell)$ the oriented cycle with two blocks, one of length k and the other of length ℓ. Observe that the oriented cycles with two blocks are the subdivisions of $C(1,1)$. As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k,\ell))) = +\infty$. We first generalise this result to every oriented cycle.

Theorem 3. *For any oriented cycle C, $\chi(\text{S-Forb}(C)) = +\infty$.*

In fact, we show the following stronger theorem.

Theorem 4. *For any positive integers b, c, there exists an acyclic digraph D_c with $\chi(D_c) \geq c$ in which all oriented cycles have more than b blocks.*

We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number.

Theorem 5. *[5, Theorem 1'] For $k, g, c \in \mathbb{N}$, there exists a k-uniform hypergraph with girth larger than g and weak chromatic number larger than c.*

We assume g is being fixed, the following construction allow us to find D_{c+1} from D_c. Let p be the number of proper c-colourings of D_c, and let those colourings be denoted by $col_c^1, ..., col_c^p$. By Theorem 5 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> g/2$. Let $X = \{x_1, \dots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct D_{c+1} from n disjoint copies $D_c^1, ..., D_c^n$ of D_c as follows. For each hyperedge $S \in \mathcal{H}$, we do the following :

- We partition S into p sets $S_1, \dots, S_p$ of cardinality c.
- For each set $S_i = \{x_{k_1}, \dots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$ such that $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \dots, v_{k_c}$ as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by $\mathcal{S}$.

Theorem 6 (Bondy [2])**.** $\chi(\text{S-Forb}(\vec{C}_k) \cap \mathcal{S}) = k - 1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

Problem 7. *Let k and ℓ be two positive integers then $\chi(\text{S-Forb}(C(k, \ell) \cap \mathcal{S})) < k + l$.*

We give evidence for this problem by showing the following weaker statement.

Theorem 8. *Let k and ℓ be two positive integers such that $k \geq \max\{\ell, 3\}$, and let D be a digraph in $\text{S-Forb}(C(k, \ell)) \cap \mathcal{S}$. Then, $\chi(D) \leq (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$.*

We need the following lemma.

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

Lemma 9. *Let D_1 and D_2 be two digraphs. $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.*

A consequence of the previous lemma is that, if we partition the arc set of D into set A_1 $\cdots$ A_k, then bounding the chromatic number of all digraphs induced by the A_i implies that D has bounded chromatic number.

Proof. Let D be a strong digraph without any copy of $C(k, \ell)$, we exhibit a colouring of D using a bounded number of colours. The proof heavily relies on the technique of *levelling.* Let u be a vertex of D. The *level* of a vertex x, noted $\text{lvl}(x)$ is the length of the shortest dipath from u to x. $L(i)$ is the set of vertices at level i.

Since D is strongly connected, it has an out-generator u. Let T be a BFS-tree with root u. We define the following sets of arcs.

$$\begin{aligned}
A_0 &= \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\
A_1 &= \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3; \\
A' &= \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}.
\end{aligned}$$

Since $k + \ell - 3 > 0$ and there is no arc xy with $\text{lvl}(y) > \text{lvl}(x) + 1$, (A_0, A_1, A') is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition A' into two sets A_2 and A_3, where $A_2 = \{xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T\}$ and $A_3 = A' \setminus A_2$. Then (A_0, A_1, A_2, A_3) is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$.

Claim 10. $\chi(D_0) \leq k + \ell - 2$.

Proof. Observe that D_0 is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \mathrm{dist}_D(u, v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leq k + \ell - 2$ for all non-negative integer i.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.

Assume now $i \geq 1$. Suppose for a contradiction $\chi(D[L_i]) \geq k + \ell - 1$. Since $k \geq 3$, by Theorem 2, $D[L_i]$ contains a copy Q of $P^+(k-1, \ell-1)$, the path on two blocks of length $k-1$ and $\ell-1$ with one vertex of indegree 2. Let v_1 and v_2 be the initial and terminal vertices of Q, and let x be the least common ancestor of v_1 and v_2. By definition, for $j \in \{1, 2\}$, there exists a dipath P_j from x to v_j in T. By definition of least common ancestor, $V(P_1) \cap V(P_2) = \{x\}$, $V(P_j) \cap L_i = \{v_j\}$, $j = 1, 2$, and both P_1 and P_2 have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k, \ell)$, a contradiction. □

Claim 11. $\chi(D_1) \leq k + \ell - 3$.

Proof. Let ϕ_1 be the colouring of D_1 defined by $\phi_1(x) = \mathrm{lvl}(x) \pmod{k+\ell-3}$. By definition of D_1, this is clearly a proper colouring of D_1. □

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

Claim 12. $\chi(D_2) \leq 2\ell + 2$.

Claim 13. $\chi(D_3) \leq k + \ell + 1$.

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result.

□

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for $\hat{C}_4$ the cycle with 4 blocks.

Theorem 14. *Let D be a digraph in* S-Forb$(\hat{C}_4)$. *If D admits an out-generator, then* $\chi(D) \leq 24$.

Proof. The general idea is the same as in the proof of Theorem 8.

Suppose that D admits an out-generator u and let T be an BFS-tree with root u. We partition $A(D)$ into three sets according to the levels of u.

$$\begin{array}{rcl} A_0 & = & \{(x, y) \in A(D) \mid \mathrm{lvl}(x) = \mathrm{lvl}(y)\}; \\ A_1 & = & \{(x, y) \in A(D) \mid |\mathrm{lvl}(x) - \mathrm{lvl}(y)| = 1\}; \\ A_2 & = & \{(x, y) \in A(D) \mid \mathrm{lvl}(y) \leq \mathrm{lvl}(x) - 2\}. \end{array}$$

For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

Claim 15. $\chi(D_0) \leq 3$.

Proof. Suppose for a contradiction that $\chi(D) \geq 4$. By Theorem 2, it contains a $P^-(1, 1)$ (y_1, y, y_2), that is (y, y_1) and (y, y_2) are in $A(D_0)$. Let x be the least common ancestor of y_1 and y_2 in T. The union of $T[x, y_1]$, (y, y_1), (y, y_2), and $T[x, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction. □

Claim 16. $\chi(D_1) \leq 2$.

Proof. Since the arc are between consecutive levels, then the colouring ϕ_1 defined by $\phi_1(x) = lvl(x) \mod 2$ is a proper 2-colouring of D_1. □

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of y in $\bigcup_{0 \leq j \leq i-1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V} \{(x, y), y \in N'(x)\}$ and $D_x = (V, A_x)$ where A_x is the set of arc inside the level and from V_i to V_{i+1} for all i. Note that $A = A' \cup A_x$ and

Claim 17. $\chi(D_2) \leq 4$.

Proof. We refer to [4] for the proof of this statement. □

Claims 15, 16, 17, and Lemma 9 implies $\chi(D) \leq 24$. □

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On the Spectra of Markov Matrices for Weighted Sierpiński Graphs

Francesc Comellas [1], Pinchen Xie [2] and Zhongzhi Zhang [2]

[1] Dep. of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain
[2] School of Computer Science, Fudan University, Shanghai, China

Extended Abstract

Relevant information from networked systems can be obtained by analyzing the spectra of matrices associated to their graph representations. In particular, the eigenvalues and eigenvectors of the Markov matrix and related Laplacian and normalized Laplacian matrices allow the study of structural and dynamical aspects of a network, like its synchronizability and random walks properties.

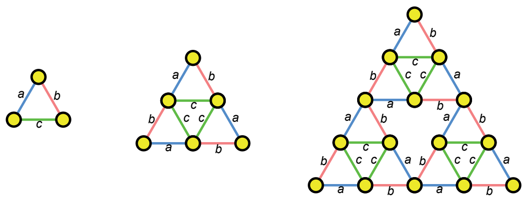
In this study we obtain, in a recursive way, the spectra of Markov matrices of a family of rotationally invariant weighted Sierpiński graphs. From them we find analytic expressions for the weighted count of spanning trees and the random target access time for random walks on this family of weighted graphs.

Construction of W_t. The rotationally invariant weighted Sierpiński graph W_t, $t \geq 0$, is constructed as follows [1]:

For $t = 0$, W_0 is K_3 (a 3-cycle) and its three edges have weights a, b, c.

For $t \geq 1$, W_t is obtained by recurrence by joining three copies of W_{t-1} and identifying two vertices of each copy with one of the vertices of each of the other two copies.

The construction process, as well as the distribution of edge weights (a, b, c), for the rotationally invariant case is shown in the next figure.



The order of W_t is $N_t = (3^{t+1} + 3)/2$ and the total number of edges is $L_t = 3^{t+1}$. At each iteration 3^t new vertices are added to the graph.

Spectrum of the probability transition matrix for random walks on W_t. A_t denotes the adjacency matrix of the weighted graph W_t and has elements $A_t(i,j) = w(i,j)$, where $w(i,j)$ is the weight of edge (i,j). The degree matrix of W_t, denoted by D_t, is a diagonal matrix such that $D_t(i,i) = \sum_j w(i,j)$. The probability transition matrix for random walks on W_t, or Markov matrix, is defined as $M_t = D_t^{-1} A_t$.

When $a = b = c$, W_t degenerates into the unweighted Sierpiński graph S_t. The spectrum of the transition matrix of S_t, denoted $\overline{M}_t$, has been determined elsewhere [2], and the resulting recursive equation for the eigenvalues is $\overline{\lambda}^{(t)} = \overline{\lambda}^{(t+1)}\left(4\overline{\lambda}^{(t+1)} - 3\right)$ where $\overline{\lambda}^{(t+1)} \neq -\frac{1}{4}, \pm\frac{1}{2}$. This equation gives a relationship between the spectra of the transition matrices at steps t and $t+1$, i.e., each eigenvalue of $\overline{M}_{t+1}$, except for the exceptional eigenvalues $\{-\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}\}$, corresponds to an eigenvalue of $\overline{M}_t$. The multiplicities of the exceptional eigenvalues are: $m_{\overline{M}_t}\left(-\frac{1}{2}\right) = \frac{3+3^t}{2}$, $m_{\overline{M}_t}\left(-\frac{1}{4}\right) = \frac{3^{t-1}-1}{2}$), $m_{\overline{M}_t}\left(\frac{1}{2}\right) = 0$, $t > 0$. Thus, in [2] the spectrum of $\overline{M}_t$ is given as $\sigma(\overline{M}_t) = \{1, -\frac{1}{2}\} \bigcup \left(\bigcup_{i=0}^{t-1} Q_{-i}\{\frac{1}{4}\}\right) \bigcup \left(\bigcup_{i=0}^{t-2} Q_{-i}\{-\frac{1}{4}\}\right)$ where $Q_{-i}A$ denotes the preimage of a set A under the i-th composition power of the function $Q(x) = x(4x-3)$.

With a similar technique, we partition the matrix M_t into blocks corresponding to the transition probabilities among old and new vertices, with respect to the last iteration, and study the Schur complement of M_t. In the next theorem, we relate the eigenvalues of $\overline{M}_t$ with those of M_t to find the complete spectrum of M_t. We have also obtained the multiplicities for all the eigenvalues of M_t.

Theorem 1 *Any eigenvalue $\overline{\lambda}^{(t-1)}$ of $\overline{M}_{t-1}$, is related to several eigenvalues of M_t, denoted $\{\lambda_i^{(t)}\}$, and they are the preimage of $\overline{\lambda}^{(t-1)}$ under the function R given by*

$$R(z) = \frac{(a+b)z(sz-2c)(sz+c) - (a^2+b^2)sz + c(a-b)^2}{2(a^2+b^2)c + 2absz}$$

where $s = a+b+2c$ and $z \notin \{-\frac{c}{s}, \frac{2c}{s}, -\frac{(a^2+b^2)c^2}{abs}\}$, the exceptional eigenvalues of M_t.

Weighted count of the spanning trees of W_t. A spanning tree of W_t is a subgraph that includes all the vertices of W_t and is a tree. Let $\Upsilon(W_t)$ denote the set of spanning trees of W_t. For $\mathcal{T} \in \Upsilon(W_t)$, we define its weight $w(\mathcal{T})$ as $\prod_{e \in \mathcal{T}} w_e$, where w_e denotes the weight of edge e. Let $\tau(W_t) = \sum_{\mathcal{T} \in \Upsilon(W_t)} w(\mathcal{T})$ denote the weighted count of the spanning trees of W_t. From [3], $\tau(W_t) = (\prod_{k=1}^{N_t-1} \gamma_k^{(t)} \prod_{i=1}^{N_t} d_i^{(t)}) / \sum_{i=1}^{N_t} d_i^{(t)}$, where $0 = \gamma_0^{(t)} < \gamma_1^{(t)} \leq \cdots \leq \gamma_{N_t-1}^{(t)}$ are the eigenvalues of $\mathcal{L}_t$, the normalized Laplacian matrix of W_t, and $d_i^{(t)} = D_t(i,i)$. Obviously, the eigenvalue spectrum of the matrix $D_t^{\frac{1}{2}} M_t D_t^{-\frac{1}{2}} = (I - \mathcal{L}_t)$ is the same as the spectrum of M_t. Thus, for each k, $(1 - \gamma_k^{(t)})$ is an eigenvalue of M_t. We find the following result:

Theorem 2 *The number of spanning trees of W_t, $t > 0$, is*

$$2^{\frac{3^{t-1}-1}{2}} 3^{\frac{3^t+2t-1}{4}} 5^{\frac{3^{t-1}-2t+1}{4}} (a+b)^{3^{t-1}} (ab+ac+bc)^{\frac{3^t+1}{2}} (a+b+3c)^{\frac{3^{t-1}-1}{2}}.$$

This result coincides with the the values obtained from generating functions by D'Angeli and Donno [1], and verifies the correctness of our computation of the spectrum of M_t.

Random target access time for random walks on W_t. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_{N_t})$ denote the stationary distribution for random walks on W_t, which is an eigenvector of M_t associated to the eigenvalue 1. Let $H_{ij}(t)$ represent the mean first passage time from vertex i to vertex j. The random target access time for random walks on W_t, denoted by H_t, is defined as the expected time for a walker starting from vertex i to reach for the first time a target vertex j, selected stochastically according to the stationary distribution. Thus, $H_t = \sum_{j=1}^{N_t} \pi_j H_{ij}(t)$. The random target access time, which reflects the structure of the the entire graph, is independent of the choice of the starting vertex. It has been proved [5] that H_t can be expressed in terms of the nonzero eigenvalues of $\mathcal{L}_t$, given as $H_t = \sum_{i=1}^{N_t-1} \frac{1}{\gamma_i^{(t)}}$. We find the following result:

Theorem 3 *The random target access time H_t for random walks on W_t is*

$$\frac{s^2-c^2}{ab+ac+bc}\left(\frac{14 \cdot 5^{t-2}}{3} - \frac{3^{t-1}}{5} + \frac{3}{5}\right) + \frac{s(3^{t-1}-1)}{2(s+c)} - \frac{s(3+3^{t-1})}{2(a+b)} + \frac{abs(3^t+1)}{2(ab+ac+bc)(a+b)} + \frac{2}{3},$$

where $s = a+b+2c$ and $t > 1$.

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Class 0 Bounds for Graph Pebbling

Daniel W. Cranston [1], *Luke Postle* [2], *Chenxiao Xue* [3] and *Carl Yerger* [3]

[1] Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, USA

[2] Department of Combinatorics and Optimization, University of Waterloo, Ontario, Canada

[3] Department of Mathematics and Computer Science, Davidson College, Davidson, NC, USA

Extended Abstract

Graph pebbling was introduced by Chung in 1989. Following a suggestion of Lagarias and Saks, she computed the pebbling number of Cartesian products of paths to give a combinatorial proof of the following number-theoretic result of Kleitman and Lemke.

Theorem 1 ([3, 10]). *Let $\mathbb{Z}_n$ be the cyclic group on n elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k\in K}$, such that $\sum_{k\in K} \frac{1}{|g_k|} \leq 1$. Here K is the set of indices of the elements in the subsequence.*

Chung developed the pebbling game to give a more natural proof of this theorem. Results of this type are important in this area of number theory, as they generalize zero-sum theorems such as the Erdős-Ginzburg-Ziv theorem [5]. Over the past two decades, pebbling has developed into its own subfield, with over 80 papers. For reference, see [8, 9] or [6, Section 11.4].

We consider a connected graph G with *pebbles* (indistinguishable markers) on some of its vertices. More precisely, a *configuration* p on a graph G is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$. A *pebbling move* removes two pebbles from some vertex and places one pebble on an adjacent vertex. A *rooted graph* is a pair (G, r) where G is a graph and $r \in V(G)$ is the *root vertex*. A pebbling configuration p is *solvable* for a rooted graph (G, r) if some configuration p' has at least one pebble on r, and p' can be obtained from p by a sequence of pebbling moves. Otherwise, p is *unsolvable* (or *r-unsolvable*, when the root r is specified.)

The *pebbling number* $\pi(G)$ is the least integer k such that, for any vertex $v \in V(G)$ and any initial configuration p of k pebbles, p is solvable for (G, v). Likewise $\pi(G, r)$ is the pebbling number of G, when the root vertex must be r. A trivial lower bound for $\pi(G)$ is $|V(G)|$: for some root r, we place one pebble on each vertex other than r, for a total of $|V(G)| - 1$ pebbles, but we cannot reach r.

The path, P_n, on n vertices has $\pi(P_n) = 2^{n-1}$. More generally, if graph G has diameter d, then $\pi(G) \geq 2^d$. Let $f(n, d)$ denote the maximum pebbling number of an n-vertex graph with diameter d. Pachter, Snevily, and Voxman [11] proved that $f(n, 2) = n + 1$, and Clarke, Hochberg, and Hurlbert [4] classified all graphs G of diameter 2 with $\pi(G) = n + 1$. Bukh [2] proved that $f(n, 3) = \frac{3}{2}n + O(1)$, and Postle, Streib, and Yerger [12] strengthened Bukh's result, proving the exact bound $f(n, 3) = \lfloor \frac{3}{2}n \rfloor + 2$. They also gave [12] an asymptotic bound for $f(n, 4)$.

In this talk, we study Class 0 graphs via a discharging-based approach. We focus on graphs with diameter at least 2 since those with diameter 0, a single vertex, and diameter 1, a complete graph, are well understood. A graph G is *Class 0* if its pebbling number is equal to its number of vertices, i.e., $\pi(G) = |V(G)|$. Recall that always $\pi(G) \geq |V(G)|$, so Class 0 graphs are those where this trivial lower bound holds with equality. For each vertex v, we write $N(v)$ for the set of vertices adjacent to v, and we write $N[v]$ to denote $N(v) \cup \{v\}$. For a graph G, let $e(G)$ denote the number of edges in G. In this talk, we prove lower bounds on $e(G)$ for all Class 0 graphs.

Blasiak et al. [1] showed that every n-vertex Class 0 graph G has $e(G) \geq \lfloor \frac{3n}{2} \rfloor$. They also conjectured (see [7, p. 19]) that for some constant C and for all sufficiently large n there exist n-vertex Class 0 graphs with $e(G) \leq \lfloor \frac{3n}{2} \rfloor + C$. In particular, they defined a family of "generalized Petersen graphs" $P_{m,d}$ on $(2^d-1)m+1$ vertices with diameter d, one vertex of some fixed degree m, and all other vertices of degree 3. They conjectured that these graphs are all Class 0. We disprove both of these conjectures.

In Corollary 4 we show that given a constant C, for sufficiently large n, all n-vertex Class 0 graphs G have $e(G) > \lfloor \frac{3n}{2} \rfloor + C$. This disproves the first conjecture. In Theorem 6, we extend this idea to show that every n-vertex Class 0 graph G, with diameter at least 3, has $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. Since the generalized Petersen graph $P_{m,d}$ has $e(P_{m,d}) = (3 + \frac{1}{2^d-1})\frac{n-1}{2}$, this shows that for $d \geq 3$ every sufficiently large generalized Petersen graph is not Class 0, which disproves the second conjecture in a strong sense. (The proof that $P_{m,2}$ is not Class 0 follows immediately from Corollary 3.)

Our main tool for proving bounds on $e(G)$ is the following lemma.

Lemma 2 (Small Neighborhood Lemma). *Let G be a graph and $u, v \in V(G)$. If $N_a[u] \cap N_b[v] = \emptyset$ and $|N_a[u] \cup N_b[v]| < 2^{a+b+1}$, then G is not Class 0. (Here $N_k[w]$ denotes the set of all vertices that are distance at most k from vertex w.)*

Proof. We assume the statement is false and construct a configuration with $|V(G)|$ vertices that is u-unsolvable. Form configuration p by putting $2^{a+b+1}-1$ pebbles on v, 0 pebbles on each vertex of $N[u] \cup N(v)$, and 1 pebble on each other vertex. Since $|N[u] \cup N[v]| \leq 2^{a+b+1}-1$, this configuration has at least $|V(G)|$ pebbles. Now no pebble can reach u, since at most 2^a-1 pebbles can leave $N_b[v]$. This contradicts that G is Class 0. $\square$

The following corollary is immediate from the Small Neighborhood Lemma.

Corollary 3. *Let G be a Class 0 graph. If $u, v \in V(G)$, $d(u) = 2$, and u and v are distance at least 3 apart, then $d(v) \geq 4$. Similarly, if $u, v \in V(G)$, $d(u) = 3$, u and v are distance at least 4 apart, and each neighbor of v is a 3-vertex, then $d(v) \geq 4$.*

Proof. Assume $d(v) \leq 3$. For the first statement take $a = b = 1$; for the second, take $a = 1$ and $b = 2$. In each case we contradict the Small Neighborhood Lemma. $\square$

Corollary 4. *For each integer C, there exists an integer n_0, such that if G is any n-vertex graph with $\delta(G) = 3$, $n \geq n_0$, and $e(G) \leq \frac{3}{2}n + C$, then G is not Class 0.*

Proof. We can choose n_0 sufficiently large so that there exists some pair of vertices u, v violating the second statement of the Small Neighborhood Lemma. Specifically, it suffices to find a 3-vertex v such that every vertex within distance four of v is a 3-vertex. To guarantee such a vertex v exists, we take, for example, $n_0 = 2C * 3^5$. $\square$

Now we use the Small Neighborhood Lemma to prove, in Theorem 6, that every n-vertex Class 0 graph G with diameter at least 3 has $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. The case $\delta(G) = 2$ is complicated, so we handle it separately, in Lemma 5. For the case $\delta(G) \leq 1$, we use a lemma from [4].

Recall that a *k-vertex* is a vertex of degree k. Similarly, a *k^+-vertex* has degree at least k and a *k-neighbor* of a vertex v is a k-vertex adjacent to v.

Lemma 5. *If an n-vertex Class 0 graph G has diameter at least 3 and $\delta(G) = 2$, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

Proof. (Sketch) Let G be an n-vertex Class 0 graph with diameter at least 3 and $\delta(G) = 2$. We assign each vertex v a charge $\mathrm{ch}(v)$, where $\mathrm{ch}(v) = d(v)$. Now we redistribute these charges, without changing their sum, so that all but a few vertices finish with charge at least $\frac{10}{3}$; the charge of each vertex v after redistributing is $\mathrm{ch}^*(v)$. If at most k vertices finish with charge less than $\frac{10}{3}$ (but all charges are nonnegative), then $e(G) = \frac{1}{2}\sum_{v \in V} \mathrm{ch}(v) = \frac{1}{2}\sum_{v \in V} \mathrm{ch}^*(v) \geq \frac{1}{2}(\frac{10}{3}(n-k)) = \frac{5}{3}n - \frac{5}{3}k$.

Choose $r \in V(G)$ such that $d(r) = 2$. For each positive integer i, let N_i denote the set of vertices at distance i from r. Also, let $N_{3^+} = \bigcup_{i \geq 3} N_i$. By the Small Neighborhood Lemma with $u = r$, if $v \in N_{3^+}$, then $d(v) \geq 4$.

We redistribute charge according to the following two discharging rules.

1. Each vertex $v \in N_2$ takes charge 1 from some neighbor in N_1. If $d(v) = 2$, then v also takes charge $\frac{1}{3}$ from its other neighbor.

2. Each vertex $v \in N_{3^+}$ with $d(v) = 4$ takes charge $\frac{1}{3}$ from each neighbor u with $d(u) \geq 3$.

We show that nearly all vertices finish with charge at least $\frac{10}{3}n$ and can bound the ones that do not to obtain the result of the lemma.

□

Theorem 6. *If G is an n-vertex Class 0 graph with diameter at least 3, then $e(G) \geq \frac{5}{3}n - \frac{11}{3}$.*

Proof. Let G be Class 0 with diameter at least 3. It is straightforward to show that $\delta(G) \geq 2$. Lemma 5 proves the bound when $\delta(G) = 2$. If $\delta(G) \geq 4$, then $e(G) \geq \frac{\delta(G)n}{2} \geq 2n$. Thus, we assume that $\delta(G) = 3$.

The proof is similar to that of Lemma 5, but easier. Choose r to be a 3-vertex with as few vertices at distance 2 as possible. For each integer i, let N_i denote the set of vertices at distance i from r. Also, let $N_{4^+} = \bigcup_{i\geq 4} N_i$. We first handle the case $|N_2| \geq 8$, which is short.

Proposition 7. *If $|N_2| \geq 8$, then $e(G) \geq \frac{5}{3}n$.*

Proof. Since r was chosen among all 3-vertices to minimize N_2, each 3-vertex has either a 5^+-neighbor or at least two 4-neighbors. Thus, we let $\mathrm{ch}(v) = d(v)$ and use the following discharging rule.

1. Each 3-vertex takes $\frac{1}{6}$ from each 4-neighbor and $\frac{1}{3}$ from each 5^+-neighbor.

If $d(v) \geq 5$, then $\mathrm{ch}^*(v) \geq d(v) - \frac{1}{3}d(v) = \frac{2}{3}d(v) \geq \frac{10}{3}$. If $d(v) = 4$, then $\mathrm{ch}^*(v) \geq d(v) - \frac{1}{6}d(v) = 4 - \frac{4}{6} = \frac{10}{3}$. If $d(v) = 3$, then $\mathrm{ch}^*(v) \geq 3 + \frac{1}{3} = \frac{10}{3}$ or $\mathrm{ch}^*(v) \geq 3 + \frac{2}{6} = \frac{10}{3}$. Hence, $e(G) = \frac{1}{2}\sum_{v\in V(G)} \mathrm{ch}(v) = \frac{1}{2}\sum_{v\in V(G)} \mathrm{ch}^*(v) \geq \frac{5}{3}n$. This proves the proposition. □

Hereafter, we assume that $|N_2| \leq 7$. Now the Small Neighborhood Lemma implies that $d(v) \geq 4$ for each vertex $v \in N_{4^+}$. Suppose instead that $d(v) = 3$ for some vertex $v \in N_{4^+}$. Let p be the configuration with 15 pebbles on r, 0 pebbles on each vertex in $N_1 \cup N_2 \cup N[v]$, and 1 pebble on each other vertex. Since $|\{r\} \cup N_1 \cup N_2 \cup N[v]| \leq 15$, the configuration has at least n pebbles, but no pebble can reach v, since at most one pebble can leave $N[r] \cup N_2$. This contradicts that G is Class 0. Thus, $d(v) \geq 4$ for each $v \in N_{4^+}$.

Now we again redistribute charge. We let $\mathrm{ch}(v) = d(v)$ and we use the following two discharging rules.

1. Each vertex in N_2 takes charge 1 from a neighbor in N_1.

2. Each vertex in N_3 takes charge $\frac{1}{3}$ from a neighbor in N_2.

We show that each vertex in $V(G) \setminus N[r]$ finishes with charge at least $\frac{10}{3}$. If $v \in N_{4^+}$, then $\mathrm{ch}^*(v) = \mathrm{ch}(v) = d(v) \geq 4$. If $v \in N_3$, then $\mathrm{ch}^*(v) \geq d(v) + \frac{1}{3} \geq \frac{10}{3}$. If $v \in N_2$, then $\mathrm{ch}^*(v) \geq d(v) + 1 - \frac{1}{3}(d(v) - 1) = \frac{2}{3}d(v) + \frac{4}{3} \geq \frac{10}{3}$. The total charge on vertices of $\{r\} \cup N_1$ is $3 + 3(1) = 6$. Thus, the sum of all final charges is at least $\frac{10}{3}(n - 4) + 6 = \frac{10}{3}n - \frac{22}{3}$. Thus, $e(G) \geq \frac{5}{3}n - \frac{11}{3}$. □

We now describe a best possible bound for the minimum number of edges in an n vertex diameter 2 Class 0 graph. Before stating this result, we describe the *only* graphs for which equality holds.

Example 8. *The following are two infinite families of Class 0 graphs. Each n-vertex graph has exactly $2n-5$ edges. To form an instance of $F_{p,q}$, begin with K_3 and replace the two edges incident to some vertex v with p parallel edges and q parallel edges (where p and q are positive); finally, subdivide each of these $p+q$ new edges. To form an instance of $G_{p,q,r}$, begin with K_4 and replace the three edges incident to some vertex v with p parallel edges, q parallel edges, and r parallel edges (where p, q, and r are positive); finally, subdivide each of these $p+q+r$ new edges.*

Note that each n-vertex graph in $F_{p,q}$ has $2n-5$ edges, since the 2-vertices induce an independent set (when $p \geq 2$ and $q \geq 2$), and the three high-degree vertices have among them a single edge. Similarly, $G_{p,q,r}$ has $2n-5$ edges, since the 2-vertices induce an independent set and the four high-degree vertices have among them 3 edges.

Clarke et al. [4, Theorem 2.4] characterized diameter 2 graphs that are not Class 0. It seems likely that we could derive our result from theirs. Our result generalizes to diameter 2 graphs with no cut-vertices.

Theorem 9. *Let G be an n-vertex graph with diameter 2. If G has no cut-vertex (in particular, if G is Class 0) then $e(G) \geq 2n-5$. Further, equality holds if and only if G is the Petersen graph or one of the graphs in Example 1.*

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Descriptions of generalized trees by logic and algebraic terms

Bruno Courcelle

LaBRI, University of Bordeaux, France
CNRS, Paris, France

Extended Abstract

We define and study countable generalized trees, called *quasi-trees,* such that the unique "path" between two nodes may be infinite and have any order type, in particular that of rational numbers. Our motivation comes from the notion of *rank-width*, a complexity measure of finite graphs investigated first in [9] and [10].

Rank-width is based on graph decompositions formalized with finite subcubic trees. In order to extend rank-width to countable graphs in such a way that *the compactness property* holds, i.e., that the rank-width of a countable graph is the least upper-bound of those of its finite induced subgraphs, we base decompositions on *subcubic quasi-trees* [4, 5] (a notion presented at the Bordeaux Graph Workshop in 2014). For a comparison, the natural extension of tree-width to countable graphs has the compactness property [8] without needing quasi-trees.

Join-trees can be seen as directed quasi-trees. A join-tree is a partial order $(N, \leq)$ such that every two elements have a least upper-bound (called their *join*) and each set $\{y \mid y \geq x\}$ is linearly ordered. The modular decomposition of a countable graph is based on an *ordered join-tree* [6, 3].

Our objective is to obtain finitary descriptions (usable in algorithms) for the following generalized trees : join-trees, ordered join-trees and quasi-trees. For this purpose we will define algebras of such generalized trees that use finitely many operations and such that the finite and infinite terms over these operations define all countable relevant generalized trees. The *regular objects* are those defined by *regular terms*, *i.e.* that have finitely many different subterms, equivalently, that are the unique solutions of certain finite equation systems [1, 2, 11].

We will prove that a generalized tree is regular if and only if it is *monadic second-order definable*, i.e., is the unique model (up to isomorphism) of a monadic second-order sentence.

A linear order whose elements are labelled by letters from an alphabet is called an *arrangement.* Regular arrangements were defined and studied in [1] and [7], and their monadic second-order definability is proved in [11]. We use the latter result for proving its extension to generalized trees.

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Identifying codes for infinite triangular grids with a finite number of rows

Rennan Dantas [1], *Frédéric Havet* [2] and *Rudini M. Sampaio* [1]

[1] PARGO, Universidade Federal do Ceará, Brazil
[2] Université Côte d'Azur, CNRS, I3S, INRIA, COATI, France

Extended Abstract

Let G be a graph G. The *neighborhood* of a vertex v in G, denoted by $N(v)$, is the set of vertices adjacent to v i G. It *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

A set $C \subseteq V(G)$ is an *identifying code* in G if

(i) for all $v \in V(G)$, $N[v] \cap C \neq \emptyset$, and

(ii) for all $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$.

The *identifier* of v by C, denoted by $C[v]$, is the set $N[v] \cap C$. Hence a identifying code is a set such that the vertices have non-empty distinct identifiers.

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G, that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C, G)$, is defined by

$$d(C, G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} ,$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code in G.

The problem of finding low-density identifying codes was introduced in [9] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their neighbors. Thus the identifying property guarantees location of a faulty processor from the set of "complaining" controllers. Identifying codes are also used in [10] to model a location detection problem with sensor networks.

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite grids in the plane, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite hexagonal grid $\mathcal{G}_H$, the best upper bound on $d^*(\mathcal{G}_H)$ is 3/7 and comes from two identifying codes constructed by Cohen et al. [4]; these authors also proved a lower bound of 16/39. This lower bound was improved to 12/29 by Cranston and Yu [6]. Cukierman and Yu [7] further improved it to 5/12.

The *infinite square grid* $\mathcal{G}_S$ is the infinite graph with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $N((x, y)) = \{(x, y \pm 1), (x \pm 1, y)\}$. Given an integer $k \geq 2$, let $[k] = \{1, \dots, k\}$ and let S_k be the subgraph of $\mathcal{G}_S$ induced by the vertex set $\{(x, y) \in \mathbb{Z} \times [k]\}$. In [3], Cohen et al. gave a periodic identifying code of $\mathcal{G}_S$ with density 7/20. This density was later proved to be optimal by Ben-Haim and Litsyn [1]. Daniel, Gravier, and Moncel [8] showed that $d^*(S_1) = \frac{1}{2}$ and $d^*(S_2) = \frac{3}{7}$. They also showed that for every $k \geq 3$, $\dfrac{7}{20} - \dfrac{1}{2k} \leq d^*(S_k) \leq \min\left\{\dfrac{2}{5}, \dfrac{7}{20} + \dfrac{2}{k}\right\}$. These bounds were recently improved by Bouznif et al. [2] who established

$$\frac{7}{20} + \frac{1}{20k} \leq d^*(S_k) \leq \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

They also proved $d^*(S_3) = \frac{3}{7}$.

The *infinite triangular grid* $\mathcal{G}_T$ is the infinite graph with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that $N((x,y)) = \{(x, y \pm 1), (x \pm 1, y), (x-1, y+1), (x+1, y-1)\}$. Given an integer $k \geq 2$, let $[k] = \{1, \ldots, k\}$ and let T_k be the subgraph of $\mathcal{G}_T$ induced by the vertex set $\{(x,y) \in \mathbb{Z} \times [k]\}$. Karpovsky et al. [9] showed that $d^*(\mathcal{G}_T) = 1/4$. Trivially, $T_1 = S_1$. Hence $d^*(T_1) = \frac{1}{2}$ In this paper, we prove the following results regarding the density of an identifying code of T_k, $k > 1$.

Theorem 1

- $d^*(T_k) = \frac{1}{4} + \frac{1}{4k}$ *for every odd* k.
- $d^*(T_2) = \frac{1}{2}$ *and* $d^*(T_4) = d^*(T_6) = \frac{1}{3}$.
- $\frac{1}{4} + \frac{1}{4k} \leq d^*(T_k) \leq \frac{1}{4} + \frac{1}{2k}$ *for every even* k.

The upper bounds are obtained by showing periodic identifying codes with the desired density. Consider the sets (see Figure 1)

$$\begin{array}{rcl}
C_2 & = & \{(x,1) \mid x \equiv 1, 3 \mod 5\} \cup \{(x,2) \mid x \equiv 1, 2, 4 \mod 5\}; \\
C_{2k-1} & = & \{(x,y) \mid x \mid x, y \text{ odd and } 1 \leq x, y \leq 2k-1\}; \\
C_4 & = & \{(x,2) \mid x \equiv 0, 3 \mod 3\} \cup \{(x,3) \mid x \equiv 0, 1 \mod 3\}; \\
C_{2k} & = & \{(x,y), (x,y) \mid x, y \text{ odd }, 1 \leq x \leq 2k \text{ and } 1 \leq y \leq 2k-3\} \cup \{(x, 2k-1) \mid x \in \mathbb{Z}\}.
\end{array}$$

It is easy to check that the above defined sets C_2 is an identifying codes of T_2 with density $1/2$, C_3 is identifying codes of T_3 with density $1/3$, C_4 is an identifying code of T_4 with density $3/10$. and C_{2k-1} is an identifying code of T_{2k-1} with density $\frac{1}{4} + \frac{1}{4k}$ and C_{2k} is an identifying code of T_{2k} with density $\frac{1}{4} + \frac{1}{2k}$.

Our lower bounds are obtained via the Discharging Method. The general idea is the following. We consider any identifying code C of T_k. The vertices in C receive a certain value $q_k > 0$ of charge and the vertices not in C receive charge 0. Then we apply some local discharging rules. Here local means that there is no charge transfer from a vertex to a vertex at distance more than d_k for some fixed constant d_k, and that the total charge sent by a vertex is bounded by some fixed value m_k. Finally, we prove that after the discharging, every vertex v has final charge $\text{chrg}^*(v)$ at least p_k for some fixed $p_k > 0$. We claim that it implies $d(C, G) \geq \frac{p_k}{q_k}$. Since a vertex sends charge at most m_k to vertices at distance at most d_k, a charge of at most $m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \leq 2d_k \cdot k \cdot m_k$ enters $B_r(v_0)$ during the discharging phase. Thus

$$\begin{aligned}
|C \cap B_r(v_0)| = \frac{1}{q_k} \sum_{v \in B_r(v_0)} \text{chrg}_0(v) & \geq \frac{1}{q_k} \left(\sum_{v \in B_r(v_0)} \text{chrg}^*(v) - m_k \cdot |B_{r+s}(v_0) \setminus B_r(v_0)| \right) \\
& \geq \frac{p_k |B_r(v_0)| - 2d_k \cdot k \cdot m_k}{q_k} .
\end{aligned}$$

But $|B_r(v_0)| \geq 2(k+1)r - k^2$, thus $d(C, \mathcal{S}_k) \geq \limsup_{r \to +\infty} \left(\frac{p_k}{q_k} - \frac{1}{q_k} \cdot \frac{2d_k \cdot k \cdot m_k}{2(k+1)r - k^2} \right) = \frac{p_k}{q_k}$. This proves our claim. As the claim holds for any identifying code, we have $d^*(T_k) \geq p_k / q_k$.

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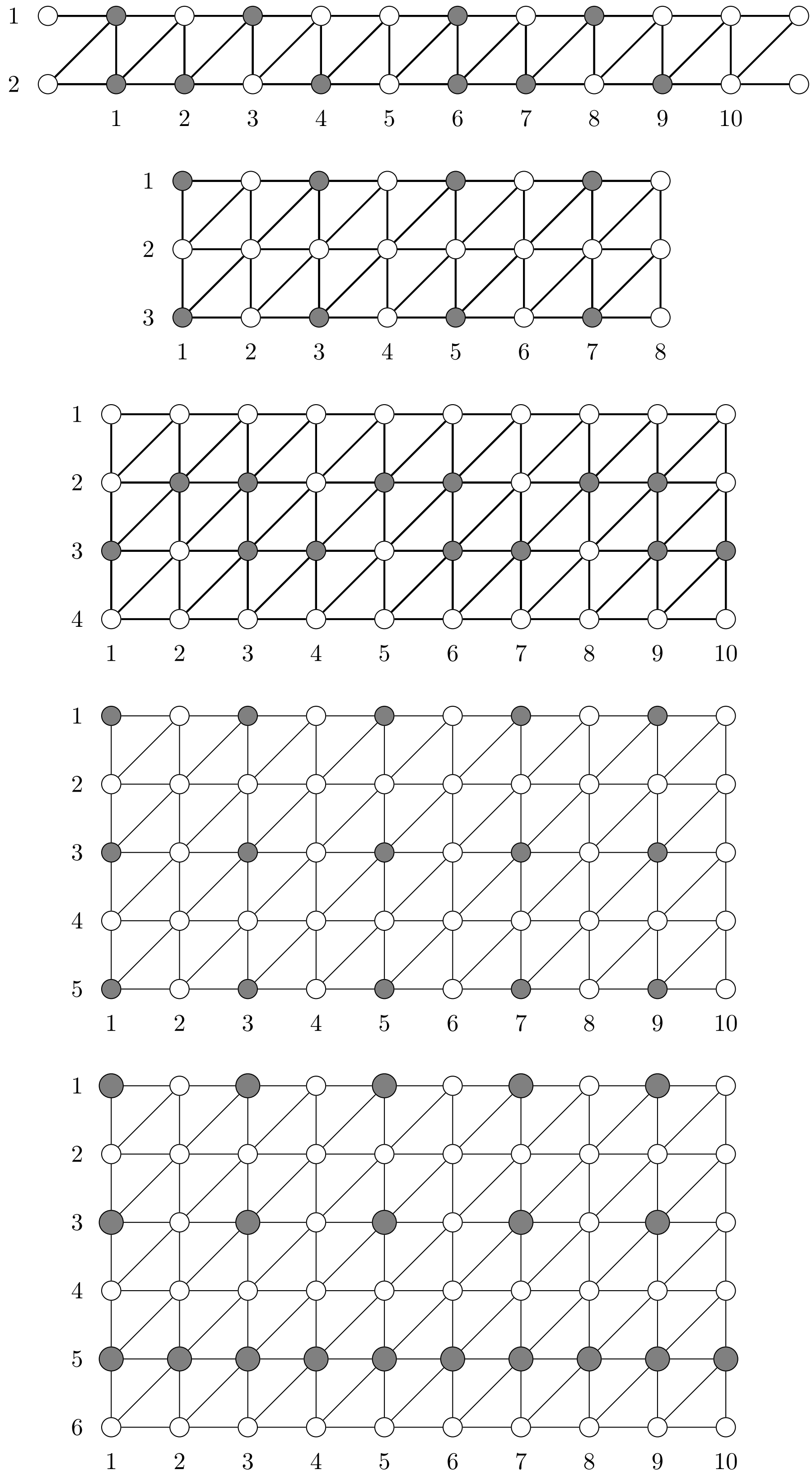

Figure 1: Optimal identifying codes of T_2, T_3, T_4, T_5 and T_6

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Almost disjoint spanning trees

Benoit Darties, *Nicolas Gastineau*, and *Olivier Togni*

LE2I UMR6306, CNRS, Arts et Métiers, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France

Extended Abstract

In this extended abstract, the graphs considered are assumed to be connected. Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. A vertex is said to be an *inner vertex* in a tree T if it has degree at least 2 in T. We denote by $I(T)$ the set of inner vertices of tree T. The spanning trees $T_1, \ldots, T_k$ are *completely independent spanning trees* if any vertex from G is an inner vertex in at most one tree among $T_1, \ldots, T_k$ and the trees $T_1, \ldots, T_k$ are pairwise edge-disjoint.

Completely independent spanning trees were introduced by Hasunuma [4] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [4], maximal planar graphs [5], Cartesian product of two cycles [6] and k-trees [10]. Moreover, determining if there exist two completely independent spanning trees in a graph G is a NP-hard problem [5]. Recently, sufficient conditions inspired by the sufficient conditions for hamiltonicity have been determined in order to guarantee the existence of several completely independent spanning trees: Dirac's condition [1] and Ore's condition [2]. Moreover, Dirac's condition has been generalized to more than two trees [7].

In this extended abstract, we introduce (i, j)-disjoint spanning trees:

Definition 0.1 *Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. We let $I(T_1, \ldots, T_k) = \{u \in V(G) | \exists \ell, \ell' \ u \in I(T_\ell) \cap I(T_{\ell'}),\ 1 \leq \ell < \ell' \leq k\}$ be the set of vertices which are inner vertices in at least two spanning trees among $T_1, \ldots, T_k$, and we let $E(T_1, \ldots, T_k) = \{e \in E(G) | \exists \ell, \ell',\ 1 \leq \ell < \ell' \leq k,\ e \in E(T_\ell) \cap E(T_{\ell'})\}$ be the set of edges which belong to at least two spanning trees among $T_1, \ldots, T_k$. The spanning trees $T_1, \ldots, T_k$ are* (i, j)-disjoint *for two positive integers i and j, if the two following conditions are satisfied:*

i) $|I(T_1, \ldots, T_k)| \leq i$;

ii) $|E(T_1, \ldots, T_k)| \leq j$.

The sets $D_1, \ldots D_k$ in a graph G are *disjoint connected dominating* sets if they are pairwise disjoint and dominating. Moreover, if $|\cup_{1 \leq i < j \leq k} D_i \cap D_j| \leq \ell$ we say that $D_1, \ldots D_k$ are *ℓ-rooted connected dominating sets.* Other works on disjoint spanning trees are about *disjoint connected dominating sets* (the disjoint connected dominating sets can be used to provide the inner vertices of $(0, E(G))$-disjoint spanning trees). The maximum number of disjoint connected dominating sets in a graph G is the *connected domatic number* [12]. An interesting result about connected domatic number concerns planar graphs, for which Hartnell and Rall have proven that, except K_4 (which has connected domatic number 4), their connected domatic number is bounded by 3 [3]. The problem of constructing a connected dominating set is often motivated by wireless ad-hoc networks [11]. Connected dominating sets are used to create a virtual backbone or spine of a wireless ad-hoc network.

By $*$ we denote a large enough integer, i.e. an integer larger than $\max(|E(G)|, |V(G)|)$, for a graph G. Remark that $(0, 0)$-disjoint spanning trees are completely independent spanning trees and that $(*, 0)$-disjoint spanning trees are edge-disjoint spanning trees. Also, $(0, *)$-disjoint spanning trees are related to connected dominating sets. Hence, we call them trees induced by disjoint connected dominating sets. For the same reason than $(0, *)$-disjoint spanning trees, $(\ell, *)$-disjoint spanning trees are trees induced by ℓ-rooted connected dominating sets. In the following sections, we illustrate that (i, j)-disjoint spanning trees provide some nuances between the existence of disjoint connected dominating sets and of completely independent spanning trees.

1 Characterizations in terms of partitions

We introduce a definition which is a generalization of CIST-partition introduced by Araki [1]. Let V_1 and V_2 be two disjoint subsets of vertices of a graph G. By $B(V_1, V_2)$ we denote the bipartite graph with vertex set $V_1 \cup V_2$ and edge set $\{uv \in E(G)|\ u \in V_1,\ v \in V_2\}$. An *$\ell$-CIST-partition* of a graph G into k sets is a partition of $V(G)$ into k sets of vertices $V_1, \ldots, V_k$ such that:

i) $G[V_i]$ is connected, for each integer i, $1 \leq i \leq k$;

ii) $B(V_i, V_j)$ contains no isolated vertex, for every two integers i, j, $1 \leq i < j \leq k$;

iii) $\sum_{1 \leq i<j \leq k} c_{i,j} \leq \ell$, where $c_{i,j}$ is the number of connected component which are trees in $B(V_i, V_j)$, $1 \leq i < j \leq k$.

Theorem 1.1 *Let G be a graph. There exist k $(0, \ell)$-disjoint spanning trees $T_1, \ldots, T_k$ in G if and only if G has an ℓ-CIST-partition into k sets.*

Proof. Suppose G has an ℓ-CIST-partition into k sets V_1,..., V_k. We are going to construct $(0, \ell)$-disjoint spanning trees $T_1, \ldots, T_k$. We begin by setting $I(T_i) = V_i$ for each integer i, $1 \leq i \leq k$. For now, we suppose that $E(T_i)$ is empty and we progressively add edges in $E(T_i)$, for each integer i, $1 \leq i \leq k$, in order to obtain spanning trees of G. Since $G[V_i]$ is connected for each i, $1 \leq i \leq k$, we can add edges in $E(T_i)$ in order to form a tree with vertex set V_i, for each i.

Let i and j be two integers, $1 \leq i < j \leq k$, and let $D_{i,j}$ be a connected component of $B(V_i, V_j)$. We add edges in order to build a spanning tree restricted to $V_i \cup V(D_{i,j})$ and another spanning tree restricted to $V_j \cup V(D_{i,j})$ by considering two cases. Let u be a vertex of $D_{i,j} \cap V_i$. First, if $D_{i,j}$ is a tree, then we add one edge e of $D_{i,j}$ incident with u in $E(T_i)$ and in $E(T_j)$. Remark that the edge e is common to T_i and T_j. Let $D^d_{i,j}(u) = \{v \in V(D_{i,j})|\ d_{D_{i,j}}(u, v) = d\}$. We add to $E(T_i)$ the edges of the set $\{vv' \in E(D_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is even}\}$ and to $E(T_j)$ the edges of the set $\{vv' \in E(D_{i,j})|\ v \in D^d_{i,j}(u),\ v' \in D^{d+1}_{i,j}(u),\ d \text{ is odd}\}$. Second, if $D_{i,j}$ is not a tree, then we suppose that u is in a cycle of $D_{i,j}$. Let e be an edge of this cycle incident with u and let $T_{i,j}$ be a spanning tree of $D_{i,j} - e$. We define $B^d_{i,j}(u)$ as follows: $\{v \in V(D_{i,j})|\ d_{T_{i,j}}(u, v) = d\}$. We add to $E(T_i)$ the edges of the set $\{vv' \in E(T_{i,j})|\ v \in B^d_{i,j}(u),\ v' \in B^{d+1}_{i,j}(u),\ d \text{ is even}\}$ and to $E(T_j)$ the edges of the set $\{vv' \in E(T_{i,j})|\ v \in B^d_{i,j}(u),\ v' \in B^{d+1}_{i,j}(u),\ d \text{ is odd}\} \cup \{e\}$. We repeat this process for every connected component of $B(V_i, V_j)$ and every two integers i and j, $1 \leq i < j \leq k$. Since there is only one common edge between T_i and T_j for each connected component that is a tree and since $\sum_{1 \leq i<j \leq k} c_{i,j} \leq \ell$, the set $E(T_1, \ldots, T_k)$ contains at most ℓ edges. Therefore we obtain, by Property ii), k $(0, \ell)$-disjoint spanning trees.

Let us prove the converse of the previous implication. Suppose there exist k $(0, \ell)$-disjoint spanning trees $T_1, \ldots, T_k$ in G. The set $I(T_i)$, $1 \leq i \leq k$, induces a connected subgraph in G. We begin by setting $V_i = I(T_i)$, for each integer i, $1 \leq i \leq k$. If some vertices are inner vertices in no trees, we can add them to any set among $V_1, \ldots, V_k$. Thus, Property i) follows. Let i and j be two integers, $1 \leq i < j \leq k$. Suppose there exists one isolated vertex u in $B(V_i, V_j)$. Without loss of generality, suppose $u \in V_i$. By Proposition **??**.i), we obtain a contradiction since $u \notin I(T_j)$ and u has no neighbor in $I(T_j)$. Thus, Property ii) follows. Now suppose $\sum_{1 \leq i<j \leq k} c_{i,j} > \ell$. Let $D_{i,j}$ be a connected component which is a tree in $B(V_i, V_j)$ for some integers i and j and suppose that $D_{i,j}$ contains no edge from $E(T_1, \ldots, T_k)$. Since $D_{i,j}$ has $|V(D_{i,j})| - 1$ edges, it is impossible that every vertex of $V(D_{i,j}) \cap V_i$ is adjacent to a vertex of $V(D_{i,j}) \cap V_j$ in T_j and that every vertex of $V(D_{i,j}) \cap V_j$ is adjacent to a vertex of $V(D_{i,j}) \cap V_i$ in T_i, since it would require $|V(D_{i,j})|$ edges. Thus, for every two integers i and j and every connected component $D_{i,j}$ of $B(V_i, V_j)$, if $D_{i,j}$ is a tree then $V(D_{i,j}) \cap E(T_1, \ldots, T_k) \neq \emptyset$ and we obtain a contradiction since $\sum_{1 \leq i<j \leq k} c_{i,j} > \ell$ implies $|E(T_1, \ldots, T_k)| > \ell$. Thus, Property iii) follows.

□

For a graph G and subset of vertices $A \subseteq V(G)$, let $N(A) = \{u \in V(G) \setminus A | \ uv \in E(G),\ v \in A\}$. In a similar way than Zelinka [12], we prove that the notion of ℓ-rooted connected dominating sets is equivalent to a notion of partition. An *ℓ-rooted partition* of a graph G into $k+1$ sets is a partition of $V(G)$ into $k+1$ sets of vertices $V_1, \ldots, V_k, A$ such that:

i) $|A| \leq \ell$;

ii) $G[V_i \cup A]$ is connected, for each i, $1 \leq i \leq k$;

ii) $B(V_i, V_j) - N(A)$ contains no isolated vertex, for every i and j, $1 \leq i < j \leq k$.

Theorem 1.2 *Let G be a graph. There exist k ℓ-rooted connected dominating sets $D_1, \ldots, D_k$ in G if and only if G has an ℓ-rooted partition into $k+1$ sets.*

Sketch of Proof. The proof is similar to the one of Theorem1.1.

2 Computational complexity and connectivity

We define the following decision problem:

k-(i,j)-**DSP**
Instance : A graph G.
Question: Does there exist k (i,j)-disjoint spanning trees in G ?

Theorem 2.1 *Let i and j be non negative integers. The problem 2-(i,j)-DSP is a NP-complete problem for every pair of integers (i,j).*

Sketch of proof. The proof uses a reduction from 3-SAT similar to the reduction used by Hasunuma [5].

□

Moreover, since the presence of a k-cut in a graph G implies that there do not exist $k+1$ disjoint connected dominating set, it is natural to ask whether k-connected graph, for k sufficiently large, contains at least two (i,j)-disjoint spanning trees or not.

Theorem 2.2 *Let i, j and k be integers. For any positive integer k, there exist a k-connected graph which does not contain two (i,j)-disjoint spanning trees.*

Sketch of proof. The considered graphs are the same than the graphs introduced by Kriesell [9] (they are the incidence graphs of complete k-uniform hypergraphs). The proof consists in proving that the existence of two (i,j)-disjoint spanning trees implies that there exists a vertex u in this graph for which the vertices of $N(u)$ are all inner vertices of the same tree. Moreover, no vertex of $N(u)$ should be inner vertex of both trees. These facts imply that u cannot be in one of the two spanning trees and thus a contradiction.

□

3 Some simple classes of graphs

We finish this extended abstract by giving some results for square of graphs, complete graphs and square grids.

Theorem 3.1 *Let G be graph. There exists two $(0,1)$-disjoint spanning trees in G^2 and there do not exist two completely independent spanning trees in G^2 if and only if G is a tree from the family described by Araki [1].*

Sketch of proof. Let V_1 and V_2 be the set of vertices induced by a bipartition of a spanning tree of G. It easy to prove that V_1 and V_2 form a 1-CIST-partition of G. Moreover, Araki has characterized the trees containing two completely independent spanning trees. It suffice to prove that a connected graph G which is not a tree contains two completely independent spanning trees to complete the proof. That is the case since the set of vertices induced by a bipartition of a spanning tree of $G - \{e\}$ for e an edge of a cycle of G, is a 0-CIST-partition.

□

Remark that there are n disjoint connected dominating sets in K_n and that there are $\lfloor n/2 \rfloor$ completely independent spanning trees in K_n [10] and that there does not exist two disjoint connected dominating sets in a sufficiently large square grid [3]. We give the following intermediate result about (i, j)-disjoint spanning trees (the proof is not given):

Theorem 3.2 *Let n be an integer. There are at most $\lfloor n/2 \rfloor + \max(\lfloor \ell/(n-1) + 1_{odd}(n)/2 \rfloor, \lceil n/2 \rceil)$ $(0, \ell)$-disjoint spanning trees in K_n, where $1_{odd}(n) = 1$ if n is odd and 0 otherwise.*

Also, there exist two 1-rooted connected dominating sets in $P_{n_1} \square P_{n_2}$, for every $n_1 \geq 3$ and $n_2 \geq 3$.

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Characterization of Stable Equimatchable Graphs[1]

Zakir Deniz [1], and *Tınaz Ekim* [2]

[1] Department of Mathematics, Süleyman Demirel University, Isparta, Turkey
[2] Department of Industrial Engineering, Boğaziçi University, Istanbul, Turkey

Extended Abstract

1 Introduction

A graph G is *equimatchable* if every (inclusion-wise) maximal matching of G has the same cardinality. In this paper, this widely studied class of graphs (see e.g. [4, 6, 1, 2, 3]) is considered from a new perspective; we deal with the stability of the property of being equimatchable with respect to edge or vertex removals. Formally, we call an equimatchable graph G *edge-stable* if $G \setminus e$ is equimatchable for each $e \in E(G)$. Edge-stable equimatchable graphs are denoted ESE-*graphs* as a shorthand. Similarly, an equimatchable graph G is called *vertex-stable* (VSE for short) if $G - v$ is equimatchable for each $v \in V(G)$.

It is well-known that a graph is equimatchable if and only if its line graph is *well-covered*, that is, every maximal independent set of it has the same size. In [8], well-covered graphs which remain well-covered upon removal of any vertex, called *1-well-covered*, are introduced and some basic properties of these graphs are studied. More recently, C_4-free 1-well-covered graph are characterized in [5]. Besides, well-covered graphs which remain well-covered upon removal of any edge, called *strongly well-covered*, are studied in [7] where these graphs are shown to be 3-connected and strongly well-covered graphs with independence number 2 are characterized. Note that a graph is ESE if and only if its line graph is 1-well-covered. Consequently, in this paper, we characterize all 1-well-covered line graphs.

Given a graph $G = (V, E)$ and a subset of vertices I, $G[I]$ denotes the subgraph of G induced by I, and $G \setminus I = G[V \setminus I]$. When I is a singleton $\{v\}$, we denote $G \setminus I$ by $G - v$. We also denote by $G \setminus e$ the graph $G(V, E \setminus \{e\})$. For a subset I of vertices, we say that I is *complete* to another subset I' of vertices (or by abuse of notation, to a subgraph H) if all vertices of I are adjacent to all vertices of I' (respectively H). The size of a maximum matching of a graph G is denoted by $\nu(G)$. A matching M is said to *saturate* a vertex v if v is the endvertex of some edge in M, otherwise it leaves a vertex *exposed*. If $G - v$ has a perfect matching for each $v \in V(G)$, then G is called *factor-critical*. Factor-critical equimatchable graphs are denoted EFC-*graphs*. We notice that a graph G is ESE if and only if every connected component of G is ESE. Therefore, we only consider connected ESE-graphs.

The following consequence of the results in [6] will guide us through our characterization.

Theorem 1 *[6] A 2-connected equimatchable graph is either factor-critical or bipartite or K_{2t} for some $t \geq 1$.*

Clearly K_{2t} is not ESE, therefore as suggested by Theorem 1, we study ESE-graphs in Section 2 under three categories: 2-connected factor-critical ESE-graphs, 2-connected bipartite ESE-graphs and ESE-graphs with a cut-vertex (it turns out that these are all bipartite ESE-graphs). These results provide a full characterization of all ESE-graphs yielding an $O(\min(n^{3.376}, n^{1.5}m))$ time recognition algorithm which is better than the most natural way of recognizing ESE-graphs by checking the equimatchability of $G \setminus e$ for every $e \in E$. In the same line as edge-stability, in Section 3, we study vertex-stable equimatchable graphs; their characterization is much simpler compared to the one of ESE-graphs.

[1] See http://arxiv.org/abs/1602.09127 for the full version.

2 Characterization and Recognition of ESE-graphs

Factor-critical ESE-graphs

Let us first underline that although we seek to characterize factor-critical ESE-graphs which are 2-connected, all the results in this section are valid for any factor-critical ESE-graph. Note that factor-critical graphs are connected but not necessarily 2-connected. However, it follows from their characterization that factor-critical ESE-graphs are also 2-connected (See Corollary 7). The following two lemmas are frequently used for our further results.

Lemma 2 *Let G be a factor-critical graph. G is equimatchable if and only if there is no independent set I with 3 vertices such that $G \setminus I$ has a perfect matching.*

Lemma 3 *Let G be an* EFC*-graph. Then G is edge-stable if and only if there is no induced $\overline{P_3}$ in G such that $G \setminus \overline{P_3}$ has a perfect matching.*

The following lemma is fundamental for the characterization of factor-critical ESE-graphs. Its proof is based on the existence of a perfect matching M_v in $G - v$ for any vertex $v \in V$. We define three sets of vertices according to M_v: let N_1 be the set of neighbors of v which are matched with non-neighbors of v, N_1' be the set of vertices matched to N_1, and N_2 be the set of neighbors of v matched to each other. By Lemma 3, we have $V(M_v) = V(G) \setminus \{v\} = N_1 \cup N_1' \cup N_2$ for a factor-critical ESE-graph.

Lemma 4 *Let G be a factor-critical graph with at least* 7 *vertices. If G is an* ESE*-graph which is not an odd clique, then there is a nontrivial independent set S which is complete to $G \setminus S$.*

Moreover, we can show that there is a nontrivial independent set S complete to $G \setminus S$ and having a special form with respect to the decomposition N_1, N_1' and N_2 for some vertex v.

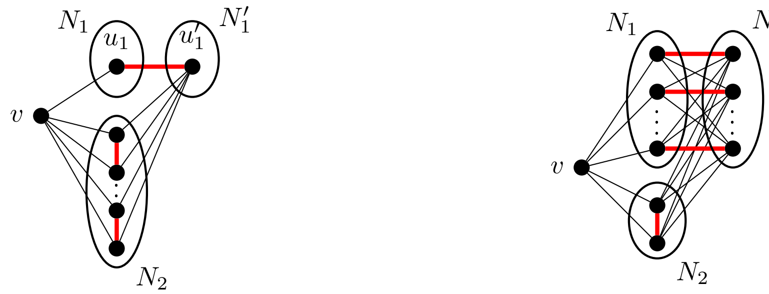
Corollary 5 *Let G be a factor-critical graph with at least 7 vertices. If G is an* ESE*-graph which is not an odd clique, then for some $v \in V(G)$, G has a decomposition into the sets N_1, N_1' and N_2 where $S = N_1' \cup \{v\}$ is a nontrivial independent set which is complete to $G \setminus S$, and N_1 is an independent set.*

Let us define two graph families $\mathcal{G}_1$ and $\mathcal{G}_2$ illustrated in Figure 1 corresponding to the cases where the nontrivial independent set S described in Corollary 5 has respectively 2 or more vertices. The matching M_v defining the sets N_1, N_1' and N_2 is shown by bold (red) edges in Figure 1. A graph G belongs to $\mathcal{G}_1$ if $G \cong K_{2r+1} \setminus M$ for some nonempty matching M containing vu_1' and $r \geq 3$. A graph G of $\mathcal{G}_1$ is illustrated in Figure 1a where the edges in $G[N_1 \cup N_2] \cong K_{2r-1} \setminus M$ are not drawn, and $S = \{v, u_1'\}$ is complete to $G \setminus S$. Besides, $\mathcal{G}_2$ is defined as the family of graphs G admitting an independent set S of size at least 3 which is complete to $G \setminus S$ and such that $\nu(G \setminus S) = 1$. In Figure 1b, we show an illustration of a graph G in $\mathcal{G}_2$ where $S = N_1' \cup \{v\}$ with $|S| \geq 3$ and $\nu(G \setminus S) = 1$. Again, the edges in $G[N_1 \cup N_2]$ are not drawn but just described by the property $\nu(G \setminus S) = 1$.

Theorem 6 *Let G be a factor-critical graph with at least* 7 *vertices. Then, G is* ESE *if and only if either G is an odd clique or it belongs to $\mathcal{G}_1$ or $\mathcal{G}_2$.*

In addition, we determined all factor-critical ESE-graphs whose orders are at most 5 by using computer programming (Python-Sage). There are just 7 such graphs: K_1, K_3, K_5, C_5, $K_5 \setminus e$, $K_5 \setminus E(2K_2)$ and $K_5 \setminus E(P_2 + P_3)$ where $+$ denotes the disjoint union of two graphs. By observing that all these graphs and all the graphs described in Theorem 6 are 2-connected, we obtain as a byproduct the following:

Corollary 7 *Factor-critical* ESE*-graphs are 2-connected.*



(a) Illustration of a graph G in $\mathcal{G}_1$ with $S = \{v, u_1'\}$ and $G[N_1 \cup N_2] \cong K_{2r-1} \setminus M$ for some matching M

(b) Illustration of a graph G in $\mathcal{G}_2$ with $S = N_1' \cup \{v\}$ and $\nu(G \setminus S) = 1$

Figure 1: Factor-critical ESE-graph families $\mathcal{G}_1$ and $\mathcal{G}_2$ where the bold (red) edges are M_v.

Bipartite ESE-graphs

The following is obtained by using the Gallai-Edmonds Decomposition of equimatchable graphs [6] and Lemmas 2 and 3.

Theorem 8 ESE-*graphs with a cut vertex are bipartite.*

Having characterized all 2-connected factor-critical ESE-graphs (Theorem 6) and having shown that ESE-graphs with a cut vertex are bipartite (Theorem 8), we now consider bipartite ESE-graphs to complete our characterization. We will see that bipartite ESE-graphs can be characterized in a way very similar to bipartite equimatchable graphs:

Lemma 9 *[6] A connected bipartite graph $G = (U \cup W, E)$, $|U| \leq |W|$ is equimatchable if and only if for every $u \in U$, there exists $S \subseteq N(u)$ such that $S \neq \emptyset$ and $|N(S)| \leq |S|$.*

Lemma 9, together with the well-known Hall's condition implies in particular that a connected bipartite graph $G = (U \cup W, E)$ with $|U| \leq |W|$ is equimatchable if and only if every maximal matching of G saturates U (then we say that every vertex in U is *strong*). A strong vertex v is called *square-strong* if for every $u \in N(v)$, v is strong in $G - u$. Noting that if a connected bipartite graph $G = (U \cup W, E)$ is ESE then $|U| < |W|$, we have the following characterization:

Proposition 10 *Let $G = (U \cup W, E)$ be a connected bipartite graph with $|U| < |W|$. Then the followings are equivalent.*

- (i) *G is* ESE.
- (ii) *Every vertex of U is square-strong.*
- (iii) *For every $u \in U$, there exists nonempty $S \subseteq N(u)$ such that $|N(S)| \leq |S| - 1$.*
- (iv) *For every $u \in U$, every maximal matching of $G - u$ leaves at least two vertices of $N(u)$ exposed.*

The following is a consequence of Proposition 10 and Hall's condition:

Corollary 11 *A connected bipartite graph $G = (U \cup W, E)$ with $|U| < |W|$ is not* ESE *if and only if there exists $u \in U$ such that $N(u)$ is saturated by some (maximal) matching of G.*

Recognition of ESE-graphs

The recognition of ESE-graphs is trivially polynomial since checking equimatchability can be done in time $O(n^2 m)$ for a graph with n vertices and m edges (see [1]) and it is enough to repeat this check for every edge removal. This trivial procedure gives a recognition algorithm for ESE-graphs in time $O(n^2m^2)$. However, using the characterization of ESE-graphs, we can improve this time complexity in a significant way. The following time complexity is due to the recognition of bipartite ESE-graphs which requires n times the computation of a maximum matching in a bipartite graph.

Theorem 12 ESE-*graphs can be recognized in time* $O(\min(n^{3.376}, n^{1.5}m))$.

3 Vertex-Stable Equimatchable Graphs

If we require not only $G - v$ for some $v \in V(G)$ to remain equimatchable, but also all induced subgraphs of G, then this coincides with the notion of hereditary equimatchable graphs which are shown in [2] to be only complete graphs and complete bipartite graphs. Then, complete graphs and complete bipartite graphs are clearly VSE. Subsequently, we show that factor-critical VSE-graphs are only odd cliques. Moreover, using the Gallai-Edmonds decomposition, we show that if G is a VSE-graph which is not a complete graph, then G is bipartite. Finally, by showing that a bipartite graph apart from complete bipartite is VSE if and only if it is ESE, we obtain the following characterization of VSE-graphs.

Theorem 13 *A graph G is* VSE *if and only if G is either a complete graph or a complete bipartite graph or a bipartite* ESE-*graph.*

It follows that VSE-graphs can also be recognized in time $O(\min(n^{3.376}, n^{1.5}m))$. Let us conclude with Figure 2 which summarizes our findings.

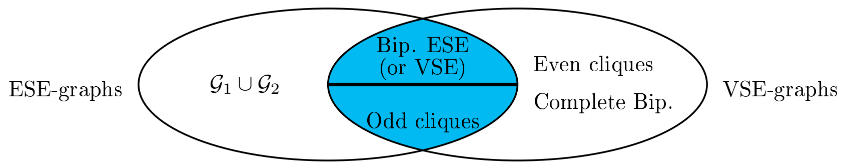


Figure 2: Illustration of the classes of ESE-graphs and VSE-graphs.

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3-Flows with Large Support

Matt DeVos [1], *Jessica McDonald* [2], *Irene Pivotto* [3], *Edita Rollová* [4], and *Robert Šámal* [5]

[1] Department of Mathematics, Simon Fraser University [2] Department of Mathematics and Statistics, Auburn University [3] School of Mathematics and Statistics, University of Western Australia [4] NTIS, Faculty of Applied Sciences, University of West Bohemia [5] Computer Science Institute of Charles University

Extended Abstract

We prove that every 3-edge-connected graph G has a 3-flow ϕ with the property that $|supp(\phi)| \geq \frac{5}{6}|E(G)|$. The graph K_4 demonstrates that this $\frac{5}{6}$ ratio is best possible.

1 Introduction

Throughout we permit graphs to have both multiple edges and loops. If G is an oriented graph and Γ is an additive abelian group, then we define a function $\phi : E(G) \to \Gamma$ to be a *flow* if it satisfies the following rule at every vertex $v \in V(G)$. Here $\delta^+(v)$ ($\delta^-(v)$) denotes the set of edges directed away from (toward) the vertex v

$$\sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) = 0$$

The flow ϕ is *nowhere-zero* if $0 \notin \phi(E(G))$ and it is called a *k-flow* for a positive integer k if $\Gamma = \mathbb{Z}$ and $|\phi(e)| \leq k - 1$ for every $e \in E(G)$. Tutte initiated the study of nowhere-zero flows by proving the following duality theorem: If G and G^* are dual plane graphs, then G^* has a proper k-colouring if and only if G has a nowhere-zero k-flow.

Based in part on this duality, Tutte made three lovely conjectures concerning the existence of nowhere-zero flows. These conjectures, known as the 5-Flow, 4-Flow, and 3-Flow conjectures have motivated a great deal of research on this subject, but despite this all three remain unsolved.

Our approach here will be to relax the notion of nowhere-zero and instead look for flows which have large support. Our main theorem is the following bound for 3-flows in 3-edge-connected graphs.

Theorem 1 *Every 3-edge-connected graph G has a 3-flow ϕ satisfying*

$$|supp(\phi)| \geq \tfrac{5}{6}|E(G)|.$$

Since the graph K_4 does not have a nowhere-zero 3-flow, the ratio $\frac{5}{6}$ in this theorem is best possible. When we restrict our attention to planar graphs, our theorem has the following corollary.

Corollary 2 *If G is a simple planar graph, then there exists a function $f : V(G) \to \{1, 2, 3\}$ so that the number of edges uv with $f(u) = f(v)$ is at most $\frac{1}{6}|E(G)|$.*

Note that the graph K_4 also demonstrates that the above corollary gives a best possible bound. In fact, this corollary is not a new result – it is also an easy corollary of the Four Colour Theorem.

Although the $\frac{5}{6}$ ratio in Theorem 1 is best possible, it seems quite possible that the same bound holds more generally for graphs which are 2-edge-connected. Unlike most results in the realm of nowhere-zero flows, our theorem on 3-edge-connected graphs does not obviously give a similar result for 2-edge-connected graphs. The best bound we have for 3-flows in 2-edge-connected graphs is the following result due to Král'.

Theorem 3 (Kráľ) *Every 2-edge-connected graph G has a 3-flow ϕ with*

$$|supp(\phi)| \geq \tfrac{3}{4}|E(G)|.$$

Our main theorem has a somewhat stronger "choosability" form related to group-connectivity as introduced by Jaeger, Linial, Payan, and Tarsi. Instead of insisting that the function ϕ be a flow, we may instead ask for the sum involved at the vertex v to take on certain prescribed values at each vertex v. Let us return to a general setting to put these definitions in place. So, we assume that G is an oriented graph, let Γ be an abelian group (written additively) and let $\phi : E(G) \to \Gamma$. The *boundary* of ϕ is the function $\partial\phi : V(G) \to \Gamma$ given by the following rule for every $v \in V(G)$.

$$\partial\phi(v) = \sum_{e \in \delta^-(v)} \phi(e) - \sum_{e \in \delta^+(v)} \phi(e)$$

If we think of ϕ as indicating a circulation of fluid, then $\partial\phi(v)$ tells us how much is accumulating at v or exiting the network at v. Note that, by definition, the function ϕ is a flow if $\partial\phi$ is identically zero. If we sum the boundary function $\partial\phi$ over every vertex, then whatever value x is assigned to an edge e will get added once and subtracted once, so it has no effect. This gives the following useful identity

$$\sum_{v \in V(G)} \partial\phi(v) = 0$$

which holds for every function $\phi : E(G) \to \Gamma$. The general form of our main theorem may now be stated as follows (a function $\mu : V(G) \to \mathbb{Z}_3$ is *zero-sum* if $\sum_{v \in V(G)} \mu(v) = 0$).

Theorem 4 *If G is an oriented 3-edge-connected graph and $\mu : V(G) \to \mathbb{Z}_3$ is zero-sum, there exists $\phi : E(G) \to \mathbb{Z}_3$ so that*

1. $\partial\phi = \mu$
2. $|supp(\phi)| \geq \frac{5}{6}|E(G)|$

Unlike our earlier theorem, in the case of Theorem 4 the assumption of 3-edge-connectivity is necessary. To see this, take an arbitrary oriented 3-edge-connected graph G and modify it by subdividing every edge twice (thus forming a directed path of three edges). Now define $\mu : V(G) \to \mathbb{Z}_3$ by the following rule:

$$\mu(v) = \begin{cases} 1 & \text{if } deg(v) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For every 3-edge path P with both interior vertices of degree 2, a straightforward check reveals that every function $\phi : E(G) \to \mathbb{Z}_3$, which satisfies $\partial\phi = \mu$, will have the property that ϕ assigns all three edges of P distinct values. Therefore, every such function ϕ will satisfy $|supp(\phi)| = \frac{2}{3}|E(G)|$.

2 Ears

Although the theorem we wish to prove concerns 3-edge-connected graphs, our proof will involve a reductive process that encounters graphs which are only 2-edge-connected. In preparation for this, we will establish some terminology and tools for working with 2-edge-connected graphs.

Ear Decomposition

We define an *ear* of a graph G to be a subgraph $P \subseteq G$ which satisfies one of the following:

- P is a nontrivial path, all interior vertices of P have degree 2 in G, but both endpoints have degree ≥ 3 in G.
- P is a cycle of G containing exactly one vertex with degree $\neq 2$ in G.
- $P = G$ is a cycle.

For an arbitrary graph H we let $H^\times$ denote the graph obtained from H by deleting all isolated vertices (i.e., vertices of degree 0). If P is an ear of G, then *removing* P brings us to the new graph $(G - E(P))^\times$. We define a *partial ear decomposition* of a graph G to be a list $P_1, P_2, \ldots, P_\ell$ of subgraphs of G satisfying the following:

1. $E(P_i) \cap E(P_j) = \emptyset$ whenever $i \neq j$.
2. P_j is an ear of the graph obtained from G by removing $P_\ell, \ldots, P_{j+1}$.

Weighted Graphs and Ear Labellings

We define a *weighted graph* to be a graph G equipped with a function $\mu_G : V(G) \to \mathbb{Z}_3$, and we call G a *zero-sum* weighted graph if μ_G is zero-sum. In preparation for the proof of our main theorem, we now introduce a framework to move from one weighted graph to another by removing ears.

For our main theorem we have an oriented zero-sum weighted graph G and we are interested in finding a function $\phi : E(G) \to \mathbb{Z}_3$ with boundary μ_G and large support. Let us take a moment to consider the possible behaviours of such a function ϕ on an ear.

If P is an ear of G, a function $\psi : E(P) \to \mathbb{Z}_3$ is called an *ear labelling* if $\partial\psi(v) = \mu(v)$ for every vertex v which is in the interior of the path P. The following observation is straightforward but crucial.

Observation 5 *Let P be an ear of the oriented zero-sum weighted graph G. Then there are exactly three ear labellings ϕ_1, ϕ_2, ϕ_3 of P, and every $e \in E(P)$ has the value 0 in exactly one of these labellings.*

Since we are looking to construct functions $\phi : E(G) \to \mathbb{Z}_3$ which have large support, we will naturally be interested in ears P of G which have an ear labelling with large support. If ψ_1, ψ_2, ψ_3 are the ear labellings of P, then by the above discussion, the average size of the support of an ear labelling ψ_i will be precisely $\frac{2}{3}|E(P)|$. When $|E(P)|$ is a multiple of 3, we may have $|supp(\psi_i)| = \frac{2}{3}|E(P)|$ for $1 \leq i \leq 3$. In this extreme case we say that P is *equitable*, and in all other cases we call P *inequitable*. When $|E(P)|$ is not a multiple of 3 (or more generally when P is inequitable), there exists at least one ear labelling ψ_i with $|supp(\psi_i)| > \frac{2}{3}|E(P)|$ and our proof will frequently exploit this. Indeed, the key to our argument is getting a small advantage for each inequitable ear.

In preparation for this we now introduce a general definition. If H is a subgraph of G and $\psi : E(H) \to \mathbb{Z}_3$, we define the *gain* of ψ to be

$$\mathrm{gain}(\psi) = 24|supp(\psi)| - 16|E(H)|.$$

The following lemma gives our basic tool for finding good ear labellings. We will use this extensively in the remainder of the paper.

Lemma 6 *Let G be a weighted graph, and assume that P is an inequitable ear of G. If $|E(P)| = 3k+i$ where $1 \leq i \leq 3$, then P has an ear labelling $\psi : E(P) \to \mathbb{Z}_3$ with $\mathrm{gain}(\psi) \geq 8i$.*

Next we introduce some terminology to facilitate the process of deciding on a particular ear labelling, and then removing this ear from the (weighted graph). If P is an ear of G and $\psi : E(P) \to \mathbb{Z}_3$ is an ear-labelling, we define the *ψ-removal* of P to be the weighted graph $G' = (G \setminus E(P))^{\times}$ equipped with the weight function $\mu_{G'} : V(G') \to \mathbb{Z}_3$ given by the following rule (we put $\partial\psi(v) = 0$ if $v \notin V(P)$)

$$\mu_{G'}(v) = \mu_G(v) - \partial\psi(v).$$

Since μ_G and $\partial\psi$ both sum to zero, the same holds for the function $\mu_G - \partial\psi$. However, this latter function is identically zero on the interior vertices of the path P. It follows that the function $\mu_{G'}$ will be zero-sum. The following straightforward observation shows that we can combine a function $\phi' : E(G') \to \mathbb{Z}_3$ with boundary $\mu_{G'}$ with our ear labelling ψ to obtain a function on $E(G)$ with boundary μ_G.

Observation 7 *Let G be an oriented zero-sum weighted graph, let $P \subseteq G$ be an ear, let ψ be an ear labelling of P, and let G' be the ψ-removal of P. If $\phi' : E(G') \to \mathbb{Z}_3$ satisfies $\partial\phi' = \mu_{G'}$, then the following function $\phi : E(G) \to \mathbb{Z}_3$ has $\partial\phi = \mu_G$.*

$$\phi(e) = \begin{cases} \phi'(e) & \text{if } e \in E(G') \\ \psi(e) & \text{if } e \in E(P) \end{cases}$$

3 Framework

We close this extended abstract by stating our workhorse lemma that is used to prove our main theorem. As seen above, ears with different lengths modulo 3 will behave differently for us when constructing our desired function. To deal with this behaviour we will introduce a bonus function which assigns each ear a value which indicates in some sense the amount we expect to gain for it. For an ear P we define the *bonus* of P as follows:

$$\text{bonus}(P) = \begin{cases} 0 & \text{if } P \text{ is equitable} \\ 3 & \text{if } |E(P)| \equiv 2 \pmod 3. \\ 4 & \text{otherwise} \end{cases}$$

For a subgraph $H \subseteq G$ which is a union of disjoint ears $H = \cup_{i=1}^{\ell} P_i$ we define $\text{bonus}(H) = \sum_{i=1}^{\ell} \text{bonus}(P_i)$. So $\text{bonus}(G)$ is the sum of the bonuses of all of the ears. With this terminology in place, we are finally ready to state the workhorse lemma which will imply our main theorem.

Lemma 8 *Let G be an oriented zero-sum weighted graph and assume that G is a subdivision of a 3-edge-connected graph. Then there exists $\phi : E(G) \to \mathbb{Z}_3$ satisfying:*

- $\partial\phi = \mu_G$,
- $\text{gain}(\phi) \geq \text{bonus}(G)$.

It is straightforward to deduce Theorem 4 from Lemma 8. The proof of the lemma is rather extensive. It is based on studying properties of possible minimal counterexample and limiting its properties step-by-step.

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Power domination of triangulations

Paul Dorbec [1], *Antonio González* [2] and <u>*Claire Pennarun*</u> [1]

[1] LaBRI-CNRS, University of Bordeaux

[2] Department of Applied Mathematics I, University of Seville

Extended Abstract

Power dominating a graph consists in choosing some vertices of the graph such that, while applying a simple set of rules, all the vertices of the graph are monitored. The problem of determining a power dominating set of minimum size has been first described in terms of monitoring an electrical network with measurement devices (see for example [1]) and has then been transposed in terms of a graph-theoretical problem by Haynes et al. in a seminal paper [3]. After some classical definitions and notations on graphs, we present the power domination problem in more details, along with our main result.

We use classical notations of graph theory. A planar graph G together with such an embedding is called a *plane graph*. The edges of a plane graph partition the plane into regions called *faces*, and the only unbounded face is called the *outerface*. A *triangulation* (or *maximal planar graph*) is a plane graph whose faces are all triangles. A triangle composed of vertices u, v, w (i.e., $uv, vw, uw \in E(G)$) is denoted as $[uvw]$, and such a triangle is *facial* if it is the boundary of some face of G. An *octahedron* is a triangulation composed of two triangles $[x_0x_1x_2]$ and $[y_0y_1y_2]$ so that x_i is adjacent to y_i and y_{i+1} (being indices taken modulo 3). A *facial octahedron* of a triangulation G is any subgraph $H \subseteq G$ isomorphic to an octahedron whose faces are facial in G, except for the outer triangle.

Given a subset $S \subseteq V(G)$, the set of vertices *monitored by S after i steps* is defined as $P^i_G(S) = P^{i-1}_G(S) \cup \{u \in N(v) \mid v \in P^{i-1}_G(S) \text{ and } N(v) \setminus P^{i-1}_G(S) = \{u\}\}$, and $P^0_G(S) = N[S]$. Clearly, there is i^* so that $P^{i^*}_G(S) = P^j_G(S)$ for each $j \geq i^*$, and so we can define the set of vertices *monitored by S* as $M_G(S) = P^{i^*}_G(S)$. We say that G is *monitored* when all its vertices are monitored; the set of *non-monitored* vertices is denoted by $\overline{M}_G(S) = V(G) \setminus M_G(S)$. When the graph G is clear from the context, we simplify the notations $P^i_G(S)$, $M_G(S)$ and $\overline{M}_G(S)$ into respectively $P^i(S)$, $M(S)$ and $\overline{M}(S)$. The set S is said to be a *power dominating set* (PD-set for short) for G if $M(S) = V(G)$, and the minimum cardinality of such a set is the *power dominating number* of G, denoted by $\gamma_P(G)$.

The decision problem associated to power domination (i.e. "Given a graph G and an integer k, does $\gamma_P(G) \leq k$?") has been proven to be NP-complete, even when restricted to special classes of graphs, such as planar graphs [2], but is fixed-parameter tractable with respect to tree-width [2]. Linear-time algorithms have been given to determine $\gamma_P(G)$ if the graph is a tree [3], a block graph [5], or an interval graph [4] (if endpoints are sorted), among other classes. Bounds are known for some classes of graphs, including connected graphs [6] and trees [3]. We are here studying power domination in maximal planar graphs.

The main result of this paper is the following theorem:

Theorem 1 *Let G be a maximal planar graph of order $n \geq 6$. Then $\gamma_P(G) \leq \frac{n-2}{4}$.*

To prove Theorem 1, we describe how to compute a PD-set S of order at most $\frac{n-2}{4}$. We introduce the following invariant, which we seek to maintain throughout the process.

Property $(*)$**.** *We say that a subset A of vertices of a graph G has* Property $(*)$ *in G whenever, for each induced triangulation $G' \subseteq G$ of order at least 4, if G' is monitored by A then* **(a)** $|V(G')| \geq 6$ *and* $|A \cap V(G')| \leq \frac{|V(G')|-2}{4}$, *or* **(b)** *there exists a vertex u of the outerface of G' such that $N_G[u]$ is monitored by A.*

Building the set S is done in three steps. First, Algorithm 1 produces a set S_1 monitoring the facial induced octahedra with a small number of vertices. Then, Algorithm 2 builds a set S_2 by expanding the set S_1 iteratively, while maintaining Property $(*)$. If G is not monitored after that, we show that G has a specific structure. This structure is finally used by Algorithm 3 to choose carefully the last vertices to add to S_2 to construct the set S.

Detailed description of the process

We denote by $\mathcal{O}_G$ the set of facial octahedra of G, and $G[\mathcal{O}_G]$ is the *octahedra subgraph* of G. We say that two facial octahedra $H, H' \in \mathcal{O}_G$ are *adjacent* if they share a vertex. Remark that if H is a facial octahedron of G, then a PD-set of G contains at least a vertex of H. Thus $\gamma_P(G)$ is at least the maximum number of disjoint facial octahedra of G. An extreme example is the graph formed by disjoint octahedra along with as many edges as one wants to reach a triangulation. We begin by monitoring the facial octahedra of G with Algorithm 1.

Algorithm 1: Monitoring the vertices of the octahedra induced subgraph

Input: A triangulation G of order $n \geq 6$
Output: A set $S_1 \subseteq V(G[\mathcal{O}_G])$ monitoring $G[\mathcal{O}_G]$ and $|M(S_1)| \geq 6\ |S_1|$
$S_1 := \emptyset$
while $\exists$ *adjacent non-monitored* $H, H' \in \mathcal{O}_G$ **do**
  u: a common vertex of H and H'
  $S_1 \leftarrow S_1 \cup \{u\}$
  Label internal vertices of H and H' with u
while $\exists\ H \in \mathcal{O}_G$ *with* $V(H) \cap S_1 = \emptyset$ **do**
  u: a vertex of the outerface of H
  $S_1 \leftarrow S_1 \cup \{u\}$
  Label all vertices of H with u
Return S_1

Lemma 2 *Let S_1 be the set obtained by application of Algorithm 1 to G. The following statements hold:* **(i)** $V(G[\mathcal{O}_G]) \subseteq M_G(S_1)$, **(ii)** $|M_G(S_1)| \geq 6\ |S_1|$, *and* **(iii)** S_1 *has Property* $(*)$ *in* G.

Proof. **(i)** At the end of the algorithm, all octahedra are monitored since they all have one of their exterior vertices in S_1 (recall that as soon as one of the vertices of an octahedron is in S_1 the whole octahedron is monitored). Consequently, all vertices of $G[\mathcal{O}_G]$ are in $M_G(S_1)$.

(ii) Two facial octahedra H and H' can only share vertices of their outer faces. Thus for each vertex $u \in S_1$, there are at least 6 monitored vertices labelled exclusively with u: either the interior vertices of two adjacent octahedra or the vertices of some octahedron. Hence $|M_G(S_1)| \geq 6\ |S_1|$.

(iii) Let G' be an induced triangulation of G monitored after Algorithm 1. If G' contains no facial octahedra, then $|S_1 \cup V(G')| = 0$ and thus Property $(*)$.(a) holds. Assume G' contains facial octahedra. If for every vertex $v \in S_1 \cup V(G')$, all vertices with label v are in G', then $|S_1 \cup V(G')| \leq \frac{|V(G')|}{6}$ and Property $(*)$.(a) holds. Otherwise, there is a vertex $v \in S_1$ such that some vertices labelled v are outside G'. But then v is a vertex of the outer face of G'. Thus Property $(*)$.(b) holds.

Thus Property $(*)$ holds for S_1 in G. $\square$

We can suppose $M(S_1) \neq V(G)$ after Algorithm 1: if $G[\mathcal{O}_G] = G$, then $|S_1| \leq \frac{n}{6}$, and since $n \geq 6$, then $|S_1| \leq \frac{n-2}{4}$. Algorithm 2 now builds a set $S_2 \subseteq V(G)$ by greedily adding vertices to S_1.

Algorithm 2: Greedy selection of vertices to expand S_1

Input: A triangulation G of order $n \geq 6$
Output: A set $S_2 \subseteq V(G)$ with $|S_2| \leq \frac{|M(S_2)|-2}{4}$
$S_2 :=$ Algorithm $1(G); M := M(S_2)$
while $\exists$ *u of maximum degree in G among* $\{v \in V(G) \setminus S_2 : |M(S_2 \cup \{v\})| \geq |M| + 4\}$ **do**
  $S_2 \leftarrow S_2 \cup \{u\}$
  $M \leftarrow M(S_2)$
Return S_2

After Algorithm 2, the following lemma holds:

Lemma 3 *Let S_2 be the output of Algorithm2 on a triangulation G with $|V(G)| \geq 6$. Then* **(i)** $|S_2| \leq \frac{|M(S_2)|-2}{4}$ *and* **(ii)** *Property $(*)$ holds for S_2 in G.*

(Sketch of proof) Statement **(i)** is proved by induction on the number of rounds of Algorithm 2. Statement **(ii)** is proved for any induced triangulation G' of G monitored after Algorithm 2, with three cases depending on the number of vertices of G'. $\square$

If $M(S_2) = V(G)$, then by Lemma 3, S_2 is a PD-set of G with at most $\frac{n-2}{4}$ vertices. We can thus suppose that $\overline{M}(S_2) \neq \emptyset$. We then show the following lemma (proof omitted here):

Lemma 4 *After applying Alg. 2, let G' be an induced triangulation of the graph G containing non-monitored vertices. If G' contains adjacent non-monitored vertices, then G' is isomorphic to one of the four first graphs depicted in Fig. 1. Otherwise (i.e. the non-monitored vertices are independent in G'), G' is isomorphic to the last graph depicted in Fig. 1.*

In each case, a *splitting structure* C of three non-monitored vertices divide the other vertices of G' into two triangulations G_1 and G_2, called the *associated triangulations* of C. Since the splitting structures must be disjoint, it implies that in the graph G, these structures are nested and can be considered one into an other. Algorithm 3 (applied initially to G) considers recursively the splitting structures and adds vertices to S_2 to create the set S.

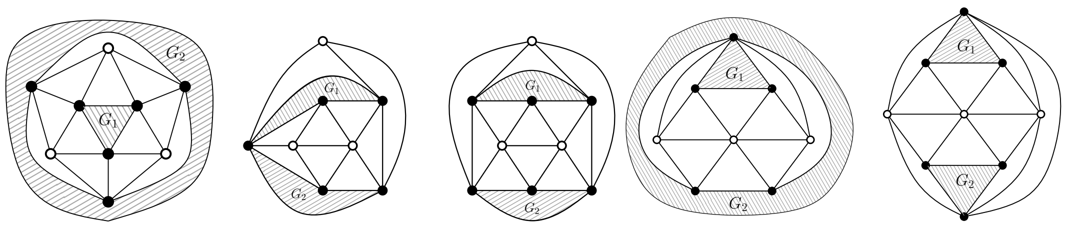

Figure 1: The five different splitting structures and their associated triangulations G_1 and G_2. White vertices belong to $\overline{M}_G(S_2)$ and black vertices to $M_G(S_2)$. All triangles are facial except for G_1 and G_2.

Lemma 5 *Suppose G is isomorphic to one of the graphs of Fig. 1 after Algorithms 1 and 2. If there exists two sets of vertices monitoring respectively G_1 and G_2, then G has a power dominating set S with $|S| \leq \frac{n-2}{4}$.*

Proof. In all five cases depicted in Fig. 1, vertices of the outerfaces of G_1 and G_2 have neighbors in $\overline{M}$. Let S_1 and S_2 be the PD-sets of respectively G_1 and G_2. By Lemmas 2 and 3, Property $(*)$.(a) holds for G. Thus G_1 and G_2 have at least 6 vertices each, $|S_1| \leq \frac{|V(G_1)|-2}{4}$ and $|S_2| \leq \frac{|V(G_2)|-2}{4}$. Then if $S = S_1 \cup S_2 \cup \{u \in V(\overline{M})\}$, then $|S| \leq \frac{|V(G_1)|+|V(G_2)|}{4}$. In all five cases, $|V(G)| \geq |V(G_1)| + |V(G_2)| + 2$. Thus $|S| \leq \frac{|V(G)|-2}{4}$. $\square$

Algorithm 3: Monitoring the last vertices

Input: A triangulation G of order $n \geq 6$, and an induced triangulation $G' \subseteq G$
$S = Algorithm\ 2(G) \cap V(G')$
if $\exists$ *a connected component* C *of* $\overline{M}_G(S_3)$ *in* $V(G')$ *isomorphic to* P_2 *or* P_3 **then**
  $G_1, G_2 \subseteq G'$ triangulations of G associated to C
  $S' \leftarrow$ Algorithm 3(G, G_1)
  $S'' \leftarrow$ Algorithm 3(G, G_2)
  $S \leftarrow S' \cup S'' \cup \{u \in C\}$
if $\exists\ u \in \overline{M}_G(S) \cap V(G')$ **then**
  $G_1, G_2 \subseteq G'$ triangulations of G associated to C
  $S' \leftarrow$ Algorithm 3(G, G_1)
  $S'' \leftarrow$ Algorithm 3(G, G_2)
  $S \leftarrow S' \cup S'' \cup \{u\}$
Return S

Lemma 6 *Let G' be an induced triangulation of G and C a splitting structure in G' with G_1 and G_2 its associated triangulations. Let u be a vertex of C. Let S' denote the set $S \cap V(G_1)$ and S'' the set $S \cap V(G_2)$. If G_1 and G_2 are monitored and Property $(*)$ holds for respectively S' in G_1 and S'' in G_2, then Property $(*)$.(a) holds for $S' \cup S'' \cup \{u\}$ in G' and G' is monitored.*

(Sketch of proof.) If a vertex of the outer face of G_1 or G_2 can propagate in G', then it can also propagate in G. Since in each case, all vertices of the outer face of G_1 and G_2 have non-monitored neighbors, then Property $(*)$.(a) holds in both G_1 and G_2. Thus $|S'| \leq \frac{|V(G_1)|-2}{4}$ and $|S''| \leq \frac{|V(G_2)|-2}{4}$. Remark that $|V(G')| \geq |V(G_1)| + |V(G_2)| + 2$ in every case. After adding a vertex $u \in C$, we have: $|S' \cup S'' \cup \{u\}| \leq \frac{|V(G_1)|-2}{4} + \frac{|V(G_2)|-2}{4} + 1$, and thus $|S' \cup S'' \cup \{u\}| \leq \frac{|V(G_1|+|V(G_2)|}{4} \leq \frac{|V(G')|-2}{4}$. Considering the cases depending on which splitting structure is found and which vertex is added, we prove that in each case, vertices u_1, u_2, u_3 are monitored. Thus G' is monitored. $\square$

Theorem 7 *At the end of Algorithm 3, $M = V(G)$ and $|S| \leq \frac{|V(G)|-2}{4}$.*

Proof. Since the splitting structures are nested, then Algorithm 3 considers a binary tree-like hierarchy of structures, such that all P_1 structures are considered after P_2 and P_3 structures. By induction on the depth of the structure considered, Lemma 6 guarantees that the bound holds and that the graph is completely monitored after Algorithm 3. Thus $|S| \leq \frac{|V(G)|-2}{4}$ and $M = V(G)$. $\square$

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Graph minor operations for the marking game

Paul Dorbec [1], *Gabriela Paris* [2], *Éric Sopena* [1]

[1] LaBRI-CNRS, Univ. Bordeaux, France
[2] LIRIS, University of Lyon 1, France

Extended Abstract

The marking game is a 2-player game played on a graph, that was introduced in 1999 by Zhu [3].The players, called Alice and Bob, alternate turns marking a yet unmarked vertex. When a vertex is marked, we define its score as one plus the number of its marked neighbors. The score of the whole game is the maximum of the scores of the vertices of the graph, independently of who marked the vertex. We say Alice *has a strategy with score k* if Alice has a strategy that ensures all vertices get score at most k. We say Bob has a strategy with score k if Bob has a strategy to ensure at least one vertex gets score more than k.

In the following, we call A-marking game the game where Alice starts, and B-marking game when Bob starts. In order to be able to consider games under progress, we denote $G|M$ the game played on G where the set of vertices M is considered already marked (and we ignore their score). The A-marking number (resp. B-marking number) denoted $col_A(G|M)$ (resp. $col_B(G|M)$) is the minimum k such that Alice has a strategy with score k on the graph $G|M$ in the A-marking game (resp. the B-marking game).

In the following, we study how the A- and B-marking numbers change when canonical operations are applied to the graph. We first prove that the A- and B-marking numbers of a graph differ by at most one. We then give some bounds on the marking game numbers of a graph under three operations: vertex removal, edge removal, and edge contraction.

1 Marking numbers

In this section we study and compare the A-marking number and the B one. The results on this section also involve some work by Elżbieta Sidorowicz [4].

For a graph $G = (V, E)$ and an integer s, we let $A_s(G) = \{v \in V | d(v) \geq s\}$ and $B_s(G) = V \setminus A_s(G)$.

Proposition 1 ([4]) *Let $G = (V, E)$ be a graph, s an integer, and $M \subseteq V$ a set of marked vertices in G. We then have:*

- *if $|A_s \setminus M| > |B_s \setminus M|$, $col_A(G|M) > s$*
- *if $|A_s \setminus M| \geq |B_s \setminus M|$, $col_B(G|M) > s$.*

A strategy for Bob to ensure the previous proposition is to play only vertices in $|B_s \setminus M|$. Then the last vertex to be marked is necessarily in A_s and thus the score is at least $s + 1$.

We improve this result by showing that in general, Alice has no advantage playing on B_s if she wants to ensure a score s. This is similar to saying it is never useful for Alice to let Bob play first.

Lemma 2 *Let $G(V, E)$ be a graph and M a set of marked vertices. We have :*

$$col_A(G|M) \leq col_B(G|M) \leq col_A(G|M) + 1.$$

Proof: We first prove that $col_A(G|M) \leq col_B(G|M)$ by using the imagination strategy argument (see [1]). Consider a strategy for Alice to ensure a score at most s in the B-marking game. Alice adapts that same strategy to play the A-marking game. Before her first move, she imagines Bob played on any vertex $x \in B_s \setminus M$, then she plays the vertex y she would have played as answer in the B-marking game. As the game goes on, she plays as if x

was marked, and she follows her strategy step by step to the answers of Bob. If Bob happens to mark the vertex x, then she imagines Bob played any other unplayed vertex $x' \in B_s \setminus M$ and continues as in the imagined game.

Each time a vertex is marked in the A-marking game, it has no more marked neighbors than in the imagined B-marking game, since Alice uses the same strategy. Moreover, the imagined vertex that is played at the end of the game belongs to B_s so it has less than s marked neighbors. Hence the maximum score is at most s. (This statement requires some induction that is omitted here.)

We now prove that $col_B(G|M) \leq col_A(G|M) + 1$. Assume Alice has a strategy in the A-marking game with score s. Playing the B-marking game, Alice uses the same imagination strategy. Bob starts playing some vertex x and Alice plays as if Bob did not play that move. If at some point she is supposed to mark the vertex x marked by Bob, Alice marks any vertex in A_s and imagines she played the vertex x (since that vertex would be played later on, it has no more than $s-1$ marked neighbors in the imagined game. If there is no vertex left in A_s, there are only vertices of B_s unmarked, and we can immediately conclude the proof). At each step there is one more vertex marked in the A-marking game than in the imagined B-marking game. Thus when a vertex is marked, it has at most s marked neighbors. So the maximum score is at most $s+1$.

Both bounds are tight. We consider the graphs $K_n \vee S_m$, where K_n denotes the complete graph on n vertices, S_m the edgeless graph on m vertices, and $G \vee H$ the joint of two graphs (see Fig. 1 for examples). For any integers $n > m$, we have that $col_A(K_n \vee S_m) = col_B(K_n \vee S_m) = n+m$. Indeed, playing only the vertices of S_m, Bob enforces that the last vertex of K_n played has all its $n+m-1$ neighbors marked. This tighten the lower bound. On the other hand, for $n = m$, $col_A(K_n \vee S_m) = n+m-1$ while $col_B(K_n \vee S_m) = n+m$. Optimal strategies for Alice and Bob are respectively to mark vertices from K_n and from S_m. □

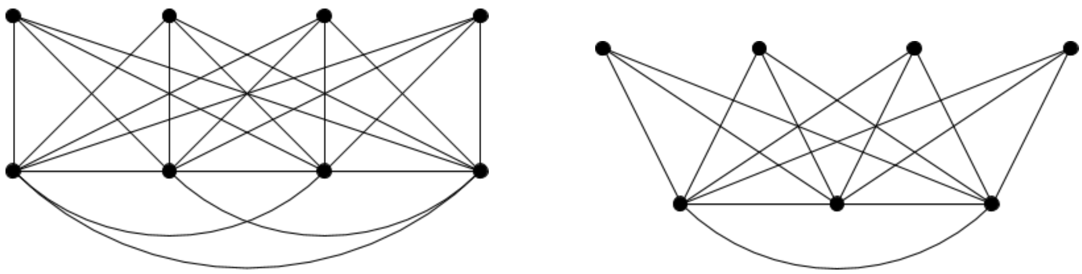

Figure 1: Graphs $K_4 \vee S_4$ and $K_3 \vee S_4$ that show tightness

2 Basic operations

In this section, we describe the possible effect of deleting a vertex, deleting an edge or contracting an edge on the marking number of a graph. We first propose the following result:

Theorem 3 (Vertex deletion) *Let v be a vertex of $V \setminus M$ (and $V \setminus M \neq \{v\}$). Then we have:*

$$\begin{array}{rcccl} col_A(G|M) - 2 & \leq & col_A(G - \{v\}|M) & \leq & col_A(G|M), \\ col_B(G|M) - 2 & \leq & col_B(G - \{v\}|M) & \leq & col_B(G|M), \end{array}$$

and these bounds are tight.

The proof is omitted here, but it is also based on some imagination argument. We observed earlier that $col_A(K_n \vee S_n) = 2n-1$ while $col_A(K_{n-1} \vee S_n) = 2n-3$, tightening the lower bound. For the B-marking game, we have $col_B(K_n \vee S_n) = 2n$ while $col_A(K_{n-1} \vee S_n) = 2n-2$. Removing a vertex of S_m in $K_n \vee S_m$ where $m \geq n+2$ does not change any of its marking numbers, tightening the upper bound.

Theorem 4 (Edge deletion) *Let e be an edge of G. Then we have:*

$$\begin{array}{rcccl} col_A(G|M) - 1 & \leq & col_A(G \setminus \{e\}|M) & \leq & col_A(G|M), \\ col_B(G|M) - 1 & \leq & col_B(G \setminus \{e\}|M) & \leq & col_B(G|M), \end{array}$$

and these bounds are tight.

The tightness is obtained by taking $G = (S_n \vee K_n)$ minus a perfect matching. We have then : $col_A(G) = 2n - 1$ and $col_B(G) = 2n - 1$. When an edge in K_n is removed we obtain : $col_A(G \setminus \{e\} = col_B(G \setminus \{e\}) = 2n - 2$.

As a corollary of Theorems 3 and 4, we get:

Theorem 5 *For H a subgraph of G, we have:*

$$\begin{array}{rcl} col_A(H) & \leq & col_A(G), \\ col_B(H) & \leq & col_B(G). \end{array}$$

This is a nice behavior of the marking game for subgraphs. We could expect that the parameter behave less nicely with the edge contraction. Indeed, we prove the following result.

Theorem 6 (Edge contraction) *Denote by G/e the graph G where the edge e has been contracted. Then we have:*

$$col_A(G|M) - 2 \leq col_A(G/e|M) \leq col_A(G|M) + 2.$$

Both bounds are tight, as show the following examples. First consider the family of graphs G_n obtained as follows. Remove from $K_n \vee S_n$ a perfect matching. Duplicate one of the vertices of K_n, changing a vertex w into two adjacent twins u and v. Then add a vertex adjacent to all the vertices of K_n (including u and v). We can check that $col_A(G_n) = 2n+1$, but if we contract the edge uv, then $col_A(G_n/uv) = 2n - 1$. This tighten the lower bound. Similarly, for G obtained from $C_n \vee S_n$ minus a matching, we have $col_A(G) = n + 2$ but if we split a vertex of the cycle in two (distributing evenly the neighbors), we get a A-marking number of n.

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Partitioning sparse graphs into an independent set and a forest of bounded degree

François Dross [1], Mickaël Montassier [1] and Alexandre Pinlou [2]

[1] Université de Monptellier, CNRS, LIRMM
[2] Université Paul-Valéry Montpellier 3, CNRS, LIRMM

Extended Abstract

Abstract

An $(\mathcal{I},\mathcal{F}_d)$-partition of a graph is a partition of the vertices of the graph into two sets I and F, such that I is an independent set and F induces a forest of maximum degree at most d. We showed that, for all $2 \leq M < 3$ and $d \geq \frac{2}{3-M} - 2$, a graph with maximum average degree less than M admits an $(\mathcal{I},\mathcal{F}_d)$-partition. Additionally, we proved that, for all $\frac{8}{3} \leq M < 3$ and $d \geq \frac{1}{3-M}$, a graph with maximum average degree less than M admits an $(\mathcal{I},\mathcal{F}_d)$-partition.

1 Introduction

In this abstract, all the considered graphs are simple graphs, without loops or multi-edges. For i classes of graphs $\mathcal{G}_1,\dots,\mathcal{G}_i$, a $(\mathcal{G}_1,\dots,\mathcal{G}_i)$-*partition* of a graph G is a partition of the vertices of G into i sets $V_1,\dots,V_i$ such that, for all $1 \leq j \leq i$, the graph $G[V_j]$ induced by V_j belongs to $\mathcal{G}_j$. In the following we will consider the following classes of graphs:

- $\mathcal{F}$ the class of forests,
- $\mathcal{F}_d$ the class of forests with maximum degree at most d,
- Δ_d the class of graphs with maximum degree at most d,
- $\mathcal{I}$ the class of empty graphs (i.e. graphs with no edges).

For example, an $(\mathcal{I},\mathcal{F},\Delta_2)$-partition of G is a vertex-partition into three sets V_1, V_2, V_3 such that $G[V_1]$ is an empty graph, $G[V_2]$ is a forest, and $G[V_3]$ is a graph with maximum degree at most 2. Note that $\Delta_0 = \mathcal{F}_0 = \mathcal{I}$ and $\Delta_1 = \mathcal{F}_1$. The *average degree* of a graph G with n vertices and m edges, denoted by $\mathrm{ad}(G)$, is equal to $\frac{2m}{n}$. The *maximum average degree* of a graph G, denoted by $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over all subgraphs H of G. The *girth* of a graph G is the length of a smallest cycle in G, and infinity if G has no cycle. Many results on partitions of sparse graphs appear in the literature, where a graph is said to be sparse if it has a low maximum average degree, or if it is planar and has a large girth. The study of partitions of sparse graphs started with the Four Colour Theorem [1, 2], which states that every planar graph admits an $(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$-partition. Borodin [3] proved that every planar graph admits an $(\mathcal{I},\mathcal{F},\mathcal{F})$-partition, and Borodin and Glebov [4] proved that every planar graph with girth at least 5 admits an $(\mathcal{I},\mathcal{F})$-partition. Poh [9] proved that every planar graph admits an $(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$-partition. More recently, Borodin and Kostochka [7] showed that for all $j \geq 0$ and $k \geq 2j+2$, every graph G with $\mathrm{mad}(G) < 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ admits a (Δ_j,Δ_k)-partition. In particular, every graph G with $\mathrm{mad}(G) < \frac{8}{3}$ admits an $(\mathcal{I},\Delta_2)$-partition, and every graph G with $\mathrm{mad}(G) < \frac{14}{5}$ admits an $(\mathcal{I},\Delta_4)$-partition. With Euler's formula, this yields that planar graphs with girth at least 7 admit $(\mathcal{I},\Delta_4)$-partitions, and that planar graphs with girth at least 8 admit $(\mathcal{I},\Delta_2)$-partitions. Borodin and Kostochka [6] proved that every graph G with $\mathrm{mad}(G) < \frac{12}{5}$ admits an $(\mathcal{I},\Delta_1)$-partition, which implies that that every planar graph with girth at least 12 admits an $(\mathcal{I},\Delta_1)$-partition. This last result was improved by Kim, Kostochka and Zhu [8], who proved that every triangle-free graph with maximum average degree at most $\frac{11}{9}$ admits an $(\mathcal{I},\Delta_1)$-partition, and thus that every planar graph with girth at least 11 admits an $(\mathcal{I},\Delta_1)$-partition. In contrast with these results, Borodin, Ivanova,

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$ $(\mathcal{I},\mathcal{F},\mathcal{F})$ $(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$	The Four Color Theorem [1, 2] Borodin [3] Poh [9]
Planar graphs with girth 5	$(\mathcal{I},\mathcal{F})$	Borodin and Glebov [4]
Planar graphs with girth 6	no $(\mathcal{I},\Delta_d)$	Borodin et al. [5]
Planar graphs with girth 7	$(\mathcal{I},\Delta_4)$ $(\mathcal{I},\mathcal{F}_5)$	Borodin and Kostochka [7] **Present paper**
Planar graphs with girth 8	$(\mathcal{I},\Delta_2)$ $(\mathcal{I},\mathcal{F}_3)$	Borodin and Kostochka [7] **Present paper**
Planar graphs with girth 10	$(\mathcal{I},\mathcal{F}_2)$	**Present paper**
Planar graphs with girth 11	$(\mathcal{I},\Delta_1)$	Kim, Kostochka and Zhu [8]

Table 1: Known results on planar graphs.

Montassier, Ochem and Raspaud [5] proved that for every d, there exists a planar graph of girth at least 6 that admits no $(\mathcal{I},\Delta_d)$-partition.

In this paper, we focus on $(\mathcal{I},\mathcal{F}_d)$-partitions of sparse graphs ; this is a follow-up of the previous studies. Note that, if a graph admits an $(\mathcal{I},\mathcal{F}_d)$-partition, then it admits an $(\mathcal{I},\Delta_d)$-partition, and that an $(\mathcal{I},\mathcal{F}_1)$-partition is the same as an $(\mathcal{I},\Delta_1)$-partition. Therefore the previous results imply the following for $(\mathcal{I},\mathcal{F}_d)$-partitions:

- for every d, there exists a planar graph of girth at least 6 that admits no $(\mathcal{I},\mathcal{F}_d)$-partition;
- every planar graph with girth at least 11 admits an $(\mathcal{I},\mathcal{F}_1)$-partition.

Here are the main results of our paper:

Theorem 1 *Let $2 \le M < 3$ be a real number and $d \ge \frac{2}{3-M} - 2$ be an integer. Every graph G with $\mathrm{mad}(G) < M$ admits an $(\mathcal{I},\mathcal{F}_d)$-partition.*

Theorem 2 *Let $\frac{8}{3} \le M < 3$ be a real number and $d \ge \frac{1}{3-M}$ be an integer. Every graph G with $\mathrm{mad}(G) < M$ admits an $(\mathcal{I},\mathcal{F}_d)$-partition.*

By a direct application of Euler's formula, every planar graph with girth at least g has maximum average degree less than $\frac{2g}{g-2}$. That yields the following corollary:

Corollary 3 *Let G be a planar graph with girth at least g.*

1. *If $g \ge 7$, then G admits an $(\mathcal{I},\mathcal{F}_5)$-partition.*
2. *If $g \ge 8$, then G admits an $(\mathcal{I},\mathcal{F}_3)$-partition.*
3. *If $g \ge 10$, then G admits an $(\mathcal{I},\mathcal{F}_2)$-partition.*

Corollaries 3.2 and 3.3 are obtained from Theorem 2, whereas Corollary 3.1 is obtained from Theorem 1. See Table 1 for an overview of the results on vertex partitions of planar graphs presented above.

2 Sketch of the proofs

The proofs of Theorems 1 and 2 work on the same model, though the proof of Theorem 2 has some additional arguments. We will focus here on the proof of Theorem 1. The proof uses the discharging method. Let G be a counter-example to Theorem 1 with the minimum number of

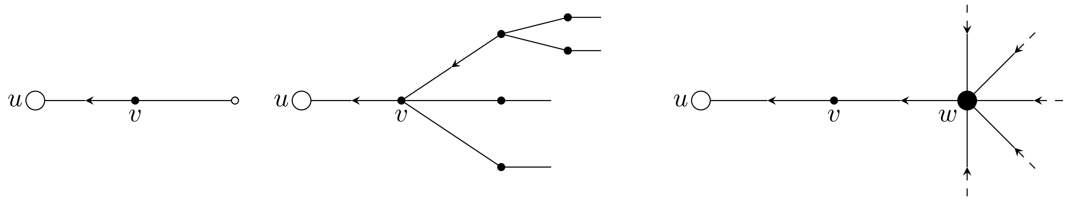

Figure 1: The construction of the light forest L. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented. The dashed lines are the continuation of the light forest. The arrows point from son to father in L.

vertices. For all k, a vertex of degree k, at least k, or at most k in G is a k-vertex, a k^+-vertex, or a k^--vertex respectively. A $(d+1)^-$-vertex is a *small* vertex, and a $(d+2)^+$-vertex is a *big* vertex. We prove some properties on the structure of G.

Lemma 4 *There are no 1^--vertices in G.*

Lemma 5 *There are no 2-vertices adjacent to two small vertices in G.*

Lemma 6 *There are no sets B of small 3^+-vertices in G inducing a tree such that every vertex that is not in B and has a neighbour in B is of degree 2.*

Note that Lemma 6 allows us to forbid configurations of unbounded size. Let us define the notion of *light forest* in the graph G that is useful both in defining some forbidden configurations and in the discharging procedure. Let B be a (maximal) set of small 3^+-vertices such that $G[B]$ is a tree with only one edge that links a vertex of B to a 3^+-vertex $u \in V(G) \setminus B$, and u is a big vertex. Such a set B is called a *bud* with father u. A 2-vertex adjacent to a small vertex is called a *leaf*. Let us build the *light forest* L, by the following three steps:

1. While there are leaves that are not in L, do the following. Pick a leaf v, and let u be the big vertex adjacent to v (that exists by Lemma 5). Add to L the vertex v, the edge uv, and the vertex u (if it is not already in L). Also set that u is the *father* of v (and v is a *son* of u). See Figure 1, left.

2. While there are buds that are not in L, do the following. Pick a bud B. Let u be the father of B, and let v be the vertex of B adjacent to u. Add $G[B]$ to L, as well as the edge uv, and the vertex u (if it is not already in L). The vertex u is the father of v, and the father/son relationship in B is that of the tree $G[B]$ rooted at v. See Figure 1, middle.

3. While, for some k, there exists a big k-vertex $w \in L$ that has $k-1$ sons in L and whose last neighbour is a 2-vertex that is not in L, do the following. Let v be the 2-vertex adjacent to w that is not in L, and let u be the neighbour of v distinct from w. Note that v is a non-leaf 2-vertex (since it was not added to L in Step 1), therefore u is a big vertex. Add to L the vertex v, the edges uv and vw, and the vertex u (if it is not already in L). We set that v is the father of w, and that u is the father of v. See Figure 1, right.

The following lemma uses the notion of light forest to define forbidden configurations of unbounded size.

Lemma 7 *For all k, the light forest L of G does not contain a big k-vertex with k sons.*

Discharging procedure:

Let $\epsilon = 3 - M$. Recall that $d \geq \frac{2}{3-M} - 2 = \frac{2}{\epsilon} - 2$, therefore $\epsilon \geq \frac{2}{d+2} > 0$. We start by assigning to each k-vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G, the sum of the charges of the vertices is negative. The initial charge of each 3^+-vertex is at least ϵ, and thus is positive.

Every big vertex gives charge $1 - \epsilon$ to each of its neighbours that are its sons in L, does not give anything to its father in L (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other neighbours with degree 2.

One can prove that, at the end of the procedure, every vertex has a non-negative charge, that is a contradiction.

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A Study of k-dipath Colourings of Oriented Graphs

Christopher Duffy [1], Gary MacGillivray [2] and Éric Sopena [3]

[1] Department of Mathematics and Statistics, Dalhousie University, Canada
[2] Department of Mathematics and Statistics, University of Victoria, Canada
[3] Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France.
CNRS, LaBRI, UMR5800, F-33400 Talence, France

Extended Abstract

Recall that an *oriented graph* is a digraph obtained from a simple, undirected graph by giving each edge one of its two possible orientations. The *directed girth* of an directed graph H is defined to be the minimum length of a directed cycle in H, or infinity if H has no directed cycle. For a pair of vertices $u, v \in V(G)$, the *weak distance from u to v*, denoted $d_{weak}(u, v)$, is the least k such that there is a directed path of length k from u to v or from v to u, or infinity if no such path exists.

Recall, also, that if G and H are oriented graphs, then a *homomorphism of G to H* is a function ϕ from the vertices of G to the vertices of H such that $\phi(x)\phi(y) \in E(H)$ whenever $xy \in E(G)$. If G and H are oriented graphs such that there is a homomorphism ϕ of G to H, then we write $\phi : G \to H$, or $G \to H$ if the name of the function ϕ is not important.

Let k and t be a positive integers, and let G be an oriented graph. Chen and Wang [2] defined a *k-dipath t-colouring* of G to be an assignment of t colours to the vertices of G so that any two vertices joined by a directed path of length at most k are assigned different colours. A 1-dipath t-colouring of an oriented graph G is a colouring of the underlying undirected graph of G. Figure 1 gives of a 3-dipath 4-colouring of an oriented graph.

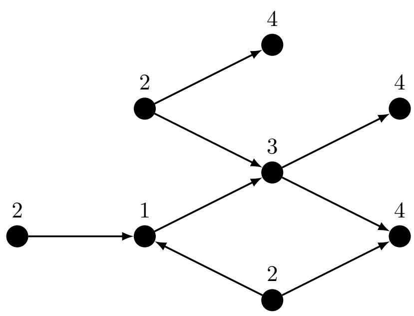

Figure 1: A 3-dipath 4-colouring.

The *k-dipath chromatic number of G*, denoted by $\chi_{kd}(G)$, is the smallest positive integer t such that there exists a k-dipath t-colouring of G. Chen and Wang showed that any orientation of a Halin graph has 2-dipath chromatic number at most 7, and there are infinitely many such graphs G with $\chi_{2d}(G) = 7$ [2].

For a positive integer t, an *oriented t-colouring* of an oriented graph G is a homomorphism of G to some oriented graph on t vertices. Oriented colourings were first introduced in 1994 [3], and have been a topic of considerable interest in the literature since then; see the recent survey by Sopena [8], and also [1], [4] for related topics. Since oriented graphs have no directed cycles of length two, the definition implies that any two vertices of G joined by a directed path of length at most two are assigned different colours (i.e. they have different images) in an oriented colouring of G. The 2-dipath chromatic number is of interest, in part, because it gives a lower bound for the *oriented chromatic number* $\chi_o(G)$ – the smallest positive integer t such that G admits an oriented t-colouring. It follows from the definition that any oriented colouring of G is a 2-dipath colouring of G; hence $\chi_{2d}(G) \leq \chi_o(G)$.

In her Master's thesis [9] (more recently published as [6]), Sherk (née Young) gives a *homomorphism model* for 2-dipath t-colouring. That is, for each positive integer t, she describes an oriented graph $\mathcal{G}_t$ with the property that an oriented graph G has a 2-dipath t-colouring if and only if there is a homomorphism of G to $\mathcal{G}_t$. As is common with such theorems, it is possible to use the homomorphism to $\mathcal{G}_t$ to find a 2-dipath t-colouring of G. The existence of this model implies an upper bound for the oriented chromatic number as a function of the 2-dipath chromatic number. It also leads to a proof that deciding whether a given oriented graph has a 2-dipath t-colouring is Polynomial if the fixed integer $t \leq 2$, and NP-complete if $t \geq 3$.

A natural question is whether Sherk's results can be generalized to k-dipath t-colouring. We seek a model similar to hers, where the homomorphism to the target oriented graph can be used to find a k-dipath t-colouring of the given oriented graph. However, we suggest that such model only exists for colouring oriented graphs with no directed cycles of length k or less. Consider the case of 3-dipath t-colouring, where $t \geq 3$. Suppose there exists a digraph $H_{3,t}$ with the property that an oriented graph G has a 3-dipath t-colouring if and only if there is a homomorphism of G to $H_{3,t}$. The digraph $H_{3,t}$ has no loops, otherwise all vertices of G can be assigned the same colour.

By definition, the directed 3-cycle has a 3-dipath 3-colouring: assign each vertex a different colour. Thus, there is a homomorphism of the directed 3-cycle to $H_{3,t}$. Consequently, $H_{3,t}$ has a directed 3-cycle. But, now there is a homomorphism of a directed path of length three to $H_{3,t}$ in which the two end vertices have the same image. Since the ends of a directed path of length three must be assigned different colours in a 3-dipath t-colouring, this model will not have the desired property. Similar considerations apply to k-dipath t-colouring for all pairs of positive integers k and t with $t \geq k$. Hence, a homomorphism model of the type we seek will not exist if the oriented graphs being coloured can have directed cycles of length k or less. This is consistent with Sherk's results for 2-dipath colouring of oriented graphs as these graphs have no directed cycles of length two or less.

Let G be an oriented graph. Define G^k to be the simple graph formed from G as follows:

- $V(G^k) = V(G)$, and
- $E(G^k) = \{uv | 0 < d_{weak}(u, v) \leq k\}$.

Observe that if G is an oriented graph with directed girth at least $k+1$, then there is a one-to-one correspondence between k-dipath colourings of G and proper colourings of G^k.

The main result of this work is the construction of a homomorphism model for k-dipath t-colouring of oriented graphs with no directed cycles of length k or less. That is, for all positive integers k and t we describe an oriented graph $\mathcal{G}_{k,t}$ with the property that an oriented graph G, with no directed cycle of length at most k, has a k-dipath t-colouring if and only if G admits a homomorphism to $\mathcal{G}_{k,t}$. Further, such a homomorphism is sufficient to construct an explicit k-dipath t-colouring of G. The oriented graph $\mathcal{G}_{k,t}$ is constructed by considering a set of matrices which encode the possible colours of vertices (in a k-dipath t-colouring) in an closed $(k-1)$-neighbourhood of a fixed vertex . The set of such matrices is taken as the vertex set of $\mathcal{G}_{k,t}$. Adjacency between a pair of matrices encodes that the respective fixed vertices in those matrices have consistent $(k-1)$-neighbourhoods. This construction generalises Sherk's homomorphism model for 2-dipath colouring using vectors to encode the colours of neighbours of a vertex with a fixed colour.

This homomorphism model for k-dipath t-colouring captures the following facts.

- If $t < k$, then no oriented graph with directed girth at least $k+1$ and a k-dipath t-colouring can have a directed path of length greater than t. Any such digraph is k-dipath t-colourable.

- If $t \geq k$, then any k-dipath t-colouring of an oriented graph is also a k-dipath $(t+1)$-colouring, and there exist oriented graphs with k-dipath chromatic number $t+1$.
- Every $(k+1)$-dipath t-colouring of an oriented graph is a k-dipath t-colouring

In addition to considering homomorphisms and k-dipath coloring, we determine the complexity of deciding the existence of a k-dipath t-colouring for all pairs of fixed positive integers k and t. When instances are restricted to oriented graphs with no directed cycles of length k or less, it is shown that that this problem is NP-complete whenever $t > k \geq 3$, and Polynomial if $t = k$ or $k \leq 2$. When there are no restrictions, it is shown that this problem is NP-complete whenever $k \geq 3$ and $t \geq 3$, and Polynomial whenever $k \leq 2$ or $t \leq 2$. These results generalise the result of Sherk for 2-dipath colouring; however, the here reduction is directly from the t-colouring problem, rather than following from the homomorphism model.

The problem of deciding whether the k-th power of a graph G is t-colourable is known to be NP-complete [7, 5]. Our results imply that if $t > k$, the problem of deciding whether the underlying graph of the k-th power of an oriented graph G (i.e. two vertices are adjacent if and only if they are at weak distance at most k) is t-colourable is NP-complete, even when restricted to powers of oriented graphs with directed girth at least $k+1$, and also that, if $t = k \geq 3$, the problem of deciding whether the k-th power of an oriented graph G is t-colourable is NP-complete.

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Hamiltonian Anti-Path on Sparse Graph Classes[1]

Pavel Dvořák[2], *Dušan Knop*[3], and *Tomáš Masařík*[4]

Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Extended Abstract

We study the parameterized complexity of the Hamiltonian Anti-Path problem (i.e. the problem of Hamiltonian Path in the graph complement) parameterized by tree-width. Standard problem of Hamiltonian path on graphs parameterized by tree-width admits an FPT algorithm by Courcelles meta-theorem [4]. However, complements of graphs with bounded tree-width can have tree-width arbitrarily large. Moreover, it is unlikely that there is an MSO formula expressing Anti-Path. We show an FPT algorithm for Hamiltonian Anti-Path parametrized by tree-width. Moreover, by an easy trick (i.e. add k isolated vertices, this change does not increase the tree-width) we can state stronger result for the k-Anti-Path cover problem.

k-Anti-Path Cover	
Input:	A graph G.
Question:	Is it possible to cover the complement of graph G (every vertex of G has to belong to exactly one path) by at most k paths?

In this notion Hamiltonian Anti-Path problem is exactly the 1-Anti-Path Cover problem.

Theorem 1 *Let G be a graph of tree-width at most w. The problem of k-Anti-Path cover admits an FPT algorithm parameterized by tree-width. Precisely, after the first step of the algorithm the middle-product graph has neighborhood diversity bounded by $2^{2w^2} + 2w^2$.*

Theorem 2 *Let G be a graph of tree-depth most d. The problem of k-Anti-Path cover admits an FPT algorithm parameterized by tree-width. Precisely, the algorithm middle-product graph has neighborhood diversity bounded by $2^d + d$.*

The notion of *tree-width* was introduced by Bertelé and Brioshi in 1972 [1].

Definition 3 (Tree decomposition) *A* tree decomposition *of a graph G is a pair (T, X), where $T = (I, F)$ is a tree, and $X = \{X_i \mid i \in I\}$ is a family of subsets of $V(G)$ (called bags) such that:*

- *the union of all X_i, $i \in I$ equals V,*
- *for all edges $\{v, w\} \in E$, there exists $i \in I$, such that $v, w \in X_i$ and*
- *for all $v \in V$ the set of nodes $\{i \in I \mid v \in X_i\}$ forms a subtree of T.*

The *width* of the tree decomposition is $\max(|X_i| - 1)$. The *tree-width* of a graph $\mathrm{tw}(G)$ is the minimum width over all possible tree decompositions of the graph G.

Sparser tree-like graph parameter is *tree-depth.*

Definition 4 (Tree-depth [7]) *The closure $Clos(F)$ of a forest F is the graph obtained from F by making every vertex adjacent to all of its ancestors. The tree-depth $\mathrm{td}(G)$ of a graph G is one more than the minimum height of a rooted forest F such that $G \subseteq Clos(F)$.*

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The last graph parameter is the *neighborhood diversity* introduced by Lampis in 2012 [6].

Definition 5 (Neighborhood diversity) *The* neighborhood diversity *of a graph G is denoted by* $\mathrm{nd}(G)$ *and it is the minimum size of a partition of vertices into classes such that all vertices in the same class have the same neighborhood, i.e.* $N(v) \setminus \{v'\} = N(v') \setminus \{v\}$, *whenever v, v' are in the same class.*

It can be easily verified that every class of neighborhood diversity is either a clique or an independent set. Moreover, for every two distinct classes C, C', either every vertex in C is adjacent to every vertex in C', or there is no edge between C and C'. If classes C and C' are connected by edges, we refer to such classes as *adjacent*.

From this definition it can be easily seen that complement of graph has exactly the same neighborhood diversity as the former graph.

All the standard parameterized complexity notion used here can be found in a nice book by Downey and Fellows [5].

We say that an algorithm for a parameterized problem L is an FPT *algorithm* if there exist a constant c and a computable function f such that the running time for input (x, k) is $f(k)|x|^c$ and the algorithm accepts (x, k) if and only if $(x, k) \in L$.

We now would like to sketch the proof of our result.

Our approach repeatedly use Bondy-Chvátal's closure theorem to increase the number of edges in the complement of a graph (i.e. to reduce edges in a graph given). This operation preserves the tree-width.

Theorem 6 (Bondy-Chvátal closure [3]) *Let $G(V, E)$ be a graph of order $|V| \geq 3$ and suppose that u and v are distinct non-adjacent vertices such that* $\deg(u) + \deg(v) \geq |V|$.

Then $G(V, E)$ has a Hamiltonian path if and only if $G'(V, E \cup \{uv\})$ has a Hamiltonian path.

Then we prove that when this process gets stuck the resulting graph has bounded neighborhood diversity and so we use an FPT algorithm for bounded neighborhood diversity by Lampis [6].

Despite being quite similar the proof of an FPT algorithm parametrized by tree-depth has easier calculations so we decided to present that one first and in more details.

Proof of Theorem 2 Lets have a graph $G(V, E)$ with tree-depth at most d. There is an FPT algorithm parameterized by tree-depth that finds a representation of a graph by a tree and its closure [7]. We denote height of a vertex in this tree as $h(v)$, height of the root is 0 and maximum height is d.

We start to remove edges from the leaves to the root using Bondy-Chvátal's closure in the complement. For all leaves of G holds that they have at least $|V| - d$ non-neighbors. We remove an edge to the next layer vertex if possible. Generally, for the leaf of height h there is at least $|V| - h$ non-neighbors. If a vertex is not a leaf we have to add number of leaves in the corresponding subtree f so it has at least $|V| - (h - 1) - f$ non-neighbors. Every time we re move an edge we increase the height of that leaf.

The algorithm gets stuck if and only if for each leaf holds $|V| - h + |V| - h + 1 - f \leq |V|$ so when $f \geq |V| - 2h + 1$ or G becomes an independent set. After this process the resulting graph is only a path of the length at most d with several additional leaves attached to each its vertex. So we have at most d special vertices that can have any possible neighborhood and other vertices forms together a clique thus the resulting graph has neighbor diversity at most $2^d + d$. $\square$

Sketch of the Proof of Theorem 1 Lets have a graph $G(V, E)$ with tree-width at most w. We can compute its tree-decomposition by an FPT algorithm parametrized by tree-width [2]. Moreover, a decomposition with at most $|V|$ bags can be found.

Then we can have at most $2k^2$ special vertices that belongs to at least $\frac{n}{2k}$ bags. Vertices belonging to less bags forms a clique again by Bondy-Chvátal's closure because each vertex there has at most $\frac{n}{2}$ neighbors and so every pair of those vertices fulfill the condition.

And so again we have $2k^2$ special vertices that can have any possible neighborhood and other vertices forms together a clique thus the resulting graph has neighbor diversity at most $2^{2k^2} + 2k^2$. $\square$

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On Strategies of Online Ramsey Game on Planar Graphs

Pavel Dvořák [1] and Tomáš Valla [2]

[1] Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic
[2] Faculty of Information Technology, Czech Technical University, Prague, Czech Republic

Extended Abstract

An online Ramsey game $(\mathcal{G}, H)$ is a game between Builder and Painter, alternating in turns. In each round Builder draws an edge and Painter colors it either red or blue. Builder wins if after some round there is a monochromatic copy of the graph H, otherwise Painter is the winner. The rule for Builder is that after each his move the resulting graph must belong to the class of graphs $\mathcal{G}$. In this abstract we consider that $\mathcal{G}$ is the class of all planar graphs. We present some new strategies for Builder and introduce a new graph class $\mathcal{H}$ such that if H is in $\mathcal{H}$ then Builder wins.

Introduction The online Ramsey game was introduced independently by Beck [1] and Friedgut et al. [4]. In this abstract we consider an extensions of the online Ramsey game $(\mathcal{G}, H)$ introduced by Grytczuk et al. [5] as follows. We are given a class of finite graphs $\mathcal{G}$ and a fixed graph H. There are two players, Builder and Painter, and the board of the game is an infinite independent set of vertices. In each round Builder draws a new edge and Painter colors it red or blue. The goal of Builder is to force Painter to create a monochromatic copy of H. However, in each round the graph induced by the drawn edges must belong to $\mathcal{G}$, otherwise Builder loses.

Definition 1 *If Builder can always win the online Ramsey game $(\mathcal{G}, H)$, we say that H is* unavoidable *on $\mathcal{G}$. A graph class $\mathcal{C}$ is* unavoidable *on $\mathcal{G}$ if every graph $C \in \mathcal{C}$ is unavoidable on $\mathcal{G}$.*

Let $R(H)$ be a *Ramsey graph* of H, that means a graph such that any two-coloring of its edges contains a monochromatic copy of H. If $R(H) \in \mathcal{G}$ then H is trivially unavoidable on $\mathcal{G}$. However, the question of unavoidability is usually highly nontrivial when $R(H) \notin \mathcal{G}$. For example, for $H = K_4$ and $\mathcal{G}$ the class of all 4-colorable graphs, no Ramsey graph $R(K_4)$ belongs to $\mathcal{G}$, but still K_4 is unavoidable on $\mathcal{G}$ by [5]. The online Ramsey theory is thus an interesting member of the family of Ramsey-related problems.

Online Ramsey theory recently gained popularity and different versions of the online Ramsey game were considered and several unavoidability results were obtained. Grytczuk et al. [5] has shown that forests are unavoidable on the class of all forests and that any k-colorable H is unavoidable on the class of k-colorable graphs. They also proved that K_3 is not unavoidable on the class of outerplanar graphs and conjectured that the class unavoidable on planar graphs is exactly the class of outerplanar graphs. This was disproved by Petříčková [7]. She showed that the class of outerplanar graphs is unavoidable on the class of planar graphs. However, she also found a class of planar graphs, which are not outerplanar, unavoidable on planar graphs.

Theorem 2 (Petříčková [7]) *Let $\theta_{i,j,k}$ be a graph consisted of three internally disjoint path of length i, j and k between connecting two vertices. The graph $\theta_{2,j,k}$ is unavoidable on planar graphs for even j, k.*

Butterfield et al. [2] presented several unavoidability results for graphs on the class of bounded-degree graphs. Rolnick [8] provided a complete characterization of trees unavoidable on the class of graphs with maximum degree 4.

Conlon [3] studied the differences between standard Ramsey theory and the online theory. He proved that there is a constant c such that for infinitely many values k Builder needs to draw less then $c^{-k}\binom{r(k)}{2}$ edges to force K_k on the infinite complete background graph, where $r(k)$ is the standard Ramsey number for complete graphs, i.e., there is more efficient strategy for Builder to force K_k than using results from standard Ramsey theory.

Builder's Strategies From now we suppose that the class $\mathcal{G}$ is the class of planar graphs. Grytczuk et al. [5] and Petříčková [7] conjectured the graph K_4 is avoidable on planar graphs. We suspect the class of series-parallel graph is exactly the unavoidable class on planar graphs. We present some techniques and results which might be useful for proving the conjecture. Our strategies were inspired by the result of Grytczuk et al. [5] that forests are unavoidable on forests. As a corollary we present new graph class unavoidable on planar graphs.

The first technique is about reusing existing strategy in a tree-like structure. We define a tree-like structure where each edge of a tree is replaced by some graph. Let $T = (U, F)$ be a tree and $\mathcal{H} = \{G_f | f \in F\}$ be a set of graphs. We call a function $r : U \to \mathcal{P}(V(\mathcal{H}))$ by a *tree function*[1] if r has the following properties for every $f \in F$:

1. $|r(v) \cap V(G_f)| \leq 1$
2. $r(v) \cap V(G_f) \neq \emptyset \Leftrightarrow v \in f$

Informally, the extended tree $T(\mathcal{H}, r)$ arises by replacing every edge $f \in F$ by a graph $G_f \in \mathcal{H}$. The tree function r determined how the graphs in $\mathcal{H}$ are connected. An example of an extended tree is in Figure 1. An *extended forest* $T(\mathcal{H}, r)$ is defined in the same way but the graph T can be a forest. Let $T(\mathcal{H}, r)$ be an extended tree and R be a subtree of T. For simplicity, we use notation $R(\mathcal{H}, r)$ for extended tree such that the graph family $\mathcal{H}$ and the tree function r are restricted to the edges and the vertices of R.

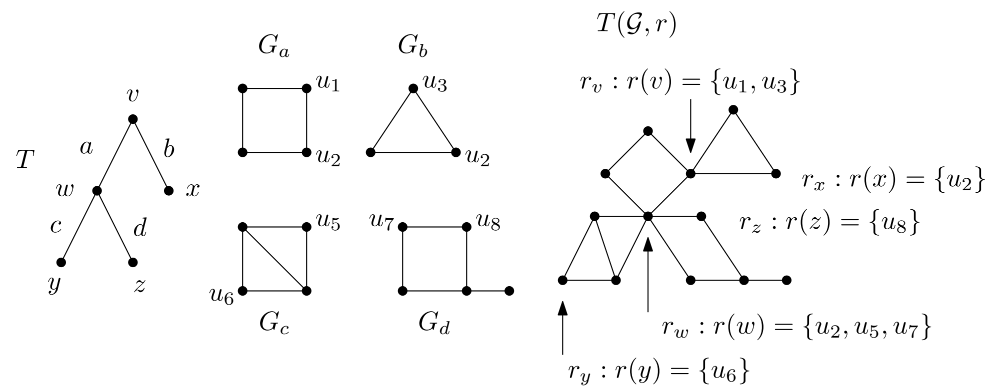

Figure 1: Example of an extended tree $T(\{G_a, G_b, G_c, G_d\}, r)$.

Definition 3 *Let $G = (V, E)$ be a connected graph and $u, v \in V$. An* uniform *extended tree $T(G, u, v, r)$ is an extended tree $T(\mathcal{H}, r)$ such that every $G_i \in \mathcal{H}$ is isomorphic to G via isomorphism $f_i : V(G_i) \to V(G)$ and*

$$\{f_i(v) | v \in V(T) : r(v) \cap V(G_i) \neq \emptyset\} = \{u, v\}.$$

Informally, the uniform extended tree arises by replacing every edge of a tree T by a copy of the fixed graph G such that all copies of G are connected by copies of two fixed vertices in $V(G)$.

Let $\tau_1 = T_1(\mathcal{H}_1, r_1)$ be an extended forest for a forest $T_1 = (U_1, F_1)$ and $\tau_2 = R(\mathcal{H}_2, r_2)$ be an extended forest for a forest $T_2 = (U_2, F_2)$. We say the extended forest τ_1 is a *proper subforest* of τ_2 if

1. the forest T_1 is a subgraph of T_2,
2. for every $f \in F_1$ it holds that $G^1_f \in \mathcal{H}_1$ is a subgraph of $G^2_f \in \mathcal{H}_2$,
3. for every $u \in U_1$ it holds that $r_1(u) = r_2(u)$.

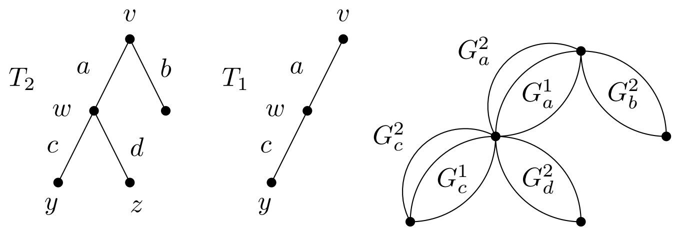

Figure 2: Example of proper subtree for trees T_1 and T_2.

An example of proper subtree is in Figure 2.

Lemma 4 *Let $G = (V, E)$ be a graph, $u, v \in V$ and $T = (U, F)$ be a tree. Let Builder has a strategy S such that he can force a monochromatic copy of G on planar graphs. Then, there exists a Builder's strategy S' such that Builder force a monochromatic copy of an uniform extended tree $\tau_1 = T(G, u, v, r)$ such that the drawn graph $D(S')$ is a planar extended forest $\tau_2 = T(\mathcal{H}, r')$ and τ_1 is a proper subtree of τ_2. Moreover, if G is forced by S such that u and v are always in the same face of the drawn graph $D(S)$ then all copies of u and v are in the same face of a drawn graph $D(S')$.*

Idea of proof of Lemma 4 is to change strategy S_f for forest by Grytczuk et al. [5]. Whenever Builder connect two vertices by an edge using the strategy S_f to force the tree T, he use the strategy S' to connect corresponding two vertices in τ_1.

Second technique is about connecting vertices with a monochromatic path. If a drawn graph has to be planar then Builder has to draw only edges between two vertices in the same face of the drawn graph. Trivially, one edge is always monochromatic. Thus, we ask a question if there are two vertices u and v in the same face of a drawn graph, can Builder force a monochromatic path between u and v? The next lemma shows our result.

Lemma 5 *Let $G = (V, E)$ be a graph and $u, v \in V$. Let S is a Builder's strategy for forcing a monochromatic copy of G such that u and v are always in the same face of the drawn graph. Then, there exists a Builder's strategy S' which forces a monochromatic copy of G and a monochromatic path P_k of the odd length connecting u and v. Moreover, u and v are in the same face of a graph drawn by Builder when using the strategy S'.*

Lemma 4 and strategy for paths by Grytczuk et al. [5] are used in a proof of Lemma 5. A monochromatic copy of extended tree for graph G is forced and then a monochromatic paths are forced as well. Note that the monochromatic copy of G and P_k do not have to be colored by the same color. A straightforward corollary of Lemma 5 gives us the new class of graphs unavoidable on planar graphs.

Corollary 6 *Let $Q(k, \ell, u, v)$ be a graph consisting ℓ internally disjoint paths of the length k between the vertices u and v. The graph $Q(k, \ell, u, v)$ is unavoidable on planar graphs for arbitrary $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$ odd.*

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Quadrangulations of the projective plane are t-perfect if and only if they are bipartite

Elke Fuchs, and *Laura Gellert*

Institute of Optimisation and OR, Ulm University, Germany

Extended Abstract

Introduction

Perfect graphs received considerous attention in graph theory. The purely combinatorial definition — that a graph is perfect if and only if for each of its induced subgraphs, the chromatic number and the clique number coincide — can be replaced by a polyhedral one (Chvàtal [3]):

A graph G is perfect if and only if its *stable set polytope* $\mathrm{SSP}(G)$, ie the convex hull of stable sets of G, is completely described by non-negativity and clique inequalities.

Modification of the considered inequalities leads to generalisations of perfection. A graph G is *t-perfect* if $\mathrm{SSP}(G)$ equals the polyhedron $\mathrm{TSTAB}(G)$ which is defined by non-negativity-, edge- and odd-cycle inequalities.

To be more precise, let $G = (V, E)$ be a graph. All the graphs mentioned here are finite and simple; we follow the notation of Diestel [4]. The *stable set polytope* $\mathrm{SSP}(G) \subseteq \mathbb{R}^V$ of G is defined as the convex hull of the characteristic vectors of stable, ie independent, subsets of V. The characteristic vector of a subset S of the set V is the vector $\chi_S \in \{0,1\}^V$ with $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. We define a second polytope $\mathrm{TSTAB}(G) \subseteq \mathbb{R}^V$ for G, given by

$$
\begin{aligned}
&x \geq 0,\\
&x_u + x_v \leq 1 \text{ for every edge } uv \in E,\\
&\sum_{v \in V(C)} x_v \leq \left\lfloor \frac{|C|}{2} \right\rfloor \text{ for every induced odd cycle } C \text{ in } G.
\end{aligned}
$$

Clearly, $\mathrm{SSP}(G) \subseteq \mathrm{TSTAB}(G)$.

The graph G is called *t-perfect* if $\mathrm{SSP}(G)$ and $\mathrm{TSTAB}(G)$ coincide. Equivalently, G is t-perfect if and only if $\mathrm{TSTAB}(G)$ is an integral polytope, ie if all its vertices are integral vectors.

It is easy to verify that vertex deletion preserves t-perfection. Another operation that keeps t-perfection was found by Gerards and Shepherd [6]: whenever there is a vertex v, so that its neighbourhood is stable, we may contract all edges incident with v simultaneously. We will call this operation a *t-contraction at v*. Any graph that is obtained from G by a sequence of vertex deletions and t-contractions is a *t-minor of G*. Let us point out that any t-minor of a t-perfect graph is again t-perfect.

A *p-wheel* W_p is a graph consisting of a cycle $(w_1, \ldots, w_p)$ and a vertex v adjacent to w_i for $i = 1, \ldots, p$ (see Figure 1). Odd wheels W_{2k+1} for $k \geq 1$ are t-imperfect. Indeed, the vector $(1/3, \ldots, 1/3)$ is contained in $\mathrm{TSTAB}(W_{2k+1})$ but not in $\mathrm{SSP}(W_{2k+1})$. Furthermore, every proper t-minor of an odd wheel is t-perfect.

As bipartite graphs are easily seen to be perfect, the above polyhedral conditions directly show that bipartite graphs are t-perfect as well. We show that non-bipartite quadrangulations of the projective plane are never t-perfect. A quadrangulation is an embedding where all faces are of size 4.

Corollary 1. *A quadrangulation of the projective plane is t-perfect if and only if it is bipartite.*

This directly follows from the fact that every non-bipartite quadrangulation of the projective plane has an odd wheel as a t-minor.

Theorem 2. *A quadrangulation G of the projective plane contains an odd wheel W_{2k+1} for $k \geq 1$ as a t-minor if and only if G is not bipartite.*

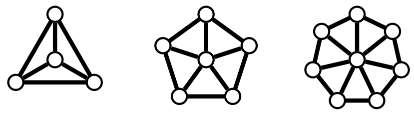

Figure 1: The odd wheels W_3, W_5 and W_7

A general treatment on t-perfect graphs may be found in Grötschel, Lovász and Schrijver [8, Ch. 9.1] as well as in Schrijver [11, Ch. 68]. Bruhn and Benchetrit showed that plane triangulations are t-perfect if and only if they do not contain a certain subdivision of an odd wheel as an induced subgraph [1]. Boulala and Uhry [2] established the t-perfection of series-parallel graphs. Gerards and Shepherd [6] characterised the graphs with all subgraphs t-perfect.

Youngs [12] showed that all non-bipartite quadrangulations of the projective plane have chromatic number equal to 4. Esperet and Stehlík [5] gave bounds for edge- and face-width of non-bipartite quadrangulations.

Embeddings in the projective plane

First let us recall several useful definitions related to surface-embedded graphs. For further background on topological graph theory, we refer the reader to Gross and Tucker [7] or Mohar and Thomassen [10].

An *embedding* of a simple graph G in the projective plane is a continuous one-to-one function from G into the projective plane where G is assumed to have the natural topology as a 1-dimensional CW-complex. For our purpose, it is convenient to abuse the terminology by referring to the image of G as the *embedded graph* G. The connected components of the complement of an embedded graph are called the *faces* of G. An embedding is a *cell embedding* if each face is homeomorphic to an open disk. A cell embedding is a *closed-cell embedding* if each face is bounded by a cycle of G.

The *size* of a face is the length of its bounding cycle. An embedding G is *even* if all faces are of even size. A *quadrangulation* is a cell embedding where all faces are of size 4. Note that a quadrangulation of a simple graph G is always a closed-cell embedding as G does not contain multiple edges.

A cycle C in the projective plane is *contractible* if C separates the projective plane into two sets S_C and $\overline{S_C}$ where S_C is homeomorphic to an open disk in $\mathbb{R}^2$. We call S_C the *interior* of C.

A quadrangulation is *nice*, if the interior of every contractible 4-cycle contains no vertex.

Since the projective plane has Euler characteristic 1, every quadrangulation G satisfies

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| = 4|V(G)| - 4. \tag{1}$$

A cycle in a non-bipartite quadrangulation of the projective plane is contractible if and only if it has even length (see e.g. [9, Lemma 3.1]). One can easily generalise this result.

Lemma 3. *A cycle in a non-bipartite even embedding in the projective plane is contractible if and only it has even length.*

Note that therefore any two odd cycles have an odd number of crossings.

Let us continue with some useful operations.

Lemma 4. *A graph obtained from an even embedding in the projective plane by a deletion of a vertex or an edge is again an even embedding.*

Lemma 5. *A graph obtained from a non-bipartite nice quadrangulation G by a t-contraction is a non-bipartite quadrangulation.*

After each t-contraction, we can delete all vertices in the interior of a 4-cycle to obtain a nice quadrangulation. This does not destroy non-bipartiteness.

Application of t-contractions (and deletion of vertices in the interior of 4-cycles) gives a graph where each vertex is contained in a triangle. The following lemma is a direct consequence of this observation:

Lemma 6. *Let G be a non-bipartite quadrangulation. Then there is a sequence of t-contractions and deletions of vertices in the interior of 4-cycles that transforms G into a non-bipartite nice quadrangulation where each vertex is contained in a triangle.*

We can easily deduce the minimal degree of nice quadrangulations.

Lemma 7. *Every nice quadrangulation has minimal degree 3.*

Finally, we consider wheels in the projective plane. *Odd wheels*, ie p-wheels where $p \geq 3$ is odd, are evenly embeddable in the projective plane; see Figure 2 for an illustration.

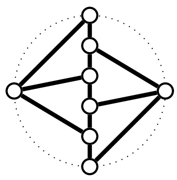

Figure 2: An even embedding of W_5

Lemma 8. *Even wheels W_{2k} for $k \geq 2$ do not have an even embedding in the projective plane.*

Proof of Theorem 2

This section is dedicated to the proof of Theorem 2 (and of Corollary 1). The theorem is based on the following lemma.

Lemma 9. *Let G be a non-bipartite nice quadrangulation of the projective plane where each vertex of G is contained in a triangle. Then G is an odd wheel.*

Proof. By Lemma 7 there exists a vertex v of degree 3 in G. Let $\{x_1, x_2, x_3\}$ be its neighbourhood and let x_1,x_2 and v form a triangle.

Recall that each two triangles intersect (see Lemma 3).

As x_3 is contained in a triangle intersecting the triangle (v, x_1, x_2) and as v has no further neighbour, we can suppose without loss of generality that x_3 is adjacent to x_1. The graph induced by the two triangles (v, x_1, x_2) and (x_1, v, x_3) is not a quadrangulation. Further, addition of the edge x_2x_3 yields a K_4. Thus, the graph contains a further vertex and this vertex is contained in a further triangle T. Since the vertex v has degree 3, it is not contained in T. If further $x_1 \notin V(T)$, then the vertices x_2 and x_3 must be contained in T. But then $x_2x_3 \in E(G)$ and, as above, v, x_1, x_2 and x_3 form a K_4. Therefore, x_1 is contained in T and consequently in every triangle of G. Since every vertex is contained in a triangle, x_1 must be adjacent to all vertices of $G' = G - x_1$. As $|E(G)| = 2|V(G)| - 2$ by (1), the graph

G' has $2|V(G)| - 2 - (|V(G)| - 1) = |V(G)| - 1 = |V(G')|$ many edges. Since G is a nice quadrangulation, G' does not contain a vertex of degree 1. Thus, G' is a cycle and G is a wheel. Since even wheels are not evenly embeddable (Lemma 8), G is an odd wheel.

□

Finally, from Lemma 9 and Lemma 6 our main result follows:

Proof of Theorem 2. As t-contraction preserves the parity of cycles, no bipartite graph has an odd wheel as a t-minor.

Let G be a non-bipartite quadrangulation. By Lemma 6, G has a non-bipartite nice quadrangulation G' where each vertex is contained in a triangle as a t-minor. By Lemma 9, G' is an odd wheel. Therefore, G has an odd wheel as a t-minor.

□

Proof of Corollary 1. Let G be a quadrangulation of the projective plane. If G is bipartite, G is also t-perfect.

Otherwise, G contains an odd wheel as a t-minor (Theorem 2). Since odd wheels are t-imperfect, G is t-imperfect as well. □

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Sharp Bounds for the Complexity of Semi-Equitable Coloring of Cubic and Subcubic Graphs[1]

Hanna Furmańczyk [1], and *Marek Kubale* [2]

[1] Institute of Informatics, University of Gdańsk, Wita Stwosza 57, 80-309 Gdańsk, Poland
[2] Department of Algorithms and System Modelling, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

Extended Abstract

In this paper we consider the complexity of semi-equitable k-coloring, $k \geq 4$, of the vertices of a cubic or subcubic graph G. In particular, we show that, given n-vertex subcubic graph G and $k \geq 4$, a semi-equitable k-coloring of G is NP-hard if $s \geq 7n/20$, and polynomially solvable if $s \leq 7n/21$, where s is the size of maximum color class of the coloring.

We say that a graph $G = (V, E)$ is *equitably k-colorable* if and only if its vertex set can be partitioned into independent sets $V_1, \ldots, V_k \subset V$ such that $|V_i| - |V_j| \in \{-1, 0, 1\}$ for all $i, j = 1, \ldots, k$. The smallest k for which G admits such a coloring is called the *equitable chromatic number* of G and denoted $\chi_=(G)$. Graph G has a *semi-equitable coloring*, if there exists a partition of its vertices into independent sets $V_1, \ldots, V_k \subset V$ such that one of these subsets, say V_1 is of size $s \notin \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$, and the remaining subgraph $G - V_1$ is equitably $(k-1)$-colorable. These two non-classical models of graph coloring have potential applications in multiprocessor scheduling of unit-execution time jobs [6, 7].

In the following we will say that graph G has $(V_1, \ldots, V_k)$ coloring to express explicitly a partition of V into k independent sets. If, however, only cardinalities of color classes are important, we will use the notation $[|V_1|, \ldots, |V_k|]$.

The problem of semi-equitable 3-coloring of connected cubic graphs was introduced in [4]. In this note we extend those results to an arbitrary number $k \geq 4$ of colors and to, possibly disconnected, subcubic graphs, where by a *subcubic graph* we mean a graph G whose vertex degrees fulfill $\deg(v) \leq 3$ for all $v \in V$. In contrast to equitable coloring not all cubic/subcubic graphs have a semi-equitable coloring (cf. Tables 1 and 2). Therefore, in the following we assume that all graphs under consideration have such a coloring.

In order to develop a polynomial time algorithm for semi-equitable k-coloring of subcubic graphs, we generalize Chen et al.'s theorem. In fact, Chen et al. [1] proved that $\chi(G) = \chi_=(G)$ for any connected cubic graph G. Their proof starts from any proper 3-coloring of G and it relies on successive decreasing the *width* of the coloring, i.e. the difference between the cardinality of the largest and smallest independent set, step by step until a coloring is equitable. Actually, this procedure works for every 3-coloring of any cubic graph, except for $K_{3,3}$.

Corollary 1 ([1]) *If G is a connected cubic graph such that $K_{3,3} \neq G \neq K_4$, then there exists an equitable 3-coloring of G.* □

We claim that their approach can be slightly modified to hold for subcubic graphs, not necessarily connected.

Theorem 2 *If G is a subcubic graph, $K_4 \neq G \neq K_{3,3}$, then there exists an equitable 3-coloring of G.*

In contrast to equitable coloring, the problem of semi-equitable coloring becomes NP-hard already if $k = 4$. More precisely, we show that computing a semi-equitable k-coloring of a subcubic graph whose maximum color class is of size at least $7n/20$ is NP-hard.

Let us consider the following combinatorial decision problem:

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IS(G,l): given a subcubic graph G on n vertices and an integer l; the question is whether G has an independent set I of size at least l,

and its subproblem IS$(G,.35n)$, i.e. IS(G,l), where $l = 35n/100$ and n is divisible by 20, in symbols $20|n$.

Note that the IS(G,l) problem is NP-complete even if G is cubic [8] and remains so if $20|n$. This is so because we can enlarge a cubic graph G by adding to it a bipartite cubic graph on j vertices, where $j = 6, 8, 10, 12, 14, 16, 18, 22$, or 24, so that the number of vertices in the new graph is divisible by 20. It is easy to see to see that G has an independent set of size at least l if and only if the new graph has an independent set of size at least $l + j/2$.

In [5] we proved that the problem of deciding whether a cubic graph has a coloring of type $[4n/10, 3n/10, 3n/10]$ is NP-complete. In the following we strenghten and generalize this result to semi-equitable k-colorings ($k \geq 4$) and subcubic graphs.

Lemma 3 *Problem* IS$(G,.35n)$ *is* NP*-complete.*

Lemma 4 *Let G be a subcubic graph such that $20|n$, where n is the number of vertices of G. The problem of deciding whether G has a semi-equitable coloring of type* $[7n/20, 13n/60, 13n/60, 13n/60]$ *is* NP*-complete.*

Theorem 5 *Given an n-vertex subcubic graph, a constant $k \geq 4$, and an integer function $s = s(n)$. Finding a semi-equitable k-coloring of G of type* $[s, \lceil \frac{n-s}{k-1} \rceil, \ldots, \lfloor \frac{n-s}{k-1} \rfloor]$ *is* NP*-hard, if $s \geq 7n/20$.*

Finally, we show how to obtain in $O(n^2)$ time a semi-equitable k-coloring of a subcubic graph whose maximum color class is at most $n/3$.

Theorem 6 *Given an n-vertex subcubic graph G, $K_4 \neq G \neq K_{3,3}$, a constant $k \geq 4$, and an integer function $s = s(n)$. Finding a semi-equitable k-coloring of G of type* $[s, \lceil \frac{n-s}{k-1} \rceil, \ldots, \lfloor \frac{n-s}{k-1} \rfloor]$ *is solvable in $O(n^2)$ time, if $s \leq \lceil n/3 \rceil$.*

One can ask about a semi-equitable 3-coloring of a subcubic graph G. The problem was discussed in [4], where we proved

Theorem 7 ([4]) *If G, n-vertex cubic graph, has an independent set I of size $|I| \geq 2n/5$, then it has a semi-equitable coloring of type* $[|I|, \lceil \frac{n-|I|}{2} \rceil, \lfloor \frac{n-|I|}{2} \rfloor]$. □

Moreover, we noticed that a cubic graph usually has such a big independent set. This is so because Frieze and Suen [3] proved that for almost all cubic graphs G their independence number $\alpha(G)$ fulfills the inequality $\alpha(G) \geq 4.32n/10 - \epsilon n$ for any $\epsilon > 0$. In practice this means that a random cubic graph is very likely to have an independent set of size $k \geq 2n/5$ and the probability of this fact increases with n.

The above result holds also for almost all subcubic graphs.

Tables 1 and 2 gather the complexity status for semi-equitable k-coloring for cubic and subcubic graps, respectively.

k	$s \le \lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil < s < \lfloor \frac{2n}{5} \rfloor$	$\lfloor \frac{2n}{5} \rfloor \le s < \frac{n}{2}$	$s = \frac{n}{2}$	$\frac{n}{2} < s$
3	-	?	NPH	$O(n)^*$/-	-
≥ 4	$O(n^2)$	?	NPH	$O(n)^*$/-	-

Table 1: The complexity of semi-equitable k-coloring of cubic graphs. The ”-” sign means that the corresponding solution does not exist. The ”*” sign means that the solution concerns bipartite cubic graphs only.

k	$s \le \lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil < s < \lfloor \frac{7n}{20} \rfloor$	$\lfloor \frac{7n}{20} \rfloor \le s \le \lfloor \frac{3n}{4} \rfloor$	$\lfloor \frac{3n}{4} \rfloor < s$
3	-	?	NPH	-*
≥ 4	$O(n^2)$	?	NPH	-*

Table 2: The complexity of semi-equitable k-coloring of subcubic graphs. The ”-” sign means that the corresponding solution does not exist. The ”*” sign means that the corresponding negative result concerns connected graphs only.

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The threshold for the appearance of a properly coloured spanning tree in an edge coloured random graph

Pu Gao [1], *Carlos Hoppen* [2] and *Juliana Sanches* [2]

[1] Monash University, Australia
[2] Universidade Federal do Rio Grande do Sul, Brazil

Extended Abstract

Given a number of colours $k \geq 1$, we consider the probability space $\mathcal{G}_{n,p}^k$ of edge-coloured random graphs, whose elements are produced by first generating a graph G in the Erdős-Rényi probability space $\mathcal{G}_{n,p}$ and then colouring each edge of G independently and uniformly with a colour from the set $[k] = \{1, \ldots, k\}$. We determine the threshold function $p = p_k(n)$ for the property that such an edge-coloured random graph contains a properly coloured spanning tree, for all fixed $k \geq 3$. It turns out to coincide with the connectivity threshold, which is $\log(n)/n$. This contrasts with the case $k = 2$, where the threshold is known to be $2\log(n)/n$ in light of recent work by Espig, Frieze and Krivelevich [6].

More generally, given positive integers n and k, and a probability space $\mathcal{G}_n$ on the set of graphs whose vertex set is labelled by $[n] = \{1, \ldots, n\}$, we may define a probability space of edge-coloured graphs $\mathcal{G}_n^k$ by first sampling an n-vertex graph G in $\mathcal{G}_n$, and then colouring each edge of G independently and uniformly with a colour from the set $[k]$. For instance, if $k = 1$, the probability space produced is precisely $\mathcal{G}_n$. Probability spaces of this type have been introduced by Cooper and Frieze [3], who considered edge-coloured graphs obtained from the standard random graph model $\mathcal{G}_n = \mathcal{G}_{n,m}$ of uniformly distributed labelled n-vertex graphs with $m = m(n)$ edges. They studied conditions on n, m and k which imply that an element of $\mathcal{G}_{n,m}^k$ a.a.s. contains a *rainbow* Hamilton cycle, that is, a Hamilton cycle such that all edges have different colours. As usual, for a sequence of probability spaces Ω_n, $n \geq 1$, an event A_n of Ω_n occurs *asymptotically almost surely*, or a.a.s. for brevity, if $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$. Also observe that results about $\mathcal{G}_{n,m}^k$ may be easily translated in terms of $\mathcal{G}_{n,p}^k$ for $p = p(n) = m/\binom{n}{2}$, as $\mathcal{G}_{n,p}$ is asymptotically equivalent to $\mathcal{G}_{n,m}$.

Since then, several authors have found tighter sufficient conditions for the occurrence of rainbow Hamilton cycles [10, 7, 1] in $\mathcal{G}_{n,p}^k$. As it turns out, the paper [1] also provides conditions on p to ensure the occurrence of a rainbow perfect matching when the number of colours is $k = n/2$. Moreover, a recent paper of Ferber, Nenadov and Peter [9] investigates the occurrence of other types of rainbow spanning subgraphs in a random coloured graph. Regarding other probability spaces, Janson and Wormald [13] studied the occurrence of rainbow Hamilton cycles in edge-coloured graphs arising from the space of random regular graphs $\mathcal{G}_{n,d}$, while Ferber, Kronenberg, Mousset and Shikhelman [8] looked for edge-disjoint rainbow copies of some fixed subgraph in random coloured percolations of graphs with sufficiently large minimum degree.

Very recently, Espig, Frieze and Krivelevich [6] determined that the threshold for the occurrence of *zebraic* Hamilton cycles (that is, Hamilton cycles whose edges alternate in the two colours) in $\mathcal{G}_{n,p}^2$ is $2\log n/n$. More precisely, they studied the random graph process where one edge is randomly added and coloured in each step, and they showed that the hitting time for a zebraic Hamilton cycle a.a.s. coincides with the hitting time for having minimum degree 1 in both colours, which is an obvious necessary condition. Their results easily imply that $p_2(n) = 2\log(n)/n$ is the threshold for the occurrence of a properly coloured spanning tree in $\mathcal{G}_{n,p}^2$.

As in previous work, we study the existence of a nicely coloured spanning substructure in an edge coloured random graph. Precisely, our objective is to find the threshold function $p = p_k(n)$ for the property that $\mathcal{G}_{n,p}^k$ contains a properly coloured spanning tree. Spanning trees were also sought in the context of *power of k choices* [2]. In that paper, every edge of a random graph was assigned u.a.r. a subset of k colours of $[n-1]$, and the objective was to find a rainbow spanning tree.

By the discussion in the previous paragraph, the work in [6] implies that $p_2(n) = 2\log(n)/n$. In the present paper, we prove that, for all $k \geq 3$, $p_k(n) = \log(n)/n$. The fact that $\mathcal{G}^k_{n,p}$ a.a.s. does not contain a properly coloured spanning tree if $p \leq (1-\epsilon)\log(n)/n$ is immediate, as $\mathcal{G}_{n,p}$ is a.a.s. disconnected in this case [4]. Our main result is the following.

Theorem 1 *Let $k \geq 3$ and $\epsilon > 0$. If $p\colon \mathbb{N} \to [0,1]$ satisfies $p \geq (1+\epsilon)\log(n)/n$, then an edge coloured random graph $G \in \mathcal{G}^k_{n,p}$ a.a.s. contains a properly coloured spanning tree.*

In this extended abstract, we restrict our discussion about the proof of Theorem 1 to the case $k = 3$. Intuitively, this is the hardest case, and the proof of the general case may be obtained with similar arguments (indeed, if there are $k = 3m$ colours, where m is a positive integer, then the result is derived directly from the case of 3-colourings by splitting the colours into three classes of size m, and by being colour-blind in each class).

Assume that the colours are red, blue and green. We view a coloured graph in this setting as the union of three random graphs using the well-known technique of two-round exposure (see [12]) to avoid problems with dependency. Our proof consists of three main steps. In the first step, we argue that a.a.s. the set of red and blue edges in $G \in \mathcal{G}^3_{n,p}$ produce a family of disjoint zebraic paths and cycles that span almost all vertices. To this end, we obtain the following lower bound on the size of a maximum matching in $\mathcal{G}_{n,q}$, where $q(n) = p(n)/3$.

Lemma 2 *If $q = (1+\epsilon)\log n/3n$ then $\mathcal{G}_{n,q}$ a.a.s. contains a matching of size*

$$\frac{1}{2}\left(n - n^{\frac{2-\epsilon}{3}}\right) + O(n^{\frac{1}{3}}).$$

Proof (Sketch). The proof of Lemma 2 uses arguments introduced in the classical proof of Erdős and Rényi [5], based on Tutte's Matching Theorem [14], for the fact that if a random graph a.a.s. has minimum degree one, then it a.a.s. contains a matching covering all but at most one vertex. Unfortunately, in the probability regimen of Lemma 2, there is a considerable number of isolated vertices and cherries (i.e. copies of $K_{1,2}$ whose leaves have degree 1 in the original graph), which clearly prevent vertices from being covered by a matching. Let H be the subgraph of G obtained by deleting isolated vertices and all degree 1 vertices that lie in cherries. With standard random graph arguments, we show that H contains at least $n - n^{\frac{2-\epsilon}{3}} + O(n^{\frac{1}{3}})$ vertices; moreover, it does not contain any isolated vertices or cherries because the following two facts hold a.a.s. in G: (i) no vertex is incident with three or more vertices of degree 1; (ii) no vertex of degree less than 4 is adjacent to two vertices of degree 1. To conclude the proof, we show that H contains a matching covering all but at most one vertex with arguments as in [5]. ∎

Henceforth, we say that a matching M in an n-vertex graph is *almost-spanning* if it covers at least $n - 2n^{\frac{2-\epsilon}{3}}$ vertices. In the second step of our proof, we randomly select almost-spanning monochromatic matchings M_1 and M_2 with colours red and blue, respectively, in the random coloured graph, and we show that they induce a reasonably small number of components. Moreover, we split the components of $M_1 \cup M_2$ into classes $\mathcal{L}$ and $\mathcal{S}$ of *large* and *small* components, respectively, satisfying $D \in \mathcal{L} \Leftrightarrow |D| \geq n^{\frac{\epsilon}{3}}$. Let $s = |\mathcal{S}|$ and $\ell = |\mathcal{L}|$.

Lemma 3 *Let M_1 and M_2 be random almost-spanning matchings in colours red and blue, respectively, in the random coloured graph $\mathcal{G}^3_{n,p}$. We split the components of $M_1 \cup M_2$ into classes $\mathcal{L}$ and $\mathcal{S}$ as above. The following statements hold with probability $1 - o(1)$:*

(a) $M_1 \cup M_2$ has $O(\log_2 n)$ cycles.

(b) $M_1 \cup M_2$ has $O(n^{\frac{1-\epsilon}{3}})$ small paths.

(c) $M_1 \cup M_2$ has $O(n^{\frac{2-\epsilon}{3}})$ components.

In particular, $s = O(n^{\frac{1-\epsilon}{3}})$ and $\ell = O(n^{\frac{2-\epsilon}{3}})$.

Proof (Sketch). One may prove this lemma by adapting arguments of Frieze and Luczak [11], who considered the distribution of cycles in two randomly chosen disjoint perfect matchings of K_n, to the current setting. Of course, here we also need to consider the occurrence of paths. ∎

To conclude the proof of Theorem 1, we show that we can choose green edges to merge red-blue zebraic paths into a properly coloured spanning tree. To this end, given matchings M_1 and M_2 as in Lemma 3, let $C_1, \dots, C_s$ be the components of $M_1 \cup M_2$ in $\mathcal{S}$ and let $D_1, \dots, D_\ell$ be the components in $\mathcal{L}$.

Lemma 4 *With probability $1-o(1)$, we find distinct vertices $u_i \in \cup_{D\in\mathcal{L}} V(D)$, $i \in \{1, \dots, s\}$, such that*

(a) there is a green edge between C_i and u_i;

(b) for every $D \in \mathcal{L}$, we have $|V(D) \cap \{u_1, \dots, u_s\}| < \frac{|D|}{2}$.

Proof (Sketch). The proof again uses standard techniques for random graphs. Recall that, to preserve independence, we expose the edges of the three colours in three rounds, and edges of later rounds must be deleted if they had already been chosen before. For part (a), we first show that every vertex in $\bigcup_{i=1}^{s} C_i$ is a.a.s. incident with at most one double edge in this process. We then fix a vertex v_i in each C_i and find the vertices u_i inductively by showing that, in the last round (corresponding to colour green), there were a.a.s. at least two exposed edges between v_i and $\cup_{D\in\mathcal{L}} V(D) - \{u_1, \dots, u_{i-1}\}$. Part (b) is a simple calculation. ∎

Lemma 4 allows us to connect all small components to large ones using green edges. Moreover, at least half the number of vertices in each large component are not incident with the edges used in this step. These vertices are called *unsaturated*. To conclude the proof, we show that its is possible to find a matching composed of green edges, all of whose endpoints are unsaturated, connecting the large components.

Lemma 5 *With probability $1-o(1)$, there exists a tree T on the vertex set $[\ell]$ satisfying the following property. For every edge $e = \{i, j\}$, there exist unsaturated vertices $u_i^e \in D_i$ and $u_j^e \in D_j$ such that $\{u_i^e, u_j^e\}$ is a green edge in the random coloured graph. Moreover, for any $i \in [\ell]$, we have $u_i^e \neq u_i^{e'}$ whenever e and e' are distinct edges of the tree.*

Proof (Sketch). Let $\tilde{D}_i$ be the set of unsaturated vertices in D_i, where $i \in [\ell]$. We use expansion properties of the random graph, as well as the number of edges between the sets $\tilde{D}_i$ and their complements to build the tree T inductively: starting from an arbitrary component in $\mathcal{L}$, say D_1, there is at least one edge between D_1 and $\bigcup_{i=2}^{\ell} D_i$ that is not a double edge and does not contain vertices that were saturated in previous steps. Let $\{u, v\}$ be such an edge and assume that $v \in D_j$. Then we add the edge $e_1 = \{1, j\}$ to T, and we set $u_1^{e_1} = u$ and $u_j^{e_1} = v$. The unsaturated vertices of D_1 and D_j are in the set of *discovered vertices*. We then apply the previous argument successively to obtain edges between the current set of discovered vertices and unsaturated vertices in its complement, until the set of discovered vertices has cardinality $\Omega(n)$. At this point, we prove that a.a.s. there is a green matching that merges the remaining large components into the required tree. ∎

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Generation of hypohamiltonian graphs

Jan Goedgebeur and *Carol T. Zamfirescu*

Department of Applied Mathematics, Computer Science and Statistics,
Ghent University, Belgium

Extended Abstract

A graph G is *hypohamiltonian* if G is non-hamiltonian and $G - v$ is hamiltonian for every $v \in V(G)$. The study of hypohamiltonicity goes back to a paper of Sousselier [10] from the early sixties. The survey of Holton and Sheehan [7] provides a good overview. For more recent results and new references not contained in the survey, we refer to [8].

In 1973, Chvátal showed [3] that if we choose n to be sufficiently large, then there exists a hypohamiltonian graph of order n. We now know that for every $n \geq 18$ there exists such a graph of order n, and that 18 is optimal, since Aldred, McKay, and Wormald showed that there is no hypohamiltonian graph on 17 vertices [1]. Their paper fully settled the question for which orders hypohamiltonian graphs exist and for which they do not exist. For more details, see [7].

They also provide a complete list of hypohamiltonian graphs with at most 17 vertices. There are seven such graphs: exactly one for each of the orders 10 (the Petersen graph), 13, and 15, four of order 16 (among them Sousselier's graph), and none of order 17. Aldred, McKay, and Wormald [1] showed that there exist at least thirteen hypohamiltonian graphs with 18 vertices, but the exact number was unknown.

In this work we will present a new algorithm to generate all non-isomorphic hypohamiltonian graphs of a given order. The algorithm is based on the work of Aldred, McKay, and Wormald from [1], but is extended with several additional bounding criteria which speed it up substantially. Furthermore, our algorithm also allows to generate planar hypohamiltonian graphs and hypohamiltonian graphs with a given lower bound on the girth efficiently.

Using our implementation of this algorithm we were able to generate complete lists of hypohamiltonian graphs of much larger orders than what was previously possible. Our program can be downloaded from [4].

Table 1 shows the counts of the complete lists hypohamiltonian graphs which were generated by the program. We generated all hypohamiltonian graphs up to 19 vertices and also went several steps further for hypohamiltonian graphs with a given lower bound on the girth. (Recall that previously the complete lists of hypohamiltonian graphs were only known up to 17 vertices.) The algorithm also allowed us to determine the smallest hypohamiltonian graph of girth 6. It is shown in Figure 1.

Table 2 gives an overview of the current bounds of the smallest hypohamiltonian graphs with given properties. The new bounds we obtained in this work are marked in bold.

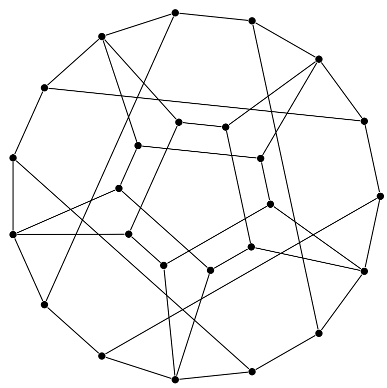


Figure 1: The smallest hypohamiltonian graph of girth 6. It has 25 vertices.

Order	# hypoham.	$g \geq 4$	$g \geq 5$	$g \geq 6$	$g \geq 7$	$g \geq 8$
$0-9$	0	0	0	0	0	0
10	1	1	1	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	1	1	1	0	0	0
14	0	0	0	0	0	0
15	1	1	1	0	0	0
16	4	4	4	0	0	0
17	0	0	0	0	0	0
18	14	13	8	0	0	0
19	34	34	34	0	0	0
20	?	?	4	0	0	0
21	?	?	85	0	0	0
22	?	?	420	0	0	0
23	?	?	?	0	0	0
24	?	?	?	0	0	0
25	?	?	?	1	0	0
26	?	?	?	0	0	0
27	?	?	?	?	0	0
28	?	?	?	≥ 2	1	0
29	?	?	?	?	0	0
30	?	?	?	?	0	0
$31-35$	?	?	?	?	?	0

Table 1: The number of hypohamiltonian graphs. The columns with a header of the form $g \geq k$ contain the number of hypohamiltonian graphs with girth at least k. The counts of cases indicated with a '$\geq$' are possibly incomplete; all other cases are complete.

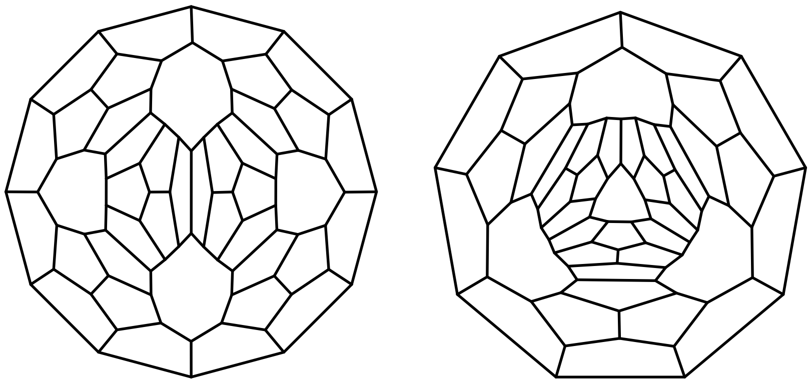
girth	3	4	5	6	7	8	9
general	18	18	10	**25** 18..28	**28** 18..28	**36**..∞ 18..∞	**61**..∞ 18..∞
cubic	–	24	10	28	28	**50**..∞ 30..∞	**66**..∞ 58..∞
planar	**23**..240 18..240	**25**..40 18..40	45	–	–	–	–
planar and cubic	–	**54**..70 44..70	76	–	–	–	–

Table 2: Bounds for the order of the smallest hypohamiltonian graph with additional properties. The bold numbers are new bounds obtained in this work; if an entry contains two lines, the upper line indicates the new bounds, while the lower line shows the previous bounds. The symbol "–" designates an impossible combination of properties and $a..b$ means that the number is at least a and at most b. $b = \infty$ signifies that no graph with the given properties is known.

McKay [9] recently showed that there exist no cubic planar hypohamiltonian graphs of girth 5 with less than 76 vertices, and exactly three such graphs of order 76. All three graphs have trivial automorphism group. In that paper the natural question is raised whether infinitely many such graphs exist. Using the program *plantri* [2] we generated all cubic planar cyclically 4-connected graphs with girth 5 with 78 vertices and tested them for hypohamiltonicity. This yielded exactly one such graph. It is shown in Figure 2 and it has D_{3h} symmetry, so it is the smallest cubic planar hypohamiltonian graph of girth 5 with a non-trivial automorphism group.

By using new theoretical and computational results we were able to answer McKay's question affirmatively:

Theorem 1 *There are infinitely many planar cubic hypohamiltonian graphs of girth* 5*. More specifically, there exist such graphs of order* n *for every* $n = 74k + a$*, where* $k \geq 2$ *and* $a \in \{2, 4\}$*.*



(a) (b)

Figure 2: The smallest cubic planar hypohamiltonian graph of girth 5 with a non-trivial automorphism group. It has 78 vertices and D_{3h} symmetry. Both Figure 2a and Figure 2b show different symmetries of the same graph.

More information can be found on the arXiv in [5] and [6].

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Obstructions to bounded cutwidth*

Archontia C. Giannopoulou [1], *Michał Pilipczuk* [2], *Jean-Florent Raymond* [2,3], *Dimitrios M. Thilikos* [3,4], and *Marcin Wrochna* [2]

[1] Technische Universität Berlin, Berlin, Germany
[2] Institute of Informatics, University of Warsaw, Poland
[3] AlGCo project team, CNRS, LIRMM, Montpellier, France
[4] Department of Mathematics, National and Kapodistrian University of Athens, Greece

Extended Abstract

1 Introduction

A few words on cutwidth. The cutwidth of a graph is defined as the minimum possible *width* of a linear ordering of its vertices, where the width of an ordering σ is the maximum, among all the prefixes of σ, of the number of edges that have exactly one vertex in a prefix. Due to its natural definition, cutwidth has various applications in a range of practical fields of computer science: whenever data is expected to be roughly linearly ordered and dependencies or connections are local, one can expect the cutwidth of the corresponding graph to be small. These applications include circuit design, graph drawing, bioinformatics, and text information retrieval; we refer to the survey of layout parameters of Díaz, Petit, and Serna [DPS02] for a broader discussion.

As finding a layout of optimum width is NP-hard [GJ79], the algorithmic and combinatorial aspects of cutwidth were intensively studied. There is a broad range of polynomial-time algorithms for special graph classes [HLMP11, HvtHLN12, Yan85], approximation algorithms [LR99], and fixed-parameter algorithms [TSB05a, TSB05b]. In particular, Thilikos, Bodlaender, and Serna [TSB05a, TSB05b] proposed a fixed-parameter algorithm for computing the cutwidth of a graph that runs† in time $2^{O(k^2)} \cdot n$, where k is the optimum width and n is the number of vertices. Their approach is to first compute the pathwidth of the input graph, which is never larger than the cutwidth. Then, the optimum layout can be constructed by an elaborate dynamic programming procedure on the obtained path decomposition. To upper bound the number of relevant states, the authors had to understand how an optimum layout can look in a given path decomposition. For this, they borrow the technique of *typical sequences* of Bodlaender and Kloks [BK96], which was introduced for a similar reason, but for pathwidth and treewidth instead of cutwidth.

Obstructions. Since the class of graphs of cutwidth at most k is closed under immersions, and the immersion order is a well-quasi ordering of graphs‡ [RS10], it follows that for each k there exists a *finite* obstruction set $\mathcal{L}_k$ of immersion-minimal graphs such that a graph has cutwidth at most k if and only if it does not admit any graph from $\mathcal{L}_k$ as an immersion. However, this existential result does not give any hint on how to generate, or at least estimate the sizes of the obstructions. The sizes of obstructions are important for efficient treatment of

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†Thilikos, Bodlaender, and Serna [TSB05a, TSB05b] do not specify the parametric dependence of the running time of their algorithm. A careful analysis of their algorithm yields the above claimed running time bound.

‡All graphs considered here may have parallel edges, but no loops.

graphs of small cutwidth; this applies also in practice, as indicated by Booth et al. [BGLR92] in the context of VLSI design.

Bounds on the size of obstructions. The estimation of sizes of minimal obstructions for graph parameters like pathwidth, treewidth, or cutwidth, has been studied before. For minor-closed parameters pathwidth and treewidth, Lagergren [Lag98] showed that any minimal minor obstruction to admitting a path decomposition of width k has size at most single-exponential in $O(k^4)$, whereas for tree decompositions he showed an upper bound double-exponential in $O(k^5)$. Less is known about immersion-closed parameters, like cutwidth. Govindan and Ramachandramurthi [GR01] showed that the number of minimal immersion obstructions for the class of graphs of cutwidth at most k is at least $3^{k-7}+1$, and their construction actually exemplify minimal obstructions for cutwidth at most k with $(3^{k-5}-1)/2$ vertices. To the best of our knowledge, nothing was known about upper bounds for the cutwidth case.

2 Obstructions to bounded cutwidth

Our main result concerns the sizes of obstructions for cutwidth.

Theorem 1 *Suppose a graph G has cutwidth larger than k, but every graph with fewer vertices or edges (strongly) immersed in G has cutwidth at most k. Then G has at most $2^{O(k^3 \log k)}$ vertices and edges.*

The above result immediately gives the same upper bound on the sizes of graphs from the minimal obstruction sets $\mathcal{L}_k$ as they satisfy the prerequisites of Theorem 1. This somewhat matches the $(3^{k-5}-1)/2$ lower bound of Govindan and Ramachandramurthi [GR01].

Overview of the proof. Our approach for Theorem 1 follows the technique used by Lagergren [Lag98] to prove that minimal minor obstructions for pathwidth at most k have sizes single-exponential in $O(k^4)$. Intuitively, the idea of Lagergren is to take an optimum decomposition for a minimal obstruction, which must have width $k+1$, and to assign to each prefix of the decomposition one of finitely many "types", so that two prefixes with the same type "behave" in the same manner. If there were two prefixes, one being shorter than the other, with the same type, then one could replace one with the other, thus obtaining a smaller obstruction. Hence, the upper bound on the number of types, being double-exponential in $O(k^4)$, gives some upper bound on the size of a minimal obstruction. This upper bound can be further improved to single-exponential by observing that types are ordered by a natural domination relation, and the shorter a prefix is, the weaker is its type. An important detail is that one needs to make sure that the replacement can be modeled by minor operations. For this, Lagergren uses the notion of *linked path decompositions*, also known as *lean path decompositions*; cf. [Tho90].

To prove Theorem 1, we perform a similar analysis of prefixes of an optimum ordering of a minimal obstruction. We show that prefixes can be categorized into a bounded number of types, each comprising prefixes that have the same "behavior". Provided two prefixes with equally strong type appear one after the other, we can "unpump" the part of the graph in their difference. To make sure that unpumping is modeled by taking an immersion, we introduce *lean orderings* for cutwidth and prove the analogue of the result of Thomas [Tho90] for treewidth: there is always an optimum-width ordering that is lean (see also [BD02]).

The proof of the upper bound on the number of types essentially boils down to the following setting. We are given a graph G and a subset X of vertices, such that at most ℓ edges have exactly one endpoint in X. The question is how X can look like in an optimum-width ordering of G. We prove that there is always an ordering where X is split into at most $O(k\ell)$ blocks, where k is the optimum width. This allows us to store the relevant information on the whole

X in one of a constant number of types (called *bucket interfaces*). The swapping argument used in this proof holds the essence of the typical sequences technique of Bodlaender and Kloks [BK96], while being, in our opinion, more natural and easier to understand.

3 Obstructions to edge-removal distance to cutwidth

As an interesting byproduct, we can also use our understanding to treat the problem of removing edges to get a graph of small cutwidth. More precisely, for parameters w, k, we say that a graph G is *at distance w from cutwidth k* if k edges can be removed from G to obtain a graph of cutwidth at most k.

We prove that for every constant k, the minimal strong immersion obstructions for this class have sizes bounded *linearly* in w.

Theorem 2 *Let k be a fixed integer. Suppose a graph G is at distance larger than w from cutwidth k, but every graph with fewer vertices or edges strongly immersed in G has distance w from cutwidth k. Then G has at most $O(w)$ vertices.*

Moreover we give an exponential lower bound to the number of these obstructions.

Theorem 3 *Let k and w be positive integers, and let us consider the set $\mathcal{C}$ of graphs that are at distance larger than w from cutwidth k, and such that every graph with fewer vertices or edges immersed in G has distance w from cutwidth k. Then $\mathcal{C}$ contains at least $\binom{3^{k-7}+w+1}{w+1}$ non-isomorphic graphs.*

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Edge-Intersection Graphs of Boundary-Generated Paths in a Grid

Martin Charles Golumbic [1], *Gila Morgenstern* [2] and *Deepak Rajendraprasad* [3]

[1] Caesarea Rothschild Institute, University of Haifa, Mount Carmel, Haifa 31905, Israel
[2] Holon Institute of Technology, Israel
[3] Indian Institute of Technology Palakkad, India.

Extended Abstract

Edge-intersection graphs of paths on a grid (or EPG graphs) are graphs whose vertices can be represented as (not necessarily distinct) simple paths on a rectangular grid such that two vertices are adjacent in the graph if and only if the corresponding paths share at least one edge of the grid. For two boundary vertices u and v on two adjacent boundaries of a rectangular grid $\mathcal{G}$, we call the unique single-bend path connecting u and v in $\mathcal{G}$ using no other boundary vertex of $\mathcal{G}$ as the path *generated* by (u, v). A path in $\mathcal{G}$ is called *boundary-generated*, if it is generated by some pair of vertices on two adjacent boundaries of $\mathcal{G}$. In this article, we study the edge-intersection graphs of boundary-generated paths on a grid or ∂EPG graphs. The motivation for studying these graphs comes from some problems in the context of circuit layout.

We show that ∂EPG graphs can be covered by two collections of vertex-disjoint cobipartite chain graphs. This leads us to a linear-time testable characterization of ∂EPG trees and also a tight upper bound on the equivalence covering number of general ∂EPG graphs. We also study the cases of two-sided ∂EPG and three-sided ∂EPG graphs, which are respectively, the subclasses of ∂EPG graphs obtained when all the boundary vertex pairs which generate the paths are restricted to lie on at most two or three boundaries of the grid. For the former case, we give a complete characterization.

We do not know yet whether one can efficiently recognize ∂EPG graphs. Though the problem is linear-time solvable on trees, we suspect that it might be NP-hard in general.

Background

EPG graphs were first introduced by Golumbic, Lipshteyn and Stern in [12] where the authors, having shown that every graph is an EPG graph, proposed limiting the number of bends (90° turns) that are permitted for each path. This restriction on EPG graphs was motivated by real-life constraints in circuit layout problems. As such, much of the current research today focuses on subclasses of EPG graphs constrained in this way, seeking both structural and algorithmic results.

The graph class B_k-EPG is defined as those graphs having an EPG representation where all paths are allowed at most k bends, i.e., direction changes. The case of $k = 1$ are the single bend EPG graphs, studied further in [3, 6, 8, 12, 13]. The B_0-EPG are easily seen to be equivalent to the well known family of interval graphs (see [10]). In [4], the authors show that for any k, only a small fraction of all labeled graphs on n vertices are B_k-EPG. Improving a result of [5], it was shown in [15] that every planar graph is a B_4-EPG graph. It is still open whether $k = 4$ is best possible. So far it is only known that there are planar graphs that are B_3-EPG graphs and not B_2-EPG graphs. It was also shown in [15] that all outerplanar graphs are B_2-EPG graphs, thus proving a conjecture of [5]. Recently, [1] have shown that circular-arc graphs are B_3-EPG graphs, and that this is best possible.

For the case of B_1-EPG graphs, Golumbic, Lipshteyn and Stern [12] showed that every tree is a B_1-EPG graph, and in [13] they showed that single bend paths on a grid have strong Helly number 4. Asinowski and Ries [3] proved that every B_1-EPG graph on n vertices contains either a clique or a stable set of size at least $n^{1/3}$. In [3], the authors also give a characterization of the B_1-EPG graphs among some subclasses of chordal graphs, namely, chordal bull-free graphs, chordal claw-free graphs, chordal diamond-free graphs, and special cases of split graphs. In [8], a characterization of the sub-family of cographs that are B_1-EPG graphs is given by a complete family of minimal forbidden induced subgraphs.

No characterization is known for B_k-EPG graphs (for any $k \geq 1$) and the recognition problems are NP-complete [14], remaining so even if just one of the four single bend shapes is allowed, the so called L-*shaped* B_1-EPG graphs [6]. Golumbic, Epstein and Morgenstern [9] proved that the minimum coloring problem and maximum independent set problem for B_1-EPG graphs are NP-complete, and then presented polynomial time approximation algorithms for them, and later for the minimum dominating set problem. The same questions may also be interesting for other subclasses B_1-EPG graphs.

In this paper, we consider a further restriction coming from applications in circuit design, where the endpoints of each path are on the boundary of a rectangular grid. This notion was first proposed for investigation in [11].

Preliminaries

All graphs considered are finite and undirected. The complement of a graph G is denoted by $\overline{G}$. Two non-adjacent vertices with the same neighborhood are called *false twins*. The *reduced graph* of a graph G is the graph obtained from G by deleting one vertex from each false twin as long as there is one.

An *equivalence graph* is a vertex disjoint union of cliques, or equivalently, the graph where the adjacency relation is an equivalence relation. The *equivalence covering number* $\mathrm{eq}(G)$ of a graph G is the minimum number of equivalence graphs whose union is G [2]. For triangle-free graphs, equivalence covering number is the same as edge-chromatic number.

The product dimension or Prague dimension of a graph is a parameter which is closely related to the equivalence covering number. A *product k-encoding* of a graph G is obtained by an associating to each vertex v a unique vector $f(v) = (v_1, \ldots, v_k)$ over the natural numbers so that for $xy \in E(G)$ the vectors $f(x)$ and $f(y)$ differ in all coordinates and for $xy \notin E(G)$ the vectors $f(x)$ and $f(y)$ agree in at least one coordinate. The *product dimension* or *Prague dimension* of a graph G, $\mathrm{pdim}(G)$, is the smallest number k such that G has a product k-encoding.

It is easy to see that

$$\mathrm{eq}(G) \leq \mathrm{pdim}(\overline{G}) \leq \mathrm{eq}(G) + 1.$$

The difference of 1 occurs because a product k-encoding needs to associate a *unique* vector to each vertex. For instance, the product dimension of the empty graph G on two vertices is 2 where as $\overline{G}$ can be covered by one clique.

The concept of product dimension of a graph was first used by Nešetřil and Rödl to prove the Galvin-Ramsey property of the class of all finite graphs [19]. Lovász, Nešetřil and Pultr showed that the product dimension of a path on $n+1$ vertices (length n) is $\lceil \lg n \rceil$ [17]. We use the same technique to show the existence of ∂EPG graphs with arbitrarily large equivalence covering number.

A bipartite graph is called a *chain graph* if, for each color class, the neighborhoods of the vertices in that color class can be linearly ordered with respect to inclusion. Equivalently it is a bipartite graph which is $2K_2$-free. A *cobipartite chain graph* is the complement of a bipartite chain graph. Note that the linear orderings of neighborhoods of vertices of each part is still preserved (there is a reversing).

Our results

Theorem 1 *If G is a ∂EPG graph then G can be covered by two graphs G_h and G_v, where both G_h and G_v are vertex-disjoint unions of cobipartite chain graphs.*

Moreover, if G is three-sided ∂EPG, then G_v can be further restricted to be an equivalence graph and if G is two-sided ∂EPG, then both G_h and G_v can be restricted to be equivalence graphs.

We explore several consequences of this theorem. Firstly, we obtain the following complete characterization for 2-sided ∂EPG graphs. Without loss of generality, we may assume that all the paths are restricted to start from the top boundary and bend towards the right boundary. Then one obtains a subclass of the L-*shaped* B_1-EPG graphs. We show that this subclass, the two-sided ∂EPG graphs, are precisely linegraphs of bipartite multigraphs. It is known that these graphs are recognizable in polynomial time and have a forbidden subgraph characterization [18].

Theorem 2 *The following conditions are equivalent for a graph G:*

(i) G is a two-sided ∂EPG graph,

(ii) G has equivalence covering number at most 2*,*

(iii) The reduced graph of $\overline{G}$ has Product dimension at most 2*.*

(iv) G is the linegraph of a bipartite multigraph

(v) G contains no claw, no gem, no 4-wheel, and no odd hole.

(vi) The clique graph of G, i.e.. the intersection graph of maximal cliques in G, is bipartite.

Once we discovered that two-sided ∂EPG graphs have equivalence covering number at most 2, it was natural to investigate the equivalence covering number of three-sided and general (i.e., four-sided) ∂EPG graphs. We establish that, unlike the case with two-sided ∂EPG graphs, the equivalence covering number of three-sided and four-sided ∂EPG graphs can be unbounded. In particular, we prove that the maximum possible equivalence covering number of an n-vertex ∂EPG graph is $\Theta(\lg n)$.

Theorem 3 *If G is a ∂EPG graph on n vertices, then* $\mathrm{eq}(G) \leq 2\lceil \lg n \rceil + 2$*. Further, if G has a* 3*-sided ∂EPG representation then* $\mathrm{eq}(G) \leq \lceil \lg n \rceil + 2$*.*

Moreover there exists n-vertex three-sided ∂EPG graphs G_n with equivalence covering number at least $\lceil \lg n \rceil - 1$*.*

The lower bound in the theorem above uses a result of Lovász, Nešetřil and Pultr [17] on product dimension. In order to establish the upper bound in the above result, apart from Theorem 1, we also needed to prove the following tight upper bound on the product dimension of bipartite chain graphs. This result may be of independent interest.

Theorem 4 *If H is a bipartite chain graph on n vertices, then the product dimension of H is at most* $\lceil \lg n \rceil + 1$*. Hence, a co-bipartite chain graph on n vertices can be covered by at most* $\lceil \lg n \rceil + 1$ *clique partitions.*

Notice that two-sided ∂EPG trees, being claw-free are simply paths. For three-sided and four-sided ∂EPG trees, we give a characterization which is similar in spirit to linear arboricity. A *linear forest* is a forest in which every connected component is a path. A *k-linear forest* is a linear forest in which every path has length at most k. The *linear k-arboricity* of a graph G is the minimum number of linear k-forests whose union is G.

Theorem 5 *A tree T is a ∂EPG graph if and only if T can be covered by two linear* 3*-forests. Moreover, T has a* 3*-sided ∂EPG representation if and only if we can restrict one of the above forests to be a disjoint collection of edges.*

One side of the above theorem is an easy consequence of Theorem 1. For the other direction we had to work more delicately with some new partial-order extension questions.

Theorem 5 tells us that the problem of recognizing ∂EPG trees is the same as deciding whether the linear 3-arboricity of a tree is at most 2. In [7], Chang et al., gives a linear time algorithm to find the linear k-arboricity of a tree. We can use the same algorithm to recognize ∂EPG trees. The case with three-sided ∂EPG trees gets a bit more dirty because of the asymmetry in the cover. Still, extending the idea in [7], we give a linear-time algorithm for recognizing three-sided ∂EPG trees.

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Connectivity and Hamiltonicity of canonical colouring graphs of bipartite and complete multipartite graphs

Ruth Haas [1], and *Gary MacGillivray* [2]

[1] Smith College, USA
[2] University of Victoria, Canada

Extended Abstract

A *k-colouring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. Thus, k-colourings f_1 and f_2 are different whenever there exists a vertex x such that $f_1(x) \neq f_2(x)$. Every k-colouring, f, is equivalent to a k-tuple, $(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k))$, in which the set of non-empty components is a partition of $V(G)$ into independent sets.

A k-colouring $f : V(G) \to \{1, 2, \ldots, k\}$ is *canonical* with respect to a vertex ordering $\pi = v_1, v_2, \ldots, v_n$ if, whenever $f(v_i) = c$, every colour less than c has been assigned to some vertex that precedes v_i in π. Thus v_1 is necessarily assigned colour 1, and colour 3 can only be assigned to some vertex after colour 2 has been assigned to a vertex that appears earlier in the sequence π. Note that canonical colourings may be very different than the colourings arising from applying the usual greedy colouring algorithm to G using the vertex ordering π.

Define an equivalence relation $\sim$ on the set of k-colourings of G by $f_1 \sim f_2$ if and only if f_1 and f_2 determine the same partition of $V(G)$ into independent sets. The set of canonical k-colourings of G with respect to π is the set of representatives of the equivalence classes of $\sim$ that are lexicographically least with respect to π.

The *canonical k-colouring graph of G* with respect to the vertex ordering π is the graph $\mathrm{Can}_k^{\pi}(G)$ with vertex set equal to the set of canonical k-colourings of G with respect to π, where two of these are adjacent if and only if they differ in the colour assigned to exactly one vertex. While every vertex ordering gives a set of representatives of the equivalence classes of k-colourings, different orderings π can lead to different canonical k-colourings graphs. When a canonical colour graph is connected, any given canonical k-colouring can be reconfigured into any other via a sequence of recolourings which each change the colour of exactly one vertex. When it is Hamiltonian, there is a cyclic list that contains all of the canonical k-colourings of G and in which consecutive elements of the list differ in the colour of exactly one vertex, that is, there is a cyclic Gray code of the canonical k-colourings of G.

The canonical k-colouring graph is an example of a *reconfiguration graph*: the vertices of any such graph are the allowed configurations and there is an edge between two of them if one can be transformed to the other by some reconfiguration rule. There is a vast literature on the complexity of reconfiguration problems, for example see [7, 8].

We now briefly mention some other colouring graphs. The *k-colouring graph of G*, denoted $\mathcal{C}_k(G)$, has vertex set equal to the set of k-colourings of G, with two k-colourings being adjacent if and only if they differ in the colour of exactly one vertex. There is a least integer $c_0 \leq col(G)+1$, such that k-colouring graph of G is connected for all $k \geq c_0$ [4]. The quantity $col(G) = 1 + \max_{H \subseteq G} \delta(H)$ is the *colouring number of G*. Results concerning connectivity of k-colouring graphs are surveyed in [7]. Hamiltonicity of the k-colouring graph was first considered in [3], wherein it was proved that there is always a least integer $k_0 \leq col(G) + 2$ such that the k-colouring graph of the graph G is Hamiltonian for all $k \geq k_0$. The number k_0 is known for a variety of graph families, see [1] for a survey. The *Bell k-colouring graph of G*, denoted $\mathcal{B}_k(G)$, has as vertices the partitions of $V(G)$ into at most k independent sets, with two of these being adjacent when there is a vertex x such that these partitions are equal when restricted to $G - x$. The Bell k-colouring graph is studied in [5], as is the *Stirling ℓ-colouring graph of G*, the subgraph of $\mathcal{B}_{|V|}(G)$ induced by the partitions with exactly ℓ cells. It is proved that $\mathcal{B}_{|V|}(G)$ is Hamiltonian for every graph G except K_n and $K_n - e$, and the quantity $|V|$ is best possible. It is also proved that the Bell k-colouring graph of a tree with at least four vertices is Hamiltonian for all $k \geq 3$, and the Stirling ℓ-colouring graph of a tree on at least

$n \geq 1$ vertices is Hamiltonian for all $\ell \geq 4$. Other colouring graphs that have been considered include the graph of $L(2,1)$-labellings [9], and colour graphs where the reconfiguration rule is a Kempe-chain exchange, see [7].

The graph $\mathrm{Can}_k^\pi(G)$ is a spanning subgraph of $\mathcal{B}_k(G)$; the restriction to canonical colourings eliminates some edges of $\mathcal{B}_k(G)$. Since at most n colours can be assigned to the vertices of an n-vertex graph G, it follows that $\mathcal{B}_k(G) = \mathcal{B}_n(G)$ and $\mathrm{Can}_k^\pi(G) = \mathrm{Can}_n^\pi(G)$ for all $k \geq n$. Notice that if $\mathrm{Can}_k^\pi(G)$ is connected for some vertex ordering π, then $\mathcal{B}_k(G)$ is connected.

Canonical k-colouring graphs were first considered in [6]. For every tree T there exists an ordering π of the vertices such that the canonical k-colouring graph of T with respect to π is Hamiltonian for all $k \geq 3$. The canonical 3-colouring graph of the cycle C_n is disconnected for all vertex orderings π, while for each $k \geq 4$ there exists an ordering π for which $\mathrm{Can}_k^\pi(C_n)$ is connected. If G is connected, but not complete then there is always a vertex ordering π such that $\mathrm{Can}_k^\pi(G)$ is disconnected for all $k \geq \chi(G)+1$ [6]. In particular, the graph $\mathrm{Can}_k^\pi(G)$ is disconnected whenever the first three vertices u, v, w of the vertex ordering π are such that $uw, vw \in E$ but $uv \notin E$. It is an open problem to find general conditions on k and π such that $\mathrm{Can}_k^\pi(G)$ is connected.

Our focus in this work is connectivity and Hamiltonicity of canonical colouring graphs of bipartite graphs, and complete multipartite graphs. In both cases, we find the best possible lower bound on the number, k, of colours needed such that there exists a vertex ordering for which the canonical k-colouring graph is connected. We then show that a canonical k-colouring graph of a complete multipartite graph rarely has a Hamilton cycle, and there is a Hamilton path only when the graph is complete or complete bipartite.

Proposition 1 *Let H be a complete multipartite graph with p parts. For any $k \geq p$, there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(H)$ is connected.*

Theorem 2 *Let G be a bipartite graph on n vertices, then there exists an ordering π of the vertices such that $\mathrm{Can}_t^\pi(G)$ is connected for $t \geq n/2+1$.*

Consider the graph $L_n = K_{n,n} - F$, where F is a perfect matching. In the n-colouring of L_n where the opposite ends of edges in F are assigned the same colour, every vertex has a neighbour of any different colour. Thus, if c is the canonical version of this colouring with respect to a vertex ordering π, then c is an isolated vertex in $\mathrm{Can}_n^\pi(L_n)$. Since L_n has $2n$ vertices, it follows that the lower bound in the above theorem is best-possible.

We now consider Hamilton paths and cycles in canonical k-colouring graphs of complete multipartite graphs. A canonical k colouring graph of a complete graph is either empty, or a single vertex. We show that it suffices to consider the case where each independent set has at least two vertices. Recall that the join of the disjoint graphs G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by inserting all possible edges with one end in $V(G_1)$ and the other in $V(G_2)$.

Proposition 3 *If $\mathrm{Can}_t^\pi(G)$ is connected (resp. has a Hamilton path, has a Hamilton cycle) then there exists an order π' such that $\mathrm{Can}_{t+r}^{\pi'}(G \vee K_r)$ is connected (resp. has a Hamilton path, has a Hamilton cycle).*

Proposition 4 *Let $G = K_{n_1,n_2,\ldots,n_r}$, where $n_i \geq 2$, for all i. Then, for all vertex orderings π and $k \geq r+1$,*

1. *$\mathrm{Can}_k^\pi(G)$ has a cut vertex and hence has no Hamilton cycle;*
2. *if $r \geq 3$ then $\mathrm{Can}_k^\pi(G)$ has no Hamilton path.*

By Proposition 4, for $m, n \geq 2$ and $k \geq 3$, the graph $\mathrm{Can}_k^\pi(K_{m,n})$ has a cut vertex, and hence no Hamilton cycle. On the other hand, for $n \geq 2$, the graph $\mathrm{Can}_k^\pi(K_{1,n})$ has a

Hamilton cycle for all $k \geq 3$ [6]. The possibility remains that the canonical k-colouring graphs of complete bipartite graphs which are not stars have a Hamilton path.

Observe that the canonical k-colouring graph of $\overline{K}_n = K_1 \cup K_1 \cup \cdots \cup K_1$ is a reconfiguration graph of partitions of an n-set into at most k parts. The number of vertices is the sum of Stirling numbers of the second kind, $S(n,1) + S(n,2) + \cdots + S(n,k)$. A Hamilton cycle in this graph corresponds a cyclic Gray code for set partitions. Many different Gray codes, cyclic and otherwise, for set partitions are known to exist [10]. The properties of the Hamilton paths in the next theorem are similar, but not identical, to those studied by various authors in the context of Gray codes for set partitions [10].

Theorem 5 *For all $n \geq 2$ and $k \geq 2$, and any vertex ordering π, the graph $\mathrm{Can}_k^\pi(\overline{K}_n)$ has a Hamilton path $x_1, x_2, \ldots, x_t$ such that:*

1. *the colouring $x_1 = 11\ldots1$, and the colouring x_t uses all k colours.*
2. *For each $1 < i < t$, the set of colours used by x_i is the same as to the set used by x_{i-1}, or x_{i+1}.*

Using these Hamilton paths, we show that $\mathrm{Can}_k^\pi(K_{n,m})$ has a Hamilton path for all admissible values of m, n, k.

Theorem 6 $\mathrm{Can}_k^\pi(K_{n,m})$ *has a Hamilton path for $n, m \geq 2$, $k \geq 3$.*

We conclude by considering the case where each independent set has size two. For $n \geq 1$, let $T_{2n,n}$ be the complete n-partite graph on $2n$ vertices in which each independent set has size two. Then $T_{2,1} \cong \overline{K}_2$, $T_{4,2} \cong K_{2,2} \cong C_4$, $T_{6,3} \cong K_{2,2,2}$, and so on. We show that a canonical k-colouring graph of $T_{2n,n}$ is either disconnected, or isomorphic to a particular tree.

Theorem 7 *Let $n \geq 1$. Then*

1. $\mathrm{Can}_n^\pi(T_{2n,n}) \cong K_1$ *for any vertex ordering π.*
2. *If $k \geq 2n$, then $\mathrm{Can}_k^\pi(T_{2n,n}) \cong \mathrm{Can}_{2n}^\pi(T_{2n,n})$ for any vertex ordering π.*
3. *If $n < k$ and the subgraph of $T_{2n,n}$ induced by the first n vertices in the vertex ordering π is not complete, then $\mathrm{Can}_k^\pi(T_{2n,n})$ is disconnected.*
4. *If $n < k$ and the subgraph of $T_{2n,n}$ induced by the first n vertices in the vertex ordering π is complete, then $\mathrm{Can}_k^\pi(T_{2n,n})$ is a tree. Further, if $\mathrm{Can}_k^\pi(T_{2n,n})$ and $\mathrm{Can}_k^\phi(T_{2n,n})$ are both trees, then $\mathrm{Can}_k^\pi(T_{2n,n}) \cong \mathrm{Can}_k^\phi(T_{2n,n})$.*

We now describe the tree which arises in point 4 of the previous theorem. In any colouring of $T_{2n,n}$, a pair of independent vertices either has the same colour, or different colours. In the latter case, each vertex in the pair is the only vertex to be assigned that colour. Suppose that the last n vertices of π are $x_1, x_2, \ldots, x_n$. A canonical $2n$-colouring with respect to π can be encoded as a binary sequence $b_1 b_2 \ldots b_n$ of length n in which the i-th element is 0 if vertex x_i is assigned the same colour as its unique non-neighbour (which is one of the first n vertices of π), and 1 if it is assigned the first colour not used on a vertex earlier in the sequence. Because of canonicity, an element b_i of the binary sequence can change (from 0 to 1, or 1 to 0) if and only if $b_j = 0$ for all $j > i$. It follows that the vertices of $\mathrm{Can}_k^\pi(T_{2n,n})$ are the binary sequences of length n with at most $t = k - n$ ones. Two such sequences are adjacent if and only if they differ in exactly one position, and all entries to the right of that position are zero. Observe that $\mathrm{Can}_k^\pi(T_{2n,n})$ is the subgraph of $\mathrm{Can}_{2n}^\pi(T_{2n,n})$ induced by the sequences with at most t ones.

For an ordering π such that the subgraph induced by the first n vertices is complete, the tree $\mathrm{Can}_6^\pi(T_{6,3})$ is shown in Figure 1. For any such ordering, the tree $\mathrm{Can}_8^\pi(T_{8,4})$ is constructed from two copies of this tree, one arising from concatenating a 1 on the left of each sequence and the other arising from concatenating a 0 on the left of each sequence, and then joining the vertices 0000 and 1000. The tree $\mathrm{Can}_{10}^\pi(T_{10,5})$ is constructed similarly, and so on.

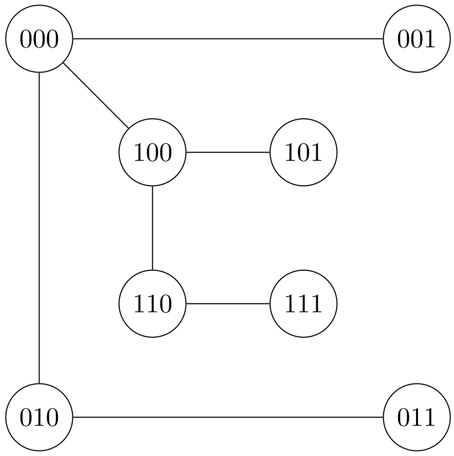

Figure 1: The tree $\mathrm{Can}_6^{\pi}(T_{6,3})$

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Bounds for the sum choice number

Jochen Harant [2], *Arnfried Kemnitz* [1], *Massimiliano Marangio* [1], and *Margit Voigt* [3]

[1] Computational Mathematics, Technical University Braunschweig, Germany
[2] Department of Mathematics, Ilmenau University of Technology, Germany
[3] Faculty of Information Technology and Mathematics, University of Applied Sciences, Dresden, Germany

Extended Abstract

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and for every vertex $v \in V(G)$ let $L(v)$ be a set (list) of available colors. The graph G is called *L-colorable* if there is a proper coloring ϕ of the vertices with $\phi(v) \in L(v)$ for all $v \in V(G)$. A function f from the vertex set $V(G)$ of G to the positive integers is called a *choice function* of G and G is said to be *f-list colorable* if G is L-colorable for every list assignment L with $|L(v)| = f(v)$ for all $v \in V(G)$. Set size$(f) = \sum_{v \in V(G)} f(v)$ and define the *sum choice number* $\chi_{sc}(G)$ as minimum of size(f) over all choice functions f of G.

For some classes of graphs the sum choice number has been determined, for example for cycles, trees, complete graphs, and all graphs with at most 6 vertices. If T is a tree then $\chi_{sc}(T) = 2|V(T)| - 1$ [1].

It is easy to see that $\chi_{sc}(G) \leq |V(G)| + |E(G)|$ for every graph G and that there is a greedy coloring of the vertices of G for the corresponding choice function f and every list assignment L with $|L(v)| = f(v)$ for all $v \in V(G)$ (see, e.g., [1]).

Obviously, if $\chi_{sc}(G) \leq k$ and H is a subgraph of G, then $\chi_{sc}(H) \leq k$. Therefore, this property is a hereditary graph property. This implies $\chi_{sc}(G) \geq 2|V(G)| - 1$ for a connected graph G since $\chi_{sc}(T) = 2|V(G)| - 1$ for a spanning tree T of G.

In this talk we will improve the above mentioned upper and lower bounds for the sum choice number. We will present several general lower and upper bounds on $\chi_{sc}(G)$ in terms of subgraphs of G, for example

Theorem 1 *Let $V(G) = V_1 \cup \cdots \cup V_p$ be a partition of $V(G)$. Then*

$$\chi_{sc}(G) \leq \sum_{i=1}^{p} \chi_{sc}(G[V_i]) + |E(G)| - \sum_{i=1}^{p} |E(G[V_i])|.$$

Theorem 2 (Harant, Kemnitz '2016 [2]) *Let G be a connected graph and $G_1, \ldots, G_r$ be nonempty, connected, and pairwise vertex-disjoint subgraphs of G. Then*

$$\chi_{sc}(G) \geq \sum_{i=1}^{r} \chi_{sc}(G_i) + (r-1) + 2(|V(G)| - \sum_{i=1}^{r} |V(G_i)|).$$

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Coloring uniform hypergraphs avoiding fixed rainbow subhypergraphs

Lucas de Oliveira Contiero [1], *Carlos Hoppen* [1], *Hanno Lefmann* [2] and *Knut Odermann* [2]

[1] Universidade Federal do Rio Grande do Sul, Brazil
[2] Technische Universität Chemnitz, Germany

Extended Abstract

We consider a hypergraph version of a graph-theoretical problem that has been initially proposed by Erdős and Rothschild [5], and which has been studied and generalized by several authors in the last few years. Originally, for a fixed number $r \geq 2$ of colors and a fixed graph F, Erdős and Rothschild were interested in n-vertex graphs that allow the largest number of edge-colorings avoiding a monochromatic copy of F. (We observe that edge-colorings in this work are not necessarily proper.) Alon, Balogh, Keevash and Sudakov [1] showed that, whenever $r \in \{2, 3\}$ and n is sufficiently large, the n-vertex ℓ-partite Turán graph $\mathcal{T}_\ell^{(2)}(n)$ admits the largest number of colorings avoiding a monochromatic copy of the complete graph $K_{\ell+1}$ (see also Yuster for the case $\ell = 2$), while Pikhurko and Yilma [16] have found the extremal graphs for K_3 and for K_4 when $r = 4$, which turn out to be neither K_3-free nor K_4-free. Very recently, two groups of authors [3, 17] have shown that, for any $r, \ell \geq 2$, the graph allowing the largest number of r-colorings avoiding monochromatic copies of $K_{\ell+1}$ is always a complete multipartite graph. In fact, the work in [3] deals with the more general problem of edge-colorings avoiding a copy of a forbidden graph F *colored according to some fixed color pattern*, which was initiated by Balogh [2]. We should also mention that three of the current authors [9] considered a version of this problem for colorings avoiding *rainbow* copies of the complete graph $K_{\ell+1}$, namely copies of $K_{\ell+1}$ in which every edge is colored differently.

The hypergraph counterpart of this problem has also been studied. The third author, along with co-authors, determined uniform n-vertex hypergraphs that allow the largest number of hyperedge-colorings avoiding monochromatic copies of some fixed hypergraph F. This includes the Fano plane [12], matchings [8], and linear hypergraphs such as expanded complete graphs and Fan(k)-hypergraphs [14], where by a *linear* hypergraph we mean a hypergraph whose hyperedges pairwise intersect in at most one vertex. Typically, the results in these papers establish that, for $r \in \{2, 3\}$ and large n, the n-vertex Turán hypergraph associated with the forbidden hypergraph F (i.e., the n-vertex hypergraph with the largest number of edges avoiding F as a subhypergraph) admits the largest number of colorings, while this is not the case for $r \geq 4$.

Here, we focus on rainbow hyperedge-colorings of hypergraphs. More precisely, for k-uniform hypergraphs F and H, and an integer r, let $c_{r,F}(H)$ denote the number of r-colorings of the set of hyperedges of H with no rainbow copy of F, that is, with no subhypergraph isomorphic to F such that the colors assigned to its hyperedges are all distinct. Let $c_{r,F}(n) = \max_{H \in \mathcal{H}_{n,k}} c_{r,F}(H)$, where the maximum runs over the *family* $\mathcal{H}_{n,k}$ of all k-uniform hypergraphs on n vertices. As usual, let $\mathrm{ex}(n, F)$ be the maximum number of hyperedges $e(H)$ of an n-vertex k-uniform hypergraph H which contains no subhypergraph isomorphic to F.

Clearly, determining $c_{r,F}(n)$ is trivial if $r \leq e(F) - 1$, as no coloring will ever produce a rainbow copy of F and the complete k-uniform n-vertex hypergraph will admit the largest number of colorings. For $r \geq e(F)$, it is clear that choosing any subset of $(e(F) - 1)$ of the r colors, and coloring the complete k-uniform n-vertex hypergraph with colors in this subset cannot produce a rainbow F, thus $c_{r,F}(n) \geq (e(F) - 1)^{\binom{n}{k}}$. On the other hand, it is also clear that every hyperedge-coloring of an F-free hypergraph H contains no rainbow copy of F, and consequently $c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)}$ for all $r \geq 2$. Note that $(e(F) - 1)^{\binom{n}{k}} \geq r^{\mathrm{ex}(n,F)}$ for smaller values of r, while the opposite is true for larger values of r. We will show that, for some classes of hypergraphs, we indeed have $c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$ for all fixed $r \geq r_0$ and n sufficiently large; in particular, this holds for the linear 3-uniform hypergraph Fano plane, and two families of linear k-uniform hypergraphs. These two families are expanded complete graphs and Fan(k)-hypergraphs.

Fano Plane. The *Fano plane* FA is the unique linear 3-uniform hypergraph on 7 vertices with 7 hyperedges. For n sufficiently large, the unique n-vertex extremal hypergraph for the Fano plane is $\mathcal{B}_n$, whose vertex set is partitioned into two classes of cardinality as equal as possible, and the set of hyperedges consists of all those 3-element sets that intersect each class in at most two vertices. The extremality of $\mathcal{B}_n$ has been proved by Keevash and Sudakov, and by Füredi and Simonovits (see, for example, [7]).

We have the following results.

Theorem 1 *Let FA be the 3-uniform Fano plane. There exists r_0 such that the following holds for all $r \geq r_0$. There exists a positive integer n_0, such that, for every 3-uniform hypergraph H on $n \geq n_0$ vertices, we have*

$$c_{r,FA}(H) \leq r^{\mathrm{ex}(n,FA)}.$$

Moreover, for $r \geq r_0$ and n sufficiently large, the only 3-uniform hypergraph H on n vertices with $c_{r,F}(H) = r^{\mathrm{ex}(n,FA)}$ is $\mathcal{B}_n$ (up to isomorphism).

For a hypergraph $H = (V, E)$ and a subset $V_1 \subseteq V$, let $e(V_1)$ be the number of hyperedges $e \in E$ with $e \subseteq V_1$. For the proof of Theorem 1 we use the following stability result for colorings.

Lemma 2 *Let $\delta > 0$ be fixed. Then there exists $r_0(\delta)$ such that the following holds for all $r \geq r_0(\delta)$. Let $H = (V, E)$ be a 3-uniform hypergraph on $n \geq n_0$ vertices with*

$$c_{r,FA}(H) \geq r^{\mathrm{ex}(n,FA)}.$$

Then, there exists a partition $V = V_1 \cup V_2$ of its vertex set such that $e(V_1) + e(V_2) \leq \delta n^3$.

For the proof of Lemma 2 we use a colored version of the weak Regularity Lemma for hypergraphs (see [4]), which is an extension of the well-known Szemerédi Regularity Lemma to hypergraphs. Moreover, we need a counting lemma associated with this type of regularity, which holds for linear hypergraphs [11], and a stability result in the sense of Simonovits for the (uncolored) 3-uniform Fano plane, which is given, for instance, in [10].

Using Lemma 2, we may prove Theorem 1 by contradiction, along the general lines of the proof of [1, Theorem 1.1] and [12, Theorem 1]. We choose n_0 appropriately and we let $H \neq \mathcal{B}_n$ be a hypergraph on $n > n_0^3$ vertices with at least $r^{\mathrm{ex}(n,FA)+m}$ rainbow-FA-free r-hyperedge colorings, for some $m \geq 0$. We show that H contains a vertex x such that the hypergraph $H - x$ obtained by deleting vertex x has at least $r^{\mathrm{ex}(n-1,FA)+m+1}$ rainbow-FA-free r-hyperedge colorings, or it contains two vertices x and y such that $H - x - y$ has at least $r^{\mathrm{ex}(n-2,FA)+m+2}$ rainbow-FA-free r-hyperedge colorings. Repeating this argument iteratively, we obtain a hypergraph on n_0 vertices whose number of rainbow-FA-free r-hyperedge colorings is at least $r^{\mathrm{ex}(n_0,FA)+m+n-n_0} > r^{n_0^3}$. However, a 3-uniform hypergraph on n_0 vertices has at most $n_0^3/6$ hyperedges and hence the number of such colorings is at most $r^{n_0^3/6}$, which is the desired contradiction.

Other Hypergraphs. Let $[\ell] := \{1, \ldots, \ell\}$ and $[\ell]^k := [\{1, \ldots, \ell\}]^k$. For integers $\ell, k \geq 2$, the so-called *expanded complete graph* $H^k_{\ell+1}$ is a k-uniform hypergraph defined as follows. Take $\binom{\ell+1}{2}$ edges of the complete graph $K_{\ell+1}$, which is called the *core* of $H^k_{\ell+1}$, and enlarge every edge of $K_{\ell+1}$ by a set of $(k-2)$ new vertices. Thus, the vertex set of $H^k_{\ell+1}$ has size $(\ell+1) + \binom{\ell+1}{2} \cdot (k-2)$ and the number of hyperedges is $\binom{\ell+1}{2}$.

Similarly, for integers $\ell, k \geq 2, \ell \geq k-1$, the *Fan(k)-hypergraph* $F^k_{\ell+1}$ is the k-uniform hypergraph that contains $(\ell+1)$ vertices $v_1, \ldots, v_{\ell+1}$, called the *core* of $F^k_{\ell+1}$. Moreover, k

vertices of this core, say the vertices $v_1, \ldots, v_k$, form a hyperedge, the *core-hyperedge,* so that for each $\{i,j\} \in [\ell+1]^2 \setminus [k]^2$ the two-element set $\{v_i, v_j\}$ is enlarged by a set of $(k-2)$ new vertices. Thus, the vertex set of $F^k_{\ell+1}$ has size $(\ell+1) + (\binom{\ell+1}{2} - \binom{k}{2}) \cdot (k-2)$ and it contains $1 + \binom{\ell+1}{2} - \binom{k}{2}$ hyperedges.

These two families of linear hypergraphs have been studied by Mubayi and Pikhurko (see [15] and the references therein), who showed that, for large n and $\ell \geq k$, the unique extremal hypergraph for $H^k_{\ell+1}$ and $F^k_{\ell+1}$ as well is the so-called *Turán hypergraph* $\mathcal{T}^{(k)}_\ell(n)$, which is a k-uniform n-vertex hypergraph, whose vertex set is partitioned into ℓ classes of cardinality as equal as possible and whose set of hyperedges consists of all those k-element sets that intersect each class in at most one vertex. Stability results in the sense of Simonovits have also been proved there.

Theorem 3 *Let $F = H^k_{\ell+1}$ be the k-uniform, expanded complete 2-graph or $F = F^k_{\ell+1}$ the Fan(k)-hypergraph, each with core of size $(\ell+1)$, where $2 \leq k \leq \ell$. There exists r_0 such that the following holds for all $r \geq r_0$. There exists a positive integer n_0, such that, for every k-uniform hypergraph H on $n \geq n_0$ vertices, we have*

$$c_{r,F}(H) \leq r^{\mathrm{ex}(n,F)}.$$

Moreover, for $r \geq r_0$, and n sufficiently large, the only hypergraph H on n vertices with $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$ is isomorphic to $\mathcal{T}^{(k)}_\ell(n)$, the k-uniform Turán hypergraph on n vertices with ℓ classes.

For $k = 2$ we have $H^k_{\ell+1} = F^k_{\ell+1} = K_{\ell+1}$ and $\mathcal{T}^{(k)}_\ell(n)$ is the usual Turán graph with ℓ classes.

We prove Theorem 3 with arguments as in the proof of Theorem 1. We first obtain a (color)-stability result of the type of Lemma 2, which establishes that the structure of any n-vertex hypergraph that admits at least $r^{\mathrm{ex}(n,F)}$ colorings is similar to that of the extremal graph. This may be proved in a more general framework, which considers families $\mathcal{F}$ of forbidden linear hypergraphs such that the vertex set of an extremal configuration admits a partition such that $e \in [n]^k$ is a hyperedge if and only if it satisfies some intersection pattern with the different partition classes. There must also be a stability result for $\mathcal{F}$, that is, a result stating that any $\mathcal{F}$-free hypergraph with a large number of hyperedges admits a partition such that only a small number of hyperedges fail to have the given intersection pattern. The second ingredient of the proof is showing that, among all hypergraphs with this structure, the (uncolored)-extremal hypergraph admits the largest number of colorings.

One natural problem is to improve the value of the lower bound r_0 in Theorems 1 and 3 obtained in this paper. Here, given a k-uniform linear hypergraph F, the size of r_0 depends on the interplay between the constants ε and δ in stability results for F-free hypergraphs, which establish that any F-free n-vertex hypergraph with at least $\mathrm{ex}(n,F) - \varepsilon n^k$ hyperedges admits a partition of its vertex set that differs from the structure of the extremal configuration by at most δn^k 'bad edges'. Much better bounds may be derived when the dependence between ε and δ is made explicit as was done for graphs in [6]. Another natural line of research would be to consider non-linear forbidden hypergraphs F, but to address this case we would require stronger versions of the Regularity Lemma for hypergraphs, as has been done in [13].

In general, the problem of determining $c_{r,F}(n)$ and the n-vertex graphs and hypergraphs that achieve this extremal value has only been solved in very special cases. One interesting feature of the monochromatic case is the emergence of extremal graphs that are neither complete nor isomorphic to the extremal graph in the uncolored case. So far, there are no known examples of such alternative extremal configurations for rainbow patterns.

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Structure of plane graphs with prescribed δ, ρ, w and w^*

Peter Hudák [1], *Mária Maceková* [2], *Tomáš Madaras* [2], and *Pavol Široczki* [2]

[1] NESS Košice Development Center, Košice, Slovakia
[2] P.J. Šafárik University, Košice, Slovakia

Extended Abstract

Throughout this contribution, we consider connected graphs without loops or multiple edges. Given a graph $G = (V, E)$, the degree $d(v)$ of a vertex $v \in V$ is the number of edges incident with v. For an edge $e = uv \in E$, the *weight* $w(e)$ of e is the sum $d(u) + d(v)$. The *minimum vertex degree* of G is the number $\delta(G) = \min\{d(v) : v \in V\}$, and the *minimum edge weight of* G is $w(G) = \min\{w(e) : e \in E\}$. The *girth* $g(G)$ of G is the length of a shortest cycle of G and the *double girth* of G is defined as the minimum sum of lengths of two distinct cycles of G which share a common edge; it will be denoted as $dg(G)$ (note that $g(G) = \infty$ if G is a tree, and $dg(G) = \infty$ if none two cycles of G share an edge).
A graph is called *planar* if it can be drawn in the plane without crossing of edges (such a drawing is called a *plane graph* and it is determined by the triple (V, E, F), where F is the set of faces). The *face size* $d(\alpha)$ of a face $\alpha \in F$ is the number of edges incident with α (incident cut-edges being counted twice). The *minimum face size* of G, denoted $\rho(G)$, is defined as $\min\{d(\alpha) : \alpha \in F\}$ and the *minimum dual edge weight* of G is the number $w^*(G) = \min\{d(\alpha) + d(\beta) : \alpha, \beta \in F, \alpha \neq \beta, \alpha, \beta \text{ have a common edge}\}$. Note that $g(G) \leq \rho(G)$ and $dg(G) \leq w^*(G)$.

For a general graph G, there are no special dependencies of the above mentioned graph invariants apart of the trivial ones: $w(G) \geq 2\delta(G)$ and $dg(G) \geq 2g(G)$. On the other hand, these invariants are strongly dependent when additional conditions are considered.
Particularly, if G is a simple plane graph, then $\min\{\delta(G), \rho(G)\} \leq 5$; additionally $\delta(G) \geq 4$ implies $\rho(G) = 3$ and $\rho(G) \geq 4$ implies $\delta(G) \leq 3$. These facts follow easily from Euler's formula for the numbers of vertices, edges and faces of a plane graph. A more subtle analysis of consequences of Euler's formula yields another dependence: if $\delta(G) \geq 3$ then $w(G) \leq 13$, whereas $\delta(G) \geq 4$ gives $w(G) \leq 11$, see [1]. By considering dual versions of these results, we obtain a dependance between the minimum face size $\rho(G)$ and the minimum dual edge weight $w^*(G)$: if $\rho(G) \geq 3$ then $w^*(G) \leq 13$ and, for $\rho(G) \geq 4$, $w^*(G) \leq 11$. Furthermore, the results of the classical paper [5] give that if $\delta(G) \geq 3$ and $\rho(G) \geq 4$, then $w(G) \leq 8$ and $\delta(G) \geq 3$ together with $\rho(G) = 5$ yield $w(G) = 6$.

The mutual dependance of all four values $\delta(G), \rho(G), w(G)$ and $w^*(G)$ for 3-connected plane graphs (i.e. $\delta \geq 3$) was studied in [3] giving the characterization of all quadruples (δ, ρ, w, w^*) for which the corresponding families $\mathcal{G}(\delta, \rho, w, w^*)$ of minimum vertex degree at least δ, girth at least ρ, minimum edge weight at least w and double girth at least w^* are empty or non-empty, respectively. Euler theorem implies that $\mathcal{G}(4, 4, 8, 8)$ is empty. From Kotzig theorem (see [6]), it follows that $\mathcal{G}(3, 3, 14, 6)$ and $\mathcal{G}(4, 3, 12, 6)$ are empty and from [5] follows the emptiness of families $\mathcal{G}(3, 4, 9, 8)$ and $\mathcal{G}(3, 5, 7, 10)$. Using the duality, we get that the families $(3, 3, 6, 14)$, $(3, 4, 6, 12)$, $(4, 3, 8, 9)$, and $(5, 3, 10, 7)$ are also empty. Ferencová and Madaras moreover proved the following:

Theorem 1 (B. Ferencová, T. Madaras [3]) *The families* $\mathcal{G}(3, 3, 7, 10)$*,* $\mathcal{G}(3, 3, 8, 9)$*, and* $\mathcal{G}(3, 4, 7, 9)$ *are empty.*

The aim of our work was to extend the results of [3] for wider families of plane graphs with $\delta = 2$. The graph $K_{2,r}$ shows that $w(G)$ is unbounded for $\rho(G) = 4$. On the other hand, recent results by Jendroľ and Maceková [4] and results from [2] show that if $g(G) \in \{5, 6\}$ then $w(G) \leq 7$ and, further, if $g(G) \in \{7, 8, 9, 10\}$, then $w(G) \leq 5$ as well as $g(G) \geq 11$ implies $w(G) = 4$. The equivalent formulation of these results is that the families $\mathcal{G}(2, 5, 8, 10), \mathcal{G}(2, 7, 6, 14)$ and $\mathcal{G}(2, 11, 5, 22)$ are empty.

We state the following:

Theorem 2 *The families* $\mathcal{G}(2,3,7,15)$, $\mathcal{G}(2,3,9,11)$, $\mathcal{G}(2,3,13,9)$, $\mathcal{G}(2,5,5,27)$, $\mathcal{G}(2,5,6,17)$, $\mathcal{G}(2,5,7,13)$, *and* $\mathcal{G}(2,7,5,23)$ *are empty.*

For the non-empty families arising from admissible quadruples, we are interested in determining the extremal ones, that is, the families $\mathcal{G}(\delta,\rho,w,w^*)$ such that the increase of any of the values δ,ρ,w and w^* results in an empty family:

Theorem 3 *The families* $\mathcal{G}(2,4,8,14)$, $\mathcal{G}(2,4,12,10)$, $\mathcal{G}(2,6,5,26)$, $\mathcal{G}(2,6,6,16)$, $\mathcal{G}(2,6,7,12)$, $\mathcal{G}(2,10,5,22)$ *are non-empty and extremal. The families* $\mathcal{G}(2,4,6,r)$ *and* $\mathcal{G}(2,4,s,8)$ *are nonempty for arbitrary* $r \geq 8$ *and* $s \geq 4$.

A possible common way how to visualize the dependance of δ,ρ,w,w^* for families of plane graphs is to construct the diagram of a partially ordered set depicting the hierarchy of all non-empty families (generated by quadruples (δ,ρ,w,w^*)) under the set inclusion partial ordering. For $\delta \geq 3$, the partially ordered set of generated families of polyhedral graphs is shown in Figure 1:

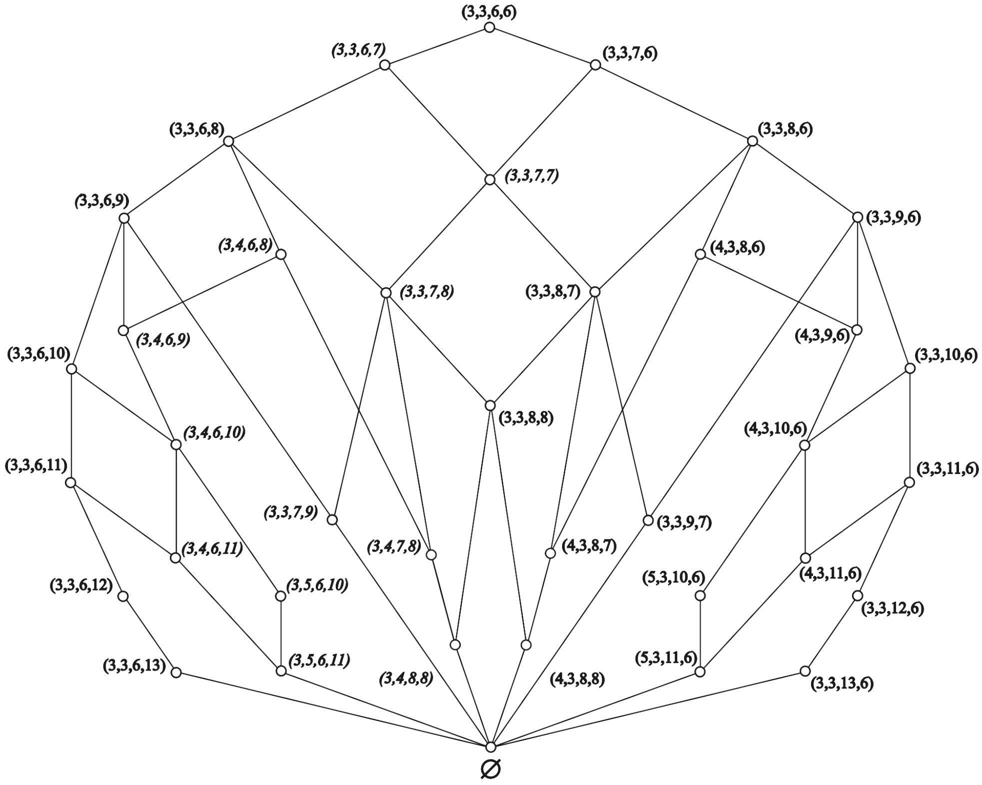

Figure 1: The hierarchy of families of polyhedral graphs generated by (δ,ρ,w,w^*)

The results for $\delta = 2$ are presented in Table 1 indexed by values of girth (rows) and edge weight (columns) such that, the corresponding table entry shows the maximal admissible value of dual edge weight. The value ∞ in the first column is due to the fact that, in the graph obtained from C_n (n arbitrarily large) by replacing every edge with two disjoint paths of length 2, the dual edge weight is unbounded. The value 8 in the last column results from the graph $K_{2,r}$ for large r. The bold values correspond to extremal families.

	4	5	6	7	8	9	10	11	12	$w \geq 13$
3	∞	∞	∞	14	14	10	10	10	10	8
4	∞	∞	$\boldsymbol{\infty}$	14	**14**	10	10	10	**10**	**8**
5	∞	26	16	12	–	–	–	–	–	–
6	∞	**26**	**16**	**12**	–	–	–	–	–	–
7	∞	22	–	–	–	–	–	–	–	–
8	∞	22	–	–	–	–	–	–	–	–
9	∞	22	–	–	–	–	–	–	–	–
10	∞	**22**	–	–	–	–	–	–	–	–
$\rho \geq 11$	$\boldsymbol{\infty}$	–	–	–	–	–	–	–	–	–

Table 1: The table of admissible values for quadruples $(2, \rho, w, w^*)$

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On Girth of Minimal Counterexample to 5-Flow Conjecture

Radek Hušek, Peter Korcsok, and *Robert Šámal*

Computer Science Institute, Charles University, Prague

Extended Abstract

Given a graph $G = (V, E)$ with fixed (but arbitrary) orientation of the edges and an abelian group Γ, we define a Γ-*flow* (alternatively just a *flow*) as a mapping $f\colon E \to \Gamma$ satisfying Kirchhoff's law at each vertex, i.e. for each vertex total in-flow equals total out-flow. More formally, f satisfies

$$\sum_{e=\bullet\to v} f(e) = \sum_{e=v\to\bullet} f(e)$$

for each vertex $v \in V$. A k-*flow* is a $\mathbb{Z}$-flow using only values strictly between $-k$ and k. Finally, a flow is *nowhere-zero* if no edge has value 0. It is an easy observation that the existence of any flow is independent on the orientation of G – whenever we reverse some edge we also change the corresponding value to its negative.

The concept of (nowhere-zero) flows was introduced by Tutte in 1954 [6] while studying a coloring of planar graphs. He observed that a plane graph G admits a nowhere-zero k-flow if and only if its dual G^* is k-colorable. Tutte also proved that the existence of a Γ-flow does not depend on the structure of Γ as long as the number of elements is the same.

Theorem 1 (Tutte '54 [6]) *For a given graph G and an abelian group Γ with k elements, G admits a k-flow if and only if G admits a Γ-flow.*

Therefore, the problems of existence of a (nowhere-zero) k-flow and a (nowhere-zero) $\mathbb{Z}_k$-flow are equivalent.

As a consequence of 4-Color Theorem [1], each bridgeless planar graph admits a nowhere-zero 4-flow. Tutte conjectured the following generalization.

Conjecture 2 (Tutte '54 [6]) *Each bridgeless graph admits a nowhere-zero 5-flow.*

This conjecture is still open and the closest result is due to Seymour.

Theorem 3 (Seymour '81 [5]) *Each bridgeless graph admits a nowhere-zero 6-flow.*

Recently, Kochol [2, 3, 4] studied a hypothetical minimal counterexample to Tutte's 5-Flow Conjecture. He introduced so-called *forbidden networks*, i.e. graphs that cannot be a subgraph of any such counterexample. He also used Tutte's contraction/deletion formula to count flows on a network using a given values on specific edges. Using these counts, he transformed the exclusion of a forbidden subgraph to the problem of equality of vector spaces and proved that such minimal counterexample cannot contain circuits of length at most 10.

We have verified the results by Kochol using an independent implementation of the computations and have improved the best known result.

Theorem 4 *Every minimal counterexample to the 5-Flow Conjecture has girth at least 12.*

It seems that we have reached the boundaries where this method can be used.

Overview of the method. Let $H = (V, E)$ be a graph and $U \subseteq V$ such that each vertex $v \in U$ has degree 1, then the pair (H, U) is a *network* and U is a set of *terminals*. We can assign a value $s(v)$ to each terminal $v \in U$ and ask if there exists a flow on (H, U) such that

Kirchhoff's law is satisfied for each non-terminal $v \in V \setminus U$ and the out-flow of any terminal $v \in U$ is $s(v)$. A necessary condition of such flow is

$$\sum_{v \in U} s(v) = 0 \tag{1}$$

Denote the number of such flows by $F_{H,U}(s)$.

A partition $P = \{Q_1, \ldots, Q_r\}$ of terminals U, i.e. a family of pairwise disjoint sets Q_i where $U = Q_1 \cup \cdots \cup Q_r$, such that each set Q_i contains at least 2 vertices is called *proper*. For such partition P and an assignment s satisfying condition (1), we define a *compatibility* function $\chi(s, P)$:

$$\chi(s,P) = \begin{cases} 1 & \text{if } \sum_{v \in Q_i} s(v) = 0 \text{ holds for each } i \in \{1, \ldots, r\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{P}$ be the set of all proper partitions and $\chi(s)$ be the vector of $\chi(s, P)$, $P \in \mathcal{P}$ in some fixed order.

Lemma 5 (Kochol '04 [2]) *Let (H, U) be a network. Then there exist integers x_P for each $P \in \mathcal{P}$ such that $F_{H,U}(s) = \sum_{P \in \mathcal{P}} x_P \chi(s, P)$ for every s satisfying condition* (1)*.*

Let (H', U') be a network such that H' is smaller than H and $|U'| = |U|$ and G be a graph containing H as a subgraph. Then we can delete all non-terminal vertices of H and identify terminal vertices of H and H' (see Figure 1) to obtain a smaller graph G'. Naturally, we require that this replacement does not create any bridge.

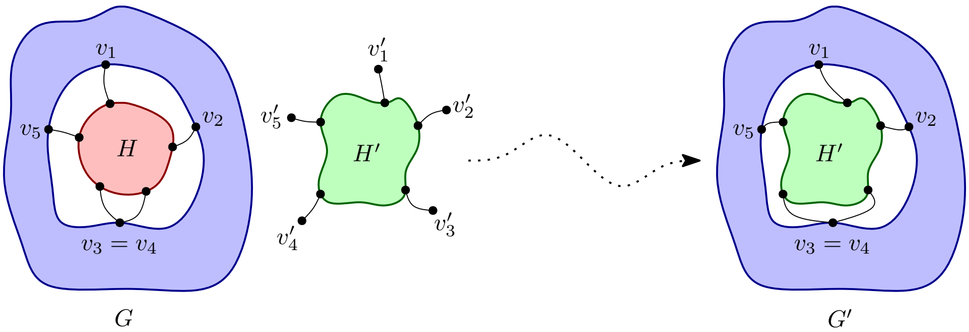

Figure 1: Graph G' obtained from G by replacing H by H'.

For the network (H, U), let S_H be the set of all assignments s satisfying condition (1) such that $F_{H,U}(s) > 0$ and V_H be the linear hull of $\{\chi(s) \colon s \in S_H\}$. Similarly, we define linear hull $V_{H'}$ for the network (H', U').

Theorem 6 (Kochol '10 [4]) *If there exists some graph H' smaller than H such that replacement of H by H' does not create any bridge in the class of cyclically 6-connected graphs and $V_{H'} \subseteq V_H$ then H cannot be a subgraph of any minimal counterexample to the 5-Flow Conjecture.*

Computations. We can formulate the problem of determining whether $V_{H'} \subseteq V_H$ in terms of matrices: Let M_H and $M_{H'}$ be the matrices where rows are exactly $\chi(s)$ for $s \in S_H$ and $s \in S_H \cup S_{H'}$, respectively. Then $V_{H'} \subseteq V_H$ if and only if the ranks of M_H and M'_H are equal.

To prove Theorem 4, we have used circuits (lengths up to 14) with a unique new edge from each vertex as a graph H and shorter circuits with one isolated edge (again with a unique new edge from each vertex) as a graph H' (see Figure 2).

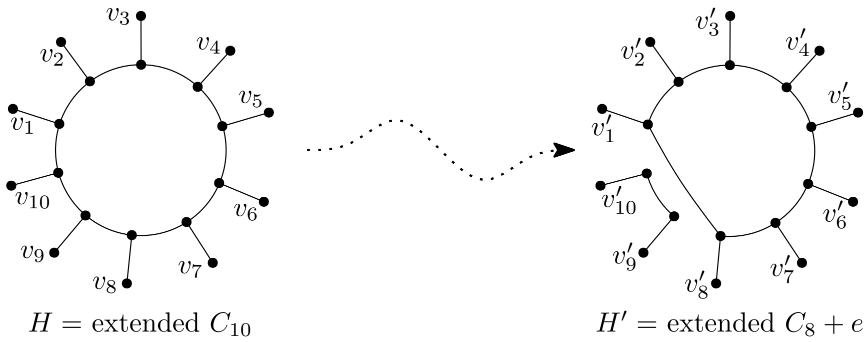

Figure 2: Example of H and H' used in computations.

Up to length of 11, the ranks of both matrices were the same but, starting with circuit C_{12}, there are some vectors in $V_{H'}$ that are not in V_H (see Table 1). We have also tried to use other graphs, e.g. a perfect matching or two shorter circuits, as H' for $H = C_{12}$ and $H = C_{13}$ but the rank of $M_{H'}$ has always exceeded the rank of M_H.

H	size of M_H	size of $M_{H'}$	rank M_H	rank $M_{H'}$	note
C_8	122×81	176×81	62	62	
C_9	262×238	430×238	151	151	[4]
C_{10}	$792 \times 1\,079$	$1\,415 \times 1\,079$	539	539	[4]
C_{11}	$1\,972 \times 4\,752$	$3\,937 \times 4\,752$	1 699	1 699	
C_{12}	$5\,697 \times 25\,421$	$12\,112 \times 25\,421$	5 550	6 226	
C_{13}	$15\,183 \times 141\,772$	$34\,969 \times 141\,772$	15 033	21 945	(a)

Table 1: The sizes and ranks of some matrices M_H and M'_H. (a) In this case, computations modulo 251 were used and matrix M'_H contained only rows for $s \in S_{H'}$.

Since the size of the matrices rises exponentially, we have used permutation group to reduce the size as presented by Kochol [4]. Moreover, we have computed larger matrices modulo appropriate prime number (instead of in rational numbers) to reduce the time and space needed for the rank computations.

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Group Connectivity: $\mathbb{Z}_4$ v. $\mathbb{Z}_2^2$

Radek Hušek, *Lucie Mohelníková*, and *Robert Šámal*

Computer Science Institute of Charles University
Prague, Czech Republic

Extended Abstract

A *flow* in a digraph $G = (V, E)$ is an assignment of values of some abelian group Γ to edges of G such that Kirchhoff's law is valid at every vertex. Formally, $f : E \to \Gamma$ satisfies

$$\sum_{e \text{ ends in } v} f(e) = \sum_{e \text{ starts in } v} f(e)$$

for every vertex $v \in V$. We say a flow is *nowhere-zero* if it does not use value 0 at any edge.

Tutte [5] started the study of nowhere-zero flows by observing, that a plane digraph G has a nowhere-zero flow in $\mathbb{Z}_k$ if and only if its plane dual G^* is k-colorable (we do not consider orientation of the edges for the coloring). This motivated several famous conjectures, we mention just the 5-flow conjecture (due to Tutte): every bridgeless graph has a nowhere-zero flow in $\mathbb{Z}_5$. A motivating feature of the theory of nowhere-zero flows are several nice properties, starting with the following discovered by Tutte. In particular, the following one:

Theorem 1 (Tutte '54 [5]) *Let Γ be an abelian group with k-elements. Then for every digraph the existence of a nowhere-zero Γ-flow is equivalent with the existence of a nowhere-zero $\mathbb{Z}_k$-flow.*

Theorem 2 (Tutte '54 [5]) *The existence of $\mathbb{Z}_k$-flow is equivalent with the existence of a nowhere-zero integer flow, that uses only values ± 1, ± 2, ..., $\pm(k-1)$.*

Jaeger et al. [3] introduced a variant of nowhere-zero flows called *group connectivity*. Among several equivalent definitions we choose the one most convenient for us. A digraph $G = (V, E)$ is Γ*-connected* if for every mapping $h : E \to \Gamma$ there is a Γ-flow f on G that satisfies $f(e) \neq h(e)$ for every edge $e \in E$. As we may choose the "forbidden values" $h \equiv 0$, every Γ-connected digraph has a nowhere-zero Γ-flow; however, the converse is false. While the notion of group connectivity is stronger than the existence of nowhere-zero flows, it is also more versatile, in particular the notion lends itself more easily to proofs by induction. This is a consequence of an alternative definition of group connectivity: instead of looking for a flow, we may check existence of a mapping $E \to \Gamma$ that has prespecified surplus at each vertex. A celebrated recent example of this is the solution to the Jaeger's conjecture by Thomassen et al. [4], but there are many more. Thus, it is worthwhile to understand the properties of group connectivity in more detail.

It is easy to see that both the existence of a nowhere-zero Γ-flow and Γ-connectivity do not change when we reverse the orientation of an edge of the digraph (we only need to change the corresponding flow value from x to $-x$). Thus, we will say that an undirected graph G has a nowhere-zero Γ-flow (is Γ-connected) if some (equivalently every) orientation of G has a nowhere-zero Γ-flow (is Γ-connected). Also, using the definition of group connectivity working with vertex surpluses, we observe that group connectivity is monotone with respect to edge addition – if G is Γ-connected then $G + e$ is Γ-connected for any edge e.

Some results on nowhere-zero flows extend to the stronger notion of group connectivity. Seymour proved that every edge 3-connected graph is $\mathbb{Z}_6$-connected. Thomassen et al. [4] proved that every edge 6-connected graph is $\mathbb{Z}_3$-connected.

However, some nice properties of group-valued flows are not shared by group connectivity. In particular [3], there is a graph that is $\mathbb{Z}_5$-connected, but not $\mathbb{Z}_6$-connected. This contrasts with the situation for flows: Suppose G has a nowhere-zero flow in $\mathbb{Z}_5$ but not in $\mathbb{Z}_6$. Using twice Theorem 2, we find that G has a nowhere-zero integer flow with values bounded in absolute value by 5, but not one bounded by 6, a clear contradiction.

An analogy of Theorem 1 is more subtle. Indeed, in Section 3.1 of [3] the authors mention: "... we do not know of any $\mathbb{Z}_4$-connected graph which is not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected, or vice versa. Neither can we prove that such graphs do not exist." Our main result is the resolution to this natural question.

Theorem 3

1. *There is a graph that is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected.*
2. *There is a graph that is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected.*

The Group Connectivity Conjecture and Results. When looking for graphs certifying Theorem 3, we only need to consider graphs that do have nowhere-zero $\mathbb{Z}_2^2$-flow (equivalently [5], nowhere-zero $\mathbb{Z}_4$-flow). We decided to try cubic graphs (and their subdivisions). In contrary to the usual case, however, we are not interested in snarks (cubic graphs that fail to be edge 3-colorable), as those do not have nowhere-zero $\mathbb{Z}_2^2$-flow.

We note that subdividing an edge has no effect on existence of a nowhere-zero flow (the new edge can have the same flow value as before). It makes the group connectivity stronger – in effect, we are forbidding one more value on an edge. This suggests the following strategy:

1. find a (random) 3-regular graph and
2. repeatedly subdivide an edge and check $\mathbb{Z}_2^2$-connectivity and $\mathbb{Z}_4$-connectivity.

This procedure yielded the graph in the Figure 1, during work for the master thesis of the second author. This graph is $\mathbb{Z}_4$- but not $\mathbb{Z}_2^2$-connected. Later, with more effective implementation (see next section) by the first author, we found graphs that are $\mathbb{Z}_2^2$- but not $\mathbb{Z}_4$-connected. The smallest among them are (threefold) subdivisions of cubic graphs on 12 vertices.

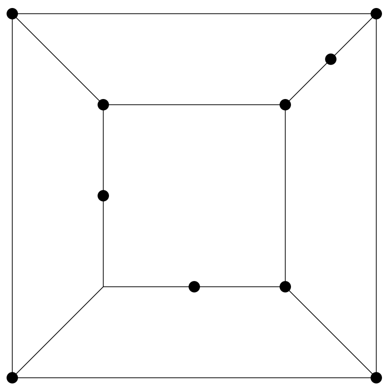

Figure 1: A subdivision of cube which is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected

Group connectivity testing. We fix a digraph $G = (V, E)$. We let n be the number of vertices and m the number of edges of G. For a fixed abelian group Γ, the set of all Γ-flows on G is denoted $\mathcal{F}$.

The most straightforward way of testing whether a graph is Γ-connected, is using the definition: We can enumerate all $h : E \to \Gamma$ assignments of forbidden values and for each of them (try to) find a satisfying flow. Finding a satisfying flow by itself is a hard problem: A cubic graph has nowhere-zero $\mathbb{Z}_4$-flow (equivalently, $\mathbb{Z}_2^2$-flow) if and only if it has an edge 3-coloring. The edge 3-colorability of cubic graphs was shown to be NP-complete by Holyer [2]. An easy observation about the structure of forbidden assignments is:

Observation 4 *Let $h, h' : E \to \Gamma$ be assignments of the forbidden values such that $h' - h = \Delta$ is a flow. Then h is satisfied by flow f if and only if h' is satisfied by $f + \Delta$.*

Definition 5 *We say that $h, h' : E \to \Gamma$ be assignments of forbidden values are flow-equivalent, denoted $h \sim_f h'$, if and only if $h' - h$ is a flow.*

Hence we can split all assignments of the forbidden values into equivalence classes of $\sim_f$ and test existence of satisfying flow only for one member of each class. This improves algorithm from finding $|\Gamma|^m$ flows to finding $|\Gamma|^{n-1}$ flows.

A bit smarter algorithm – used to find $\mathbb{Z}_2^2$-connected graphs which are no $\mathbb{Z}_4$-connected – can be obtained by looking at Observation 4 the other way around. It follows that each equivalence class of $\sim_f$ is exactly a coset generated by adding some its fixed member to all flows. Therefore if a equivalence class $[x]_{\sim_f}$ is satisfied then for every flow f there is $h \in [x]_{\sim_f}$ such that f satisfies h.

So we can fix a flow – everywhere-zero flow being the obvious candidate – and for each equivalence class we test whether some of its members is satisfied by it. This increases number of tests back to $|\Gamma|^m$ but now each test is just a simple comparison instead of an NP-complete problem.

We can also trade some space for time: We keep table of all groups and instead of enumerating members of all groups we enumerate all assignments of forbidden values that are satisfied by given flow. For each of them we determine its group and mark that group as satisfied. After enumerating them all we just check whether every group is satisfied. This decreases number of enumerated elements to $(|\Gamma| - 1)^m$ but consumes extra 2^{n-1} bits of memory.

Because we were testing subdivisions of cubic graphs we would like to optimize cases of once and twice subdivided edges. Without any additional optimization each subdivision of an edge increases number of edges by one and hence slows down the described method by factor $|\Gamma| - 1$. But a subdivision creates an edge 2-cut.

Without loss of generality we may assume that edges of a 2-cut – denote them e_1 and e_2 – are oriented in opposite directions. Value of any flow must be the same on both of them. Hence swapping forbidden values for edges e_1 and e_2 does not change set of satisfying flows. Moreover we may assume that forbidden values for e_1 and e_2 are different because it is more restrictive than the case when they are the same. This reduces number of cases from $|\Gamma|^2$ to $\binom{|\Gamma|}{2}$ (i.e., from 16 to 6 for groups of size four). Double subdivision is in our case even simpler because we have three forbidden values and again the most restrictive case is when they all are distinct. So such double-subdivided edge has only one possible value (in our case, where $|\Gamma| = 4$).

Now we need to plug this observations into above-described algorithm. Observe that the groups used in the algorithm do not have to be equivalence classes of $\sim_f$ but we can use classes of any equivalence $\sim$ which is congruence with respect to satisfiability and which is coarser than $\sim_f$. Being congruence with respect to satisfiability means that either all elements of equivalence class are satisfiable or none of them is. Being coarser than $\sim_f$ ensures that $[x]_{\sim_f} \subseteq [x]_\sim$ and so if class $[x]_\sim$ is satisfiable that for every flow f there is some $y \in [x]_\sim$ satisfied by f.

We begin with equivalence $\sim_f$. For each 2-cut we remove all classes that forbid the same value on both edges of the cut and merge classes which differ only by swapping values on edges of the cut. For double-subdivided edges we remove all classes that do not forbid 3 different values on each double-subdivided edge and than merge all classes that differ only by order of forbidden values on given subdivided edge.

Conclusions and open problems. We have found graphs that show that $\mathbb{Z}_2^2$- and $\mathbb{Z}_4$-connectivity are independent notions. All of the graphs that we have found to certify this do have vertices of degree 2. Therefore, it is natural to ask, whether such graphs exist that are edge 3-connected.

Another challenging task is to find a proof that does not use computers. The main obstacle is to find efficient techniques to show that a particular graph is Γ-connected. To prove the converse is much easier: we guess forbidden values $h : E \to \Gamma$ and then show non-existence of a flow.

Our final question is the complexity of testing group connectivity. The algorithm we have developed is fast enough for our purposes, however, the required time is exponential. To test for group connectivity seems harder than to test for existence of a nowhere-zero flow, which suggests the problem is NP-hard. In fact, we believe it is Π_2^p-complete. For choosability, an analogous notion for the dual of the graph, the Π_2^p-completeness was proved by Erdős et al. [1]. For testing group connectivity we do not know any hardness results, though.

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Collection of Codes for Tolerant Location

Ville Junnila and *Tero Laihonen*

Department of Mathematics and Statistics, University of Turku, Finland
viljun@utu.fi, terolai@utu.fi

Extended Abstract

Let us model a sensor network using a finite, simple and undirected graph $G = (V, E)$. We consider sensor networks where in each vertex we have a sensor which can be turned on or off as in [1]. Denote the set of active sensors by $C \subseteq V$. An observer (in $x \in V$) wishes to determine her location with the aid of ID packets sent from the active sensors in its closed neighbourhood $N[x]$, that is, she wants to find out x by knowing the set

$$I(x) = I(C; x) = N[x] \cap C.$$

In addition, we would like to cope with the situation that some, say at most s, of the sensors are malfunctioning. In other words, we want our system to tolerate any s broken sensors. We call a subset $C \subseteq V$ a *code* and its elements *codewords*.

Previously, in the literature, this situation has been handled using robust identifying codes (see different variants in [3, 5, 6, 10]). This approach requires from the code C that

$$|I(x) \bigtriangleup I(y)| \geq s + 1,$$

where the symmetric difference $I(x) \bigtriangleup I(y) = (I(x) \setminus I(y)) \cup (I(y) \setminus I(x))$, and

$$|I(x)| \geq s + 1.$$

These codes, however, are usually of large size, which implies signal interference and unnecessary energy consumption [8]. However, they have importance, for example, in applications where sensors are considered highly expensive.

In this paper, similar to [1, 8], we study situations where sensors are relatively cheap. Now there is a sensor in each vertex of the network and the sensor can be turned on or off. In order to minimize signal interference and energy consumption, we optimize the number of sensors that are turned on simultaneously. For this purpose, we consider a collection of codes — only one of the codes have its corresponding sensors active at any given time. Each code is required to be such that when the observer determines her location using the code it is correct. There must also be a way to determine when we need to change the code to another one in the collection if the current code has been damaged by some broken sensors. Moreover, for any s malfunctioning sensors, there has to be at least one working code in the collection. Naturally, we also wish to have as small codes as possible.

If a code C satisfies the following property

$$\bigcap_{c \in I(C;x)} N[c] = \{x\} \qquad \text{for all } x \in V, \tag{1}$$

then it gives us a way to determine when to switch to another code as well as gives a correct position for the observer if the code C is not damaged by the broken sensors. Indeed, if after receiving the set $I(C; x)$, it is deduced that $\bigcap_{c \in I(C;x)} N[c] = \{x\}$, then the unique vertex x from the intersection is the sought position of the observer. If, on the other hand,

$$\left| \bigcap_{c \in I(C;x)} N[c] \right| \geq 2,$$

it is declared that the current code C in the collection is useless and another code from the collection is needed. We also make the convention that if $I(x) = \emptyset$, then $\bigcap_{c \in I(x)} N[c] = V$.

A code satisfying (1) is called *self-identifying* (or 1^+-*identifying*). Moreover, a self-identifying code C is *minimal* if removing any element of C implies that (1) no longer holds. The size of the smallest possible self-identifying code in G is denoted by $\gamma^+ = \gamma^+(G)$. A self-identifying code attaining the size γ^+ is called *optimal*.

Definition 1 *A collection of codes $\mathcal{L} = \{C_1, C_2, \ldots, C_h\}$ is called an s-*tolerant identifying collection *in $G = (V, E)$ if*

(i) $C_i \subseteq V$ is minimal self-identifying code for all $i = 1, \ldots, h$ and

(ii) for any $S \subseteq V$ of size at most s we have

$$S \cap C_i = \emptyset \qquad \text{for at least one } i = 1, \ldots, h. \tag{2}$$

In the previous definition, a self-identifying code guarantees that the searched location can be found and a possible malfunctioning of sensors can be detected. The minimality requirement is motivated by the efforts for efficiency (regarding energy consumption). Moreover, the condition (2) guarantees that there always exists a code without malfunctioning vertices (or sensors). Naturally, we prefer a collection with less codes and also strive for as small self-identifying codes as possible (often using optimal codes).

A code $C \subseteq V$ is called 1-*identifying* if $I(x)$ is non-empty and $I(x) \neq I(y)$ for all $x \neq y$ [7]. Here the idea is that the location of the observer is determined by comparing received set $I(x)$ with the other sets. We call the code satisfying (1) *self*-identifying, because the vertex x can be uniquely determined using only the set $I(x)$ without knowledge of other $I(y)$ sets. Earlier, in [8], a collection of disjoint 1-identifying codes were used in a different (but inspiring) manner. The reason we do not utilize the usual 1-identifying codes in the collection is because then we could end up determining a wrong position for the observer (due to broken sensors) and we do not know when to change our code to a new one in the collection. In our model, we also loosen the previous restriction of the codes being disjoint by allowing them to overlap.

The following theorem connects the self-identifying codes with the $(1, \leq 1)^+$-identifying codes in [4] designed for another purpose of detecting several observers. Thus, the self-identifying codes were also called 1^+-identifying.

Theorem 2 *Let $C \subseteq V$ be a code. Then the following two conditions are equivalent:*

(i) $\bigcap_{c \in I(C;x)} N[c] = \{x\}$ for all $x \in V$

(ii) $I(C;x) \setminus I(C;y) \neq \emptyset$ for all distinct $x, y \in V$.

1 On existence of the collection

Let us start with an existence result for the s-tolerant identifying collections in a graph.

Theorem 3 *There always exists an s-tolerant identifying collection $\mathcal{L}$ with $|\mathcal{L}| \leq \binom{|V|}{s}$ for any s in a graph $G = (V, E)$ provided that*

$$s < \min_{x \neq y} |N[x] \setminus N[y]|.$$

Moreover, if $s \geq \min_{x \neq y} |N[x] \setminus N[y]|$, then no s-tolerant identifying collection exists.

A (v, k, t) *covering design* is a family of k-subsets, called *blocks*, chosen from a set of v elements, such that each t-subset is contained in at least one of the blocks. The minimum size of such covering design is denoted by $C(v, k, t)$.

In the following theorem, we present a lower bound on the size of a self-identifying collection.

Theorem 4 *For $s \geq 1$, we have a lower bound*

$$|\mathcal{L}| \geq C(|V|, |V| - \gamma^+, s).$$

2 Smallest collections for two families of graphs

Let us first consider s-tolerant identifying collections in the Cartesian product $K_n \square K_m$, where K_n and K_m are complete graphs of order n and m, respectively. The graph $K_n \square K_m$ is also known as the rook's graph. Notice that the rook's graph can be viewed as a chess board with n *rows* and m *columns*, and the neighbourhood of a vertex is determined by the movement of a rook. Previously, identification and related problems in the rook's graphs have been considered, for example, in [2, 9]. In the following theorem, we give a characterization for self-identifying codes in $K_n \square K_m$ as well as determine the sizes of optimal codes.

Theorem 5 *Let $n \geq 2$ and $m \geq 2$ be integers. Then a code C is self-identifying in $K_n \square K_m$ if and only if there exist at least two codewords in each row and column of the graph. Moreover, we have the following result for the size of an optimal self-identifying code in the graph:*

$$\gamma^{+}(K_n \square K_m) = 2 \cdot \max\{m, n\}.$$

In the following theorem, we determine the smallest number of codes in an s-tolerant identifying collection in the rook's graph, when $s \geq 2$. In the case $s = 1$, it is clear that two disjoint self-identifying codes are enough. By Theorem 3, it is immediate that no s-tolerant identifying collection exists if $s \geq \min\{m, n\} - 1$.

Theorem 6 *Let n, m and s be integers such that $n, m, s \geq 2$, $n \geq m$ and $s \leq m - 2$. Now the following statements hold:*

(i) There exists an s-tolerant identifying collection in $K_n \square K_m$ with $C(m, m-2, s)$ optimal *self-identifying codes.*

(ii) Any s-tolerant identifying collection has at least $C(m, m-2, s)$ minimal self-identifying codes.

Let us then consider s-tolerant identifying collections in binary hypercubes of length n, denoted by $\mathbb{F}^n$. In this extended abstract, we restrict our study to the lengths $n = 2^r - 1$, where $r \geq 2$ is an integer. Combining Theorem 2 and [4], we obtain the following result.

Theorem 7 *Let $n \geq 2$ be an integer and C a code in $\mathbb{F}^n$. Then C is self-identifying if and only if C is a 3-fold 1-covering, i.e., $|I(x)| = |N[x] \cap C| \geq 3$ for all $x \in \mathbb{F}^n$. Moreover, if $r \geq 2$ is an integer and $n = 2^r - 1$, then we have the following result for the size of an optimal self-identifying code in $\mathbb{F}^n$:*

$$\gamma^{+}(\mathbb{F}^n) = 3 \cdot \frac{2^n}{n+1}.$$

In the following theorem, we determine the smallest number of codes in an s-tolerant identifying collection in the binary hypercube, when $s \geq 2$. As in the case of the rook's graph, two disjoint self-identifying codes are enough if $s = 1$. By Theorem 3 (as above), it is immediate that no s-tolerant identifying collection exists if $s \geq n - 1$.

Theorem 8 *Let $r, s \geq 2$ be integers such that $n = 2^r - 1$ and $s \leq n - 2$. Now the following statements hold:*

(i) There exists an s-tolerant identifying collection in $\mathbb{F}^n$ with $C(n+1, n-2, s)$ optimal *self-identifying codes.*

(ii) Any s-tolerant identifying collection has at least $C(n+1, n-2, s)$ minimal self-identifying codes.

Comparing our new model to existing ones, we first recall that previously, in [8], collections of *disjoint* identifying codes have been studied. As stated above, self-identifying codes are special type of identifying codes with the additional requirement that the observer can determine its location without comparison to other vertices. They also have the nice property that malfunctioning sensors can be detected solely based on the sets $I(x)$. In our model, we do not restrict ourselves to collections of disjoint codes. This has the benefit that we can handle greater values s of malfunctioning sensors. Indeed, if we consider binary hypercubes with $n = 2^r - 1$ and $r \geq 2$, then using disjoint collections of self-identifying codes the maximum number of malfunctioning sensors that can be handled is $s \leq \lfloor (n+1)/3 \rfloor$. By Theorem 8, in our model, we can handle up to $s \leq n - 2$ malfunctioning sensors.

Our model can also be compared to robust identifying codes, which have been designed to handle possible malfunctioning of sensors (see [3, 5, 6, 10]). As stated earlier, a single identifying code C is robust against s malfunctions if for all $x \neq y$ we have $|I(x) \triangle I(y)| \geq s+1$ and $|I(x)| \geq s + 1$. This implies that, for example in $\mathbb{F}^n$, we have the following lower bound for identifying codes robust against $s = n - 2$ malfunctions:

$$|C| \geq \frac{n-1}{n+1} \cdot 2^n = \left(1 - \frac{2}{n+1}\right) 2^n$$

This further means that almost all the sensors have to be on; hence implying high signal interference and energy consumption. However, in our model, if $n = 2^r - 1$, the size of (optimal) self-identifying code is only $\gamma^+(\mathbb{F}^n) = 3 \cdot 2^n/(n+1)$. Thus, giving significantly smaller construction, although now we have to be ready to switch the set of operating sensors if errors occur.

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Coloring and $L(2,1)$-labeling of unit disk intersection graphs

Konstanty Junosza-Szaniawski [1], *Paweł Rzążewski* [1,2], *Joanna Sokół* [1] and *Krzysztof Węsek* [1]

[1] Faculty of Mathematics and Information Science, Warsaw University of Technology, Poland
[2] Institute of Computer Science and Control, Hungarian Academy of Sciences, Hungary

Extended Abstract

1 Introduction

Intersection graphs of families of geometric objects attracted much attention of researches both for their theoretical properties and practical applications (c.f. McKee and McMorris [10]). For example intersection graphs of families of discs, and in particular discs of unit diameter (called *unit disk intersection graphs*), play a crucial role in modeling radio networks. Apart from the classical coloring, other labeling schemes such as T-coloring and distance-constrained labeling of such graphs are applied to frequency assignment in radio networks [9, 13].

In this paper we consider the classical coloring and the $L(2,1)$-labeling. The latter asks for a vertex labeling with non-negative integers, such that adjacent vertices get labels that differ by at least two, and vertices at distance two get different labels. The *span* of an $L(2,1)$-labeling is the maximum label used. The $L(2,1)$-*span* of a graph G, denoted by $\lambda(G)$, is the minimum span of an $L(2,1)$-labeling of G (note that the number of available labels is $\lambda(G)+1$, but some may not be used).

We say that a graph coloring algorithm is *on-line* if the input graph is not known a priori, but is given vertex by vertex (with all edges adjacent to already revealed vertices). Each vertex is colored at the moment when it is presented and its color cannot be changed later. On the other hand, *off-line* coloring algorithms know the whole graph before they start assigning colors. The on-line coloring can be much harder than off-line coloring, even for paths. For an off-line coloring algorithm (off-line $L(2,1)$-labeling algorithm, resp.), by the *approximation ratio* we mean the worst-case ratio of the number of colors used by this algorithm (the largest label used by this algorithm, resp.) to the chromatic number of the graph ($\lambda(G)$, resp.). For on-line algorithms, the same value is called the *competitive ratio.*

A unit disk intersection graph G can be colored off-line in polynomial time with $3\omega(G)$ colors [12] (where $\omega(G)$ denotes the size of a maximum clique) and on-line with $5\chi(G)$ colors [11, 12]. Fiala *et al.* [3] presented an on-line algorithm that finds an $L(2,1)$-labeling of a unit disk intersection graph with span not exceeding $25\omega(G)$. The algorithm is based on a special pre-coloring of the plane, that resembles colorings studied by Exoo [2], inspired by the classical Hadwiger-Nelson problem [8]. Our main results are on-line algorithms for coloring and $L(2,1)$-labeling of unit disc intersection graphs with better competitive ratios than previous algorithms. They are inspired by [3], although a b-fold coloring of the plane (see [7]) is used instead of a classical coloring. In particular, in the case of using 1-fold coloring we obtain the algorithm from [3]. Our algorithm colors (in the classical sense) unit disc intersection graphs with large maximum clique, using less than $5\omega(G)$ colors and hence it is the best currently known approximation on-line coloring algorithm for such graphs. For $L(2,1)$-labeling, in the case of 1-fold coloring of the plane, our algorithm gives a labeling with span not exceeding $20\omega(G)$. Using b-fold coloring for $b>1$ we obtain even better results.

For general graphs, Griggs and Yeh proved that $\lambda(G) \leq \Delta(G)^2 + 2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta(G)^2$. Shao *et al.*[14] showed $\lambda(G) \leq \frac{4}{5}\Delta(G)^2 + 2\Delta(G)$ if G is a unit disk intersection graph. Actually, they gave an on-line algorithm that finds an $L(2,1)$-labeling of G with span at most $\frac{4}{5}\Delta(G)^2 + 2\Delta(G)$. We managed to improve this bound to $\frac{3}{4}\Delta(G)^2 + 3(\Delta(G)-1)$, in the off-line case. Moreover, we show that the algorithm from [3] implies the bound $18\Delta(G) + 18$, which is better for $\Delta(G) \geq 22$.

Throughout the paper we always assume that the input unit disk intersection graph is given along with its geometric representation. In practical application for mobile Wi-Fi routers, a representation can be found with methods from [5].

2 Preliminaries

For an integer n, we define $[n] := \{1, \ldots, n\}$. A function $c\colon V \to [k]$ is a k-coloring of $G = (V, E)$ if for any $xy \in E$ holds $c(x) \neq c(y)$. By $d(u, v)$ we denote the number of edges on the shortest u-v–path in G.

For a sequence of unit discs in the plane $(D_i)_{i\in[n]}$ we define its intersection graph by $G((D_i)_{i\in[n]}) = (\{v_i : i \in [n]\}, E)$, where v_i is the center of D_i for every $i \in [n]$ and $v_iv_j \in E$ iff $D_{v_i} \cap D_{v_j} \neq \emptyset$. Notice that $v_iv_j \in E$ if and only if the Euclidean distance between v_i and v_j, denoted by $\mathrm{dist}(v_1, v_2)$, is at most one. By *UDG* we mean the class of graphs that admit a representation by intersecting unit disks.

For an on-line algorithm alg, let $\mathrm{alg}(G)$ be the value of the solution given by the algorithm for instance G and $\mathrm{opt}(G)$ be the optimal solution for instance G. If alg is a minimization algorithm, we denote by $\mathrm{cr}(\mathrm{alg})$ the *competitive radio* of alg, which is the supremum of $\frac{\mathrm{alg}(G)}{\mathrm{opt}(G)}$ over all instances G. For the classic coloring, we use the fact that any coloring requires at least $\omega(G)$ colors, where $\omega(G)$ denotes the size of the largest clique of G. By $\mathcal{G}_\omega$ we denote the class of graphs with largest clique of size at least ω and by $\mathrm{cr}(\mathrm{alg}(\mathcal{G}_\omega))$ we denote the supremum of $\frac{\mathrm{alg}(G)}{\mathrm{opt}(G)}$ over all graphs $G \in \mathcal{G}_\omega$.

A *tiling* is a partition of the plane into convex polygons with partially removed boundary, called *tiles*, such that every two points from one tile are at distance less than one. If we have b tilings, then by a *subtile* we mean a non-empty intersection of b tiles, one from each tiling. We will use a hexagon as a tile and hexagon tiling, just as Fiala *et al.* [3].

A function $\varphi : \mathbb{R}^2 \to [k]$ is called a *coloring of the plane with the color set* $[k]$ if for any two points $p_1, p_2 \in \mathbb{R}^2$ with $\mathrm{dist}(p_1, p_2) = 1$ holds $\varphi(p_1) \neq \varphi(p_2)$. A function $\varphi = (\varphi_1, \ldots, \varphi_b)$ where $\varphi_i : \mathbb{R}^2 \to [k]$ for $i \in [b]$ is called a *b-fold coloring of the plane with color set* $[k]$ if:
– for any point $p \in \mathbb{R}^2$ and $i, j \in [b]$, if $i \neq j$, then $\varphi_i(p) \neq \varphi_j(p)$;
– for any two points $p_1, p_2 \in \mathbb{R}^2$ with $\mathrm{dist}(p_1, p_2) = 1$ and $i, j \in [b]$ it holds $\varphi_i(p_1) \neq \varphi_j(p_2)$.
The function φ_i for $i \in [b]$ is called an *i-th layer* of φ. Notice that a coloring of the plane is a 1-fold coloring of the plane. A coloring of the plane φ is called tiling-based if there exists a tiling such that each tile is monochromatic and adjacent tiles have different colors. A b-fold coloring of a plane $\varphi = (\varphi_1, \ldots, \varphi_b)$ is called tiling-based if for every $i \in [b]$ coloring φ_i is tiling-based.

For technical reasons, we shall consider $L(2,1)$-labelings with labels starting with one. To avoid confusion, we shall call such labelings $L(2,1)$-*colorings*. A b-fold coloring of the plane φ is called a *b-fold* $L^*(2,1)$-*coloring of the plane with color set* $[k]$ if for any two points $p_1, p_2 \in \mathbb{R}^2$ the following two statements are true:
– if $\mathrm{dist}(p_1, p_2) = 1$ then for any $i_1, i_2 \in \{1, ..., b\}$ it holds $2 \leq |\varphi_{i_1}(p_1) - \varphi_{i_2}(p_2)| < k - 1$;
– if $1 < \mathrm{dist}(p_1, p_2) \leq 2$ then for any $i_1, i_2 \in \{1, ..., b\}$ it holds $1 \leq |\varphi_{i_1}(p_1) - \varphi_{i_2}(p_2)|$.
By $L^*(2,1)$-coloring of the plane we mean 1-fold $L^*(2,1)$-coloring of the plane.

3 On-line coloring

The main idea of the algorithm is as follows. We start with some fixed tiling-based b-fold coloring $\varphi = (\varphi_1, \ldots, \varphi_b)$ of the plane with colors $[k_\varphi]$. When a disc D is read, it is assigned to one of the b layers of φ (we try to distribute discs to layers as uniformly as possible). Then a tile from this layer that contains a center of D is found. The vertex corresponding to D

is colored with the color of this tile plus k_φ multiplied by the number of vertices previously assigned to this tile.

Algorithm $Color_\varphi((D_i)_{i\in[n]})$

1. ForEach $i \in [n]$
2. Read D_i, let v_i be the center of D_i
3. ForEach $r \in [b]$ let $T_r(v_i)$ be the tile from the layer r containing v_i
4. $\ell(v_i) \leftarrow 1 + (|\{v_1, \dots v_{i-1}\} \cap \bigcap_{r\in[b]} T_r(v_i)| \pmod b)$
5. $t(v_i) \leftarrow |\{u \in \{v_1, \dots v_{i-1}\} \cap T_{\ell(v_i)}(v_i) : \ell(u) = \ell(v_i)\}|$
6. $c(v_i) \leftarrow \varphi_{\ell(v_i)}(v_i) + k_\varphi \cdot t(v_i)$
7. Return c

Theorem 1 *Let φ be a tiling-based b-fold coloring of the plane with color set $[k_\varphi]$, and $(D_i)_{i\in[n]}$ be a sequence of unit discs. Algorithm $Color_\varphi((D_i)_{i\in[n]})$ returns a coloring of $G := G((D_i)_{i\in[n]})$ with the highest color not exceeding $k_\varphi \cdot \lfloor \frac{\omega(G)+(b-1)\gamma_\varphi}{b} \rfloor$, where γ_φ is the maximum number of subtiles in one tile. Moreover, if φ is a b-fold $L^*(2,1)$-coloring of the plane, then Algorithm $Color_\varphi((D_i)_{i\in[n]})$ returns an $L(2,1)$-coloring of G.*

This shows that is it crucial to construct good b-fold colorings of the plane.

Theorem 2 ([7]) *For $h \in \mathbb{N}_+$ there exists a tiling-based h^2-fold coloring of the plane with $\left\lceil (\frac{2}{\sqrt{3}} + 1) \cdot h \right\rceil^2$ colors and $\gamma_\varphi = 6h^2$.*

Directly from Theorems 1 and 2 we obtain:

Corollary 3 *For the h^2-fold φ coloring of the plane from Theorem 2 we have*

$$\mathrm{cr}(Color_\varphi(\mathcal{G}_\omega)) \le \frac{\left\lceil (\frac{2}{\sqrt{3}} + 1) \cdot h \right\rceil^2}{\omega} \cdot \left\lfloor \frac{\omega + (h^2-1)6h^2}{h^2} \right\rfloor = 4.65 + O\left(\frac{1}{h}\right) + O\left(\frac{h^4}{\omega}\right).$$

Notice that for $h = 5$ and graphs G with $\omega(G) \ge 108901$, the competitive ratio of the algorithm is less than 5.

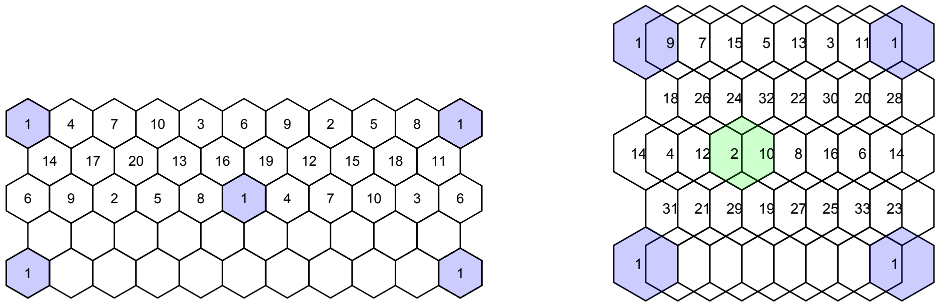
Analogously to Theorem 2, we are able to construct a good b-fold $L^*(2,1)$-coloring of the plane.

Theorem 4 *There exists b-fold tiling-based $L^*(2,1)$-coloring φ of the plane for*

1. *$b = 1$ with color set $[20]$ (see Figure 1a),*
2. *$b = 2$ with color set $[34]$ and the parameter $\gamma_\varphi = 4$ (see Figure 1b),*
3. *$b = 3$ with color set $[49]$ and the parameter $\gamma_\varphi = 6$,*
4. *$b = h^2$ for $h \in \mathbb{N}$ with $3\rho^2 + 1$ colors, where $\rho = \left\lceil h(\frac{2}{\sqrt{3}} + 1) + 1 \right\rceil$, and $\gamma_\varphi = 6h^2$*

Corollary 5 *For $b \in \mathbb{N}$ and b-fold $L^*(2,1)$-colorings φ of the plane from Theorem 4, the value $\mathrm{cr}(Color_\varphi(\mathcal{G}_\omega))$ is at most:*

1. *$10 + \frac{10}{2\omega-1}$ for φ from Theorem 4.1,*
2. *$8.5 + \frac{76.5}{2\omega-1}$ for φ from Theorem 4.2,*
3. *$8\frac{1}{6} + \frac{204.17}{2\omega-1}$ for φ from Theorem 4.3,*
4. *$(3\lceil h(\frac{2}{\sqrt{3}} + 1) + 1\rceil^2 + 1)\frac{\omega+6h^2(h^2-1)}{h^2(2\omega-1)}$
$= 6.97 + O\left(\frac{1}{h}\right) + O\left(\frac{h^4}{\omega}\right)$ for φ from Theorem 4.4.*



(a) 1-fold $L^*(2,1)$-coloring of the plane (b) 2-fold $L^*(2,1)$-coloring of the plane

Figure 1: Ilustration of Theorem 4

4 Off-line $L(2,1)$-labeling

In this section we give an improvement for a theorem by Shao *et al.* [14], which partially answers the question of Calamoneri [1, Section 4.7.1]. Namely, Shao *et al.* [14] proved that If $G \in UDG$, then $\lambda(G) \leq \frac{4}{5}\Delta^2 + 2\Delta$.

We obtained our first improvement by analysing a greedy colouring with respect to increasing x-coordinate. We showed that if $G \in UDG$ and $\Delta \geq 7$, then $\lambda(G) \leq \frac{3}{4}\Delta^2 + 3(\Delta - 1)$. Moreover, using result of Fiala *et al.* [3], we observed that a linear bound $\lambda(G) \leq 18\Delta + 18$ is true for $G \in UDG$. Combining our bounds with the bound $\lambda(G) \leq \Delta^2 + 2\Delta - 2$ by Gonçalves [4], we get the following theorem.

Theorem 6 *If $G \in UDG$, then $\lambda(G) \leq f(\Delta)$ for*

$$f(\Delta) = \begin{cases} \Delta^2 + 2\Delta - 2 & \textit{if } \Delta < 7, \\ \frac{3}{4}\Delta^2 + 3\Delta - 3 & \textit{if } 7 \leq \Delta < 22, \\ 18\Delta + 18 & \textit{if } \Delta \geq 22. \end{cases}$$

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Short cycle covers of signed graphs

Tomáš Kaiser [1], Robert Lukoťka [2], *Edita Máčajová* [2] and *Edita Rollová* [1]

[1] Department of Mathematics and NTIS, University of West Bohemia, Czech Republic
[2] FMFI, Comenius University, Slovakia

Extended Abstract

Introduction. A *circuit cover* of a bridgeless graph G is a collection of circuits such that each edge of G belongs to at least one of them. One of the most studied problems concerning circuit covers is finding a circuit cover with small total length for a given graph. A conjecture by Alon and Tarsi [1] bounds the length of the shortest circuit cover of a bridgeless graph from above.

Conjecture 1 (Short Cycle Cover conjecture) *Every bridgeless graph G has a circuit cover of total length at most* $7/5 \cdot |E(G)|$.

The best general upper bound on the length of shortest cycle cover is a result obtained independently by Bermond, Jackson, and Jaeger [2] and by Alon and Tarsi [1].

Theorem 2 *If G is a bridgeless graph, then it admits a circuit cover of total length at most* $5/3 \cdot |E(G)|$.

In parallel to the classical graph theory there is a fast-growing theory of signed graphs. A *signed graph* is a graph where each edge has either a positive or a negative sign. More formally, a signed graph (G, σ) is a graph G with a function σ called *signature* which assigns values either 1 (*positive edges*) or -1 (*negative edges*) to the edges. A *switching* at a vertex v is an operation that inverts exactly the signs on the edges incident to the vertex. Two signed graphs are *equivalent* if one can be obtained from another one by a series of switchings. Equivalent signed graphs are in many aspects considered to be the same and in the context of circuit covers we feel free to replace a signature of a signed graph with a signature of any equivalent signed graph. We say that a signature σ is a *minimal signature* of (G, σ) if there is no equivalent signed graph of (G, σ) with fewer negative edges.

We study a problem of short circuit covers for signed graphs introduced by Máčajová, Raspaud, Rollová and Škoviera in [6]. They defined a *signed circuit* of a signed graph to be one of the following subgraphs: (1) a *balanced circuit*, which is a circuit with even number of negative edges, (2) the union of two unbalanced circuits which meet at a single vertex—a *short barbell*, or (3) the union of two disjoint unbalanced circuits with a path which meets the circuits only at it ends—a *long barbell*. A *barbell* is either a short or a long barbell. It is easy to see that equivalent signed graphs have the same set of signed circuits.

The definition of signed circuit is chosen so that several important correspondences are preserved. A signed circuit in a signed graph G forms a signed circuit in the signed graphic matroid $\mathcal{M}(G)$ and vice versa [7, Theorem 5.1]. Moreover, a signed graph is *flow-admissible*, that is it admits a signed nowhere-zero flow, if and only if every edge of the signed graph belongs to a signed circuit [3, Proposition 3.1]. This implies that signed circuits are minimal graphs that are flow-admissible. All these properties are satisfied also for circuits in ordinary graphs.

Let $\mathcal{C}$ be a collection of signed circuits of a signed graph G. We say that $\mathcal{C}$ is a *signed circuit cover* if each edge of G is *covered by* $\mathcal{C}$ (i.e. it is contained in at least one subgraph from $\mathcal{C}$). The *length* of $\mathcal{C}$ is the sum $\sum_{F \in \mathcal{C}} |E(F)|$.

Máčajová et al. [6] proved that every flow-admissible signed graph with m edges has a signed circuit cover of length at most $11 \cdot m$. Cheng, Lu, Luo and Zhang [4] announced in 2015 an improvement of the bound to $14/3 \cdot m - 5/3 \cdot \epsilon - 4$ for flow-admissible signed graphs with ϵ negative edges in a minimal signature. A combination of their proof idea with our key lemmas enabled us to prove the bound $11/3 \cdot m$.

A *weighted* signed graph (G, σ) is a signed graph with a weight function that asigns each edge a non-negative real value. By $w(G)$ we denote the sum of weights of the edges of G. As the results of weighted signed graphs is slightly more general and it is technically convenient for us to use weighted graphs we state our main result as follows:

Theorem 3 *Let (G, σ) be a flow-admissible signed graph and let $w : E(G) \to \mathbb{R}^+$ be a weight. Then there is a signed circuit cover of (G, σ) with total weight at most $(3 + 2/3) \cdot w(G)$.*

To our knowledge, no weighted signed graph that requires weight more than $(1 + 2/3) \cdot w(G)$ to cover all the edges is known. The signed Petersen graph with five negative edges forming a 5-cycle and with edges of the same weight is an example of a graph that requires circuits of total weight at least $(1 + 2/3) \cdot w(G)$ to cover all the edges.

Covering auxiliary structures. We obtain the main improvement compared to the approach of Cheng, Lu, Luo and Zhang [4] by covering several auxiliary structures more effectively. Several ideas used in the proof of Theorem 3 are based on the work of Máčajová, Rollová, and Škoviera on signed circuit covers of Eulerian graphs [5].

Let $\mathcal{E}$ be a signed graph. Delete all bridges of $\mathcal{E}$ to obtain $\mathcal{E}^*$. If each component of $\mathcal{E}^*$ is either an isolated vertex or a circuit, then $\mathcal{E}$ is a *tree of circuits*. If each component of $\mathcal{E}^*$ is an Eulerian graph, then $\mathcal{E}$ is a *tree of Eulerian graphs*. Let L be the set of unbalanced loops of $\mathcal{E}^*$. The components of $\mathcal{E}^* - L$ together with loops in L are *balloons of* $\mathcal{E}$. A *leaf balloon* is a balloon of $\mathcal{E}$ that is either a loop from L or a component of $\mathcal{E}^* - L$ that is incident to either one bridge and no loops or one loop and no bridge. A balloon is an *even balloon* (*odd balloon*, respectively) if it contains an even (odd, respectively) number of negative edges. In case we know that a balloon is a circuit we use terms *leaf circuit*, *even circuit*, and *odd circuit*.

First, we examine trees of circuits.

Lemma 4 *Let $\mathcal{E}$ be a weighted tree of circuits of weight $w(\mathcal{E})$ with even number of odd circuits. Then there is a collection of signed circuits of weight at most $4/3 \cdot w(\mathcal{E})$ that covers all edges in the circuits of $\mathcal{E}$.*

Lemma 5 *Let $\mathcal{E}$ be a tree of circuits with at least two leaf circuits, and let every leaf circuit of $\mathcal{E}$ be odd. Then $\mathcal{E}$ admits a signed circuit cover that covers leaf circuits and bridges twice and all other edges once.*

We generalise the results to trees of Eulerian graphs.

Lemma 6 *Let $\mathcal{E}$ be a weighted tree of Eulerian graphs that contains even number of odd balloons. Then there is a collection of signed circuits of weight at most $4/3 \cdot w(\mathcal{E})$ that covers all edges in balloons of $\mathcal{E}$.*

The following lemma provides the crucial tool that allows us to improve the result of Cheng, Lu, Luo and Zhang [4].

Lemma 7 *Let $\mathcal{E}$ be a weighted tree of Eulerian graphs with at least two leaf balloons, and let every leaf balloon of $\mathcal{E}$ be odd. Then $\mathcal{E}$ admits a signed circuit cover of weight at most $2 \cdot w(\mathcal{E})$ that covers every loop of $\mathcal{E}$ exactly twice.*

Sketch of the proof of Lemma 7. We proceed by induction on size of $\mathcal{E}$. If $\mathcal{E}$ has a cut-vertex, we can easily split $\mathcal{E}$ as follows. If one of the parts is balanced, then we cover the part separately, and use induction hypothesis on the rest of the graph. If both parts are unbalanced, we add an unbalanced loop to each part and use induction hypothesis twice. The condition of the lemma that loops are covered twice, allows us to merge the covers of the parts to the cover of $\mathcal{E}$.

Thus we may assume that $\mathcal{E}$ is 2-vertex-connected ($\mathcal{E}$ may contains loops as we do not consider the incident vertex to be a cut-vertex). By the assumption of the lemma, $\mathcal{E}$ has at least two leaf balloons. Due to connectivity of $\mathcal{E}$, there is at most one balloon that is not a loop. Therefore $\mathcal{E}$ has at least one loop. If $\mathcal{E}$ has exactly one loop, a slight variation of Lemma 6 applies. The main part of the proof deals with the case when $\mathcal{E}$ has at least two loops.

Let a and b be two loops of $\mathcal{E}$. We find two internally vertex-disjoint paths P_1 and P_2 connecting the vertices of a and b. Let $\mathcal{E}^- = \mathcal{E} - P_1 - P_2 - a - b$. Note that the components of $\mathcal{E}^-$ are Eulerian graphs. If $\mathcal{E}^-$ is empty, then Lemma 5 applies. If the number of negative edges of a component $\mathcal{E}^-$ is even, then we cover the component separately by Lemma 6 and the rest of the graph by induction hypothesis. If a component E of $\mathcal{E}^-$ intersects one of the paths, say P_1, in two different vertices x and y, then we consider the segment of P_1 between x and y and a segment of an Eulerian tour of E between x and y containing the same number of negative edges as the path segment. The union of these subgraphs is an Eulerian graph with even number of negative edges that can be covered by Lemma 6. The rest of the graph is covered by induction hypothesis. If there are two components E_1 and E_2 of $\mathcal{E}^-$ such that E_1 intersects P_1 closer to a and P_2 closer to b compared to E_2, then, again, we are able to find an Eulerian subgraph with even number of negative edges and use the same trick. This analysis gives us a very precise characterisation of $\mathcal{E}$. By 2-vertex-connectivity, except for unbalanced loops and isolated vertices, each component of $\mathcal{E}^-$ intersects P_1 and P_2, both. We attach two loops to each component of $\mathcal{E}^-$ at the vertices where the component intersected P_1 and P_2 and we cover the components by induction hypothesis. It is straightforward to extend the covers of the modified components to the cover of $\mathcal{E}$.

Sketch of the proof of Theorem 3. Without loss of generality let G be connected. We follow the ideas of Cheng et al. [4]. Let σ be a minimal signature of G and X be the set of negative edges of (G, σ). If $X = \emptyset$, by Theorem 2 the result follows. If $|X| = 1$, then (G, σ) is not flow-admissible. Hence $|X| \geq 2$. Let B be the set of such edges b of (G, σ) that $G - b$ has two components, each of them being unbalanced. Note that since (G, σ) is flow-admissible, B is the set of all bridges of G. Moreover, as σ is minimal, we have $B \cap X = \emptyset$. Let S be the set of such edges s of (G, σ) that there exists a 2-edge-cut $\{s, t\}$, where $t \in X$. Note that $S \cap X = \emptyset$, because σ is minimal. Observe that $G - X - B - S$ is a bridgeless balanced graph. By Theorem 2, $G - X - B - S$ has a signed circuit cover of total weight at most $5/3 \cdot w(G - X - B - S)$.

Let T be a spanning tree of the connected graph $G - X$, and we denote $T \cup X$ by (G', σ'), where $\sigma' = \sigma|_{G'}$. Note that σ' may not be a minimal signature of (G', σ'). Let X', B', S' be defined on (G', σ') in the same way as X, B, S on (G, σ), respectively. We have $X' = X$, $B' \supseteq B$ and $S' \supseteq S$. We find a set of signed circuits $\mathcal{C}$ that covers the edges of $X' \cup B' \cup S'$ in the following lemma.

Lemma 8 *If $|X'| \geq 2$, then there is a set of signed circuits $\mathcal{C}'$ in G' of weight at most $2 \cdot w(G')$ that covers the edges of $X' \cup B' \cup S'$. Moreover, $\mathcal{C}'$ covers every unbalanced loop of (G', σ') exactly twice.*

The lemma can be easily reduced to 2-vertex-connected graphs, similarly as in the proof of Lemma 7. Thus $B' = \emptyset$. Let C_x be the circuit of $T \cup x$ for $x \in X'$. For $A \subseteq X'$, let C_A be the symmetric difference of all circuits C_a for $a \in A$. Note that $C_{X'}$ contains all edges of $X' \cup S'$.

Suppose first that G' contains at least two loops. We connect the components of $C_{X'}$ into several trees of Eulerian graphs in such a way that the trees that contain loops satisfy the conditions of Lemma 7 and trees without loops are Eulerian graphs with even number of negative edges. Lemma 7 and Lemma 6 give the desired circuit covers.

Suppose now that G' contains at most one loop. Cases when G' has even number of negative edges are straightforward, so we assume that $|X'|$ is odd. If G' contains one loop l, we choose two negative edges a and b. We cover $C_{X'-a}$ using Lemma 6 and we cover either $C_{a,b}$ or $C_{l,a}$ by a single signed circuit (we choose to cover $C_{a,b}$ if l is covered twice by the first cover, otherwise we choose $C_{l,a}$). This may produce a cover of length $(2 + 1/3) \cdot w(G')$, but up to several cases where $|X'|$ is small the choice of a, b, and the cover of $C_{X'-a}$ can be made in such a way that the total weight of the cover is at most $2 \cdot w(G')$. For the exceptional cases we find the covers satisfying Lemma 8 directly. The case when G' contains no loop is solved similarly.

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Chromatic number of ISK4-free graphs

Ngoc Khang Le[1]

LIP, École Normale Supérieure de Lyon, Lyon, France

Extended Abstract

We say that a graph G is H-free if G does not contain any induced subgraph isomorphic to H. For $n \geq 1$, denote by K_n the complete graph on n vertices. A graph is called *complete bipartite* (resp. *complete tripartite*) if its vertex-set can be partitioned into two (resp. three) non-empty stable sets that are pairwise complete to each other. If these two (resp. three) sets have size p, q (resp. p, q, r) then the graph is denoted by $K_{p,q}$ (resp. $K_{p,q,r}$). A subdivision of a graph G is obtained by subdividing edges into paths of arbitrary length (at least one). We say that H is *an ISK4 of a graph G* if H is an induced subgraph of G and H is a subdivision of K_4. A graph that does not contain any subdivision of K_4 is said to be *ISK4-free*. For instance, series-parallel graphs and line graph of cubic graphs are ISK4-free. A *triangle* is a graph isomorphic to K_3.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k such that G can be partitioned into k stable sets. Denote by $\omega(G)$ the size of a largest clique in G. A class of graphs $\mathcal{G}$ is *χ-bounded* with *χ-bounding function* f if, for every graph $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$. This concept was introduced by Gyárfás [2] as a natural extension of perfect graphs, that forms a χ-bounded class of graphs with χ-bounding function $f(x) = x$. The question is: which induced subgraphs need to be forbidden to get a χ-bounded class of graphs? One way to forbid induced structures is the following: fix a graph H, and forbid every induced subdivision of H. We denote by $Forb^*(H)$ the class of graphs that does not contain any induced subdivision of H. The class $\text{Forb}^*(H)$ has been proved to be χ-bounded for a number of graph H. Scott [6] proved that for any forest F, $\text{Forb}^*(F)$ is χ-bounded. In the same paper, he conjectured that $\text{Forb}^*(H)$ is χ-bounded for any graph H. Unfortunately, this conjecture has been disproved by [5]. However, there is no general conjecture on which graph H, $\text{Forb}^*(H)$ is χ-bounded. This question is discussed in [1]. We focus on the question when $H = K_4$. In this case, $\text{Forb}^*(K_4)$ is the class of ISK4-free graphs. Since K_4 is forbidden, proving that the class of ISK4-free graphs is χ-bounded is equivalent to proving that there exists a constant c such that for every ISK4-free graph G, $\chi(G) \leq c$. Remark that the existence of such constant was pointed out in [4] as a consequence of a result in [3], but it is rather large ($\geq 2^{2^{2^{25}}}$) and very far from these two conjectures:

Conjecture 1 (Lévêque, Maffray, Trotignon '2012 [4]) *Let G be an ISK4-free graph. Then $\chi(G) \leq 4$.*

Conjecture 2 (Trotignon, Vušković '2016 [8]) *Let G be an {ISK4, triangle}-free graph. Then $\chi(G) \leq 3$.*

No better upper bound is known even for the chromatic number of {ISK4, triangle}-free graphs. However, certain attempt is made toward these two conjectures. A *hole* of a graph is an induced cycle on at least four vertices. For $n \geq 4$, we denote by C_n the hole on n vertices. A *wheel* is a graph consisting of a hole H and a vertex $x \notin H$ which is adjacent to at least three vertices on H. The *girth* of a graph is the length of its smallest cycle. The optimal bound is known for chromatic number of {ISK4, wheel}-free graphs and {ISK4, triangle, C_4}-free graphs:

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Theorem 3 (Lévêque, Maffray, Trotignon '2012 [4]) *Any {ISK4, wheel}-free graph is 3-colorable.*

Theorem 4 (Trotignon, Vušković '2016 [8]) *Every ISK4-free graph of girth at least 5 contains a vertex of degree at most 2 and is 3-colorable.*

The proof of Theorems 3 and 4 relies on structural decomposition. Now, we need several definitions before stating some helpful decomposition lemmas. A *cutset* in a graph is a subset $S \subsetneq V(G)$ such that $G \setminus S$ is disconnected. For any $k \geq 0$, a k-cutset is a cutset of size k. A cutset S is a *clique cutset* if S is a clique. When $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. A *proper* 2-*cutset* of a graph G is a 2-cutset $\{a, b\}$ such that $ab \notin E(G)$, $V(G) \setminus \{a, b\}$ can be partitioned into two non-empty sets X and Y so that there is no edge between X and Y and each of $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ is not a path from a to b. In [4], it is proved that clique cutsets and proper 2-cutsets are useful for proving Conjecture 1 in the inductive sense. If we can find such a cutset in G, then we immediately have a bound for chromatic number of G, since $\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\}$, where G_1 and G_2 are two blocks of decomposition of G with respect to that cutset. Therefore, we only have to prove Conjecture 1 for the class of {ISK4, $K_{3,3}$, prism, $K_{2,2,2}$}-free graphs since the existence of $K_{3,3}$, prism or $K_{2,2,2}$ implies a good cutset by the following lemmas:

Lemma 5 (Lévêque, Maffray, Trotignon '2012 [4]) *Let G be an ISK4-free graph that contains $K_{3,3}$. Then either G is a complete bipartite or complete tripartite graph, or G has a clique cutset of size at most* 3.

Lemma 6 (Lévêque, Maffray, Trotignon '2012 [4]) *Let G be an ISK4-free graph that contains a rich square or a prism. Then either G is the line graph of a graph with maximum degree* 3*, or G is a rich square, or G has a clique cutset of size at most* 3 *or G has a proper* 2*-cutset.*

One way to prove Conjectures 1 and 2 is to find a vertex of small degree. This approach is successfully used in [8] to prove Theorem 4. Two following stronger conjectures will immediately imply the correctness of Conjectures 1 and 2:

Conjecture 7 (Trotignon '2015 [7]) *Every {ISK4, $K_{3,3}$, prism, $K_{2,2,2}$}-free graph contains a vertex of degree at most three.*

Conjecture 8 (Trotignon, Vušković '2016 [8]) *Every {ISK4, $K_{3,3}$, triangle}-free graph contains a vertex of degree at most two.*

However, we find a new bound for the chromatic number of ISK4-free graphs using another approach. Our main results are the following theorems:

Theorem 9 *Let G be an {ISK4, triangle}-free graph. Then $\chi(G) \leq 4$.*

Theorem 10 *Let G be an ISK4-free graph. Then $\chi(G) \leq 24$.*

Remark that the bounds we found are much closer to the bound of the conjectures than the known ones. The main tool that we use to prove these theorems is classical. It is often used to prove χ-boundedness results relying on the layers of neighborhood. The *distance* between two vertices x, y in $V(G)$ is the length of the shortest path from x to y in G. Let $u \in V(G)$ and i be an integer, denote by $N_i(u)$ the set of vertices of G that are of distance exactly i from u. Note that there is no edge between $N_i(u)$ and $N_j(u)$ for every i, j such that $|i - j| \geq 2$. We find a bound for $\chi(G)$ by looking at the forbidden structure in each layer $N_i(u)$ and the following lemma:

Lemma 11 *Let G be a graph and $u \in V(G)$. Then:*

$$\chi(G) \leq \max_{i\ odd} \chi(G[N_i(u)]) + \max_{j\ even} \chi(G[N_j(u)]).$$

Now, we describe briefly the proof of Theorems 9 and 10. For Theorem 9, we consider the class of {ISK4, triangle, $K_{3,3}$}-free graphs and prove that for any $i \geq 1$ and any $u \in V(G)$, $G[N_i(u)]$ does not contain any hole. Suppose that there is a hole C in some layer $N_i(u)$, we show that there must be some vertices from layer $N_{i-1}(u)$ which see very few neighbors on C (this is a bit technical). It can then be proved that connecting them through the upper layers yields an ISK4, contradiction. Therefore, there is no hole in every layer $N_i(u)$ and since our class is triangle-free, each layer induces a forest, the theorem follows by Lemma 11. Not only the bound we found is very close to the one stated in Conjecture 2, but the simple structure of each layer is also interesting. We believe that it is very promising to settle Conjecture 2 by this way of looking at our class.

Now, we introduce some definitions for the proof of Theorem 10. A *boat* is a graph consisting of a hole C and a vertex v that has exactly four consecutive neighbors in C. A 4-*wheel* is a particular boat when its hole is of length 4. In this proof, we try to look at the layers of neighborhood of our graph several times to exclude more and more structures. In the course of proving Theorem 10, we have some lemmas which might be of independent interest:

Lemma 12 *Let G be an {ISK4, $K_{3,3}$, prism, boat}-free graph. Then $\chi(G) \leq 6$.*

Lemma 13 *Let G be an {ISK4, $K_{3,3}$, prism, 4-wheel}-free graph. Then $\chi(G) \leq 12$.*

We believe that the bound 24 we found could be slightly improved by this method if we look at each layer more carefully and exclude more structures, but it seems hard to reach the desired bound mentioned in Conjecture 1.
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List star edge coloring of sparse graphs

Samia Kerdjoudj [1,2] and André Raspaud [3]

[1] University of Blida 1, Algeria.
[2] L'IFORCE, University of Sciences and Technology Houari Boumediene, Algeria.
[3] LaBRI, University of Bordeaux, France.

Extended Abstract

All the graphs we consider are finite and simple. For a graph G, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree, respectively.

A *proper* vertex (respectively, edge) coloring of G is an assignment of colors to the vertices (respectively, edges) of G such that no two adjacent vertices (respectively, edges) receive the same color. A *star coloring* of G is a proper vertex coloring of G such that the union of any two color classes induces a *star forest* in G, i.e. every component of this union is a star. The *star chromatic number* of G, denoted by $\chi_s(G)$, is the smallest integer k for which G admits a star coloring. This notion was first mentioned by Grünbaum [7] in 1973 (see also [6]).

In 2008, Liu and Deng [9] proposed to study a natural variation of the star coloring problem: coloring the edges of a graph under the same constraints (alternatively, to investigate the star chromatic number of line graphs). In this context, they defined a *star-edge coloring* of a graph G as a proper edge coloring such that every 2-colored connected subgraph of G is a path of length at most 3. In other words, we forbid bi-colored 4-cycles and 4-paths in G (by a d-path we mean a path with d edges). We say that G is star-edge k-colorable if it admits a star-edge coloring with k colors. The *star chromatic index* of G, denoted by $\chi'_{st}(G)$, is thus defined as the smallest integer k such that G is star-edge k-colorable.

A proper edge coloring of a graph G is called *acyclic* (respectively, *strong*) if there is no bi-colored cycle (respectively, 3-path) in G. The *acyclic* (respectively, *strong*) *chromatic index* of G, denoted by $\chi'_a(G)$ (respectively, $\chi'_s(G)$), is the minimum number of colors needed for a acyclic (respectively, strong) edge-coloring of G. Let us give an immediate remark : for any graph G, we have :

$$\chi'_a(G) \leq \chi'_{st}(G) \leq \chi'_s(G)$$

By using Lovász's Local Lemma [4], Liu and Deng [9] proved the following upper bound.

Theorem 1 *[9] For every G with maximum degree $\Delta \geq 7$, $\chi'_{st}(G) \leq \lceil 16(\Delta-1)^{\frac{3}{2}} \rceil$.*

The star-edge coloring has been recently investigated by many authors. Deng and Liu [2] (see also [1]) proved that any tree may be star-edge colored with $\lfloor \frac{3}{2}\Delta \rfloor$ colors. Moreover, the bound is tight. Bezegovà et al [1] showed that $\chi'_{st}(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 12$ for every outerplanar graph G, and conjectured that $\chi'_{st}(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ for every such G of maximum degree at least 3. In [3], Dvořák, Mohar and Šámal, showed that even determining the star chromatic index of the complete graph K_n with n vertices is a hard problem. They gave the following bounds:

$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log(n)}}}{\log n^{\frac{1}{4}}}.$$

They also studied the star chromatic index of *subcubic* graphs, that is, graphs with maximum degree at most 3. They proved that $\chi'_{st}(G) \leq 7$ for every subcubic graph G, and they made the following conjecture

Conjecture 2 *[3] For every subcubic graph G, $\chi'_{st}(G) \leq 6$.*

If the Conjecture 2 is confirmed, then this bound is best possible. Indeed, they saw that the cubic complete bipartie graph $K_{3,3}$ is not star-edge 5-colorable.

A natural generalization of star-edge coloring is the list star-edge coloring. An *edge list* L for a graph G is a mapping that assigns a finite set of colors to each edge of G. Given an edge list L for a graph G, we say that G is *L-star-edge colorable* if it has a star-edge coloring c such that $c(e) \in L(e)$ for every edge of G. The *list-star chromatic index*, $ch'_{st}(G)$, of a graph G is the minimum k such that for every edge list L for G with $|L(e)| = k$ for every $e \in E(G)$, G is L-star-edge colorable.

For any graph G, it is obvious that $ch'_{st}(G) \geq \chi'_{st}(G)$. Dvořák, Mohar and Šámal [3, Question 3] asked whether $ch'_s(G) \leq 7$ for every subcubic G? Kerdjoudj, Kostochka and Raspaud [8], gave a partial answer to this question. They proved that:

Theorem 3 *[8] For every subcubic graph G, $ch'_{st}(G) \leq 8$.*

They also gave sufficient conditions for the list-star chromatic index of a subcubic graph to be at most 5 and 6 in terms of the maximum average degree $\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}$.

Theorem 4 *[8] Let G be a subcubic graph.*

1. *If $mad(G) < \frac{7}{3}$ then, $ch'_{st}(G) \leq 5$.*
2. *If $mad(G) < \frac{5}{2}$ then, $ch'_{st}(G) \leq 6$.*

We complete Theorem 4 by proving:

Theorem 5 *Let G be a subcubic graph. If $mad(G) < \frac{30}{11}$ then, $ch'_s(G) \leq 7$.*

No bounds are known for general graphs except the one given for complete graphs given in [3]. We consider graphs with any maximum degree, we are able to give bounds for the chromatic index of sparse graphs in terms of the maximum degree. We prove the following Theorems:

Theorem 6 *Let G be a graph with maximum degree Δ.*

1. *If $mad(G) < \frac{7}{3}$ then $ch'_{st}(G) \leq 2\Delta - 1$.*
2. *If $mad(G) < \frac{5}{2}$ then $ch'_{st}(G) \leq 2\Delta$.*
3. *If $mad(G) < \frac{8}{3}$ then $ch'_{st}(G) \leq 2\Delta + 1$.*

The girth of a graph G is the size of a smallest cycle in G. As every planar graph with girth g satisfies $\mathrm{mad}(G) < \frac{2g}{g-2}$, Theorem 6 yields the following.

Corollary 7 *Let G be a planar graph with girth g.*

1. *If $g \geq 14$ then $ch'_{st}(G) \leq 2\Delta - 1$.*
2. *If $g \geq 10$ then $ch'_{st}(G) \leq 2\Delta$.*
3. *If $g \geq 8$ then $ch'_{st}(G) \leq 2\Delta + 1$.*

In this talk we will present our theorems and give some hints for the proofs.

Sketch of proof of the Theorem 6. We use an idea that we used already in [8]. Suppose on the contrary that Theorem 6 is not true. Let H be a minimum counterexample. In each of the cases, first of all we prove that H does not contain a vertex v adjacent to $d_H(v) - 1$ vertices of degree 1 (where $d_H(v)$ is the degree of a vertex v in H). After that, we consider a graph H^* obtained from H by deleting all vertices of degree 1. Then, we can

observe that H^* does not contains a 1-vertex and $\mathrm{mad}(H^*) < m$ (where $m \in \mathbb{R}$ is the value of the upper bound on the maximum average degree given by Theorem 6). Now, for each of the cases, we prove the non-existence of some set of subgraphs $\mathcal{C}$ in H^*. In the next step, we use a discharging procedure in H^* to get a contradiction and then prove the theorem.
We define the weight function $\omega : V(H^*) \longrightarrow \mathbb{R}$ by $\omega(x) = d(x) - m$. It follows from hypothesis on the maximum average degree, the total sum of weights must be strictly negative. We define discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we show that $\omega^*(x) \geq 0$ for all $x \in V(H^*)$. This lead us to the following contradiction :

$$0 \leq \sum_{x \in V(H^*)} \omega^*(x) = \sum_{x \in V(H^*)} \omega(x) < 0.$$

Therefore, such a counterexample cannot exist.

□

We will also give in this talk a bound for k-degenerate graphs.
Let $k \in \mathbb{N}$. A graph G is k-degenerate if every subgraph has minimum degree at most k. Recently, Wang [10] proved:

Theorem 8 *[10] The strong chromatic index for each k-degenerate graph with maximum degree Δ is at most $(4k-2)\Delta - k^2 + 2$.*

As a corollary, it holds immediately that for every integer k, a k-degenerate graph G is star-edge $((4k-2)\Delta - k^2 + 2)$-colorable. We improve this bound by proving:

Theorem 9 *The list star chromatic index for each k-degenerate graph with maximum degree Δ is at most $(3k-2)\Delta - k^2 + 2$.*

Since every K_4-minor free graph is 2-degenerate, we deduce the following corollary

Corollary 10 *If G is a K_4-minor free graph, then $ch'_{st}(G) \leq 4\Delta - 2$, where Δ is maximum degree of G.*

Remark 11 *In [5], it is proven that the strong chromatic index of any planar graph G is bounded by $4\Delta(G) + 4$. This gives a bound for the star chromatic index of planar graphs. It leads to the following question: Is-it possible to find a constant C (not too big) such that for any planar graphs $\chi'_{st}(G) \leq 2\Delta(G) + C$?*

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Automorphism Groups of Comparability Graphs

Pavel Klavík [1], and <u>*Peter Zeman*</u> [234]

[1] Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

[2] Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

[3] Department of Mathematics, University of West Bohemia, 306 14 Pilsen, Czech Republic

Extended Abstract

Comparability Graphs. A *comparability graph* X is derived from a poset: $xy \in E(G)$ if and only if $x < y$ or $x > y$. Equivalently, the edges of X have a transitive orientation $\rightarrow$: if $x \rightarrow y$ and $y \rightarrow z$, then $xz \in E(X)$ and $x \rightarrow z$; see Fig 1a. This class was first studied by Gallai [8] and we denote it by COMP.

An important structural parameter of a poset P is its *Dushnik-Miller dimension* [5]. It is the least number of linear orderings $L_1, \dots, L_k$ such that $P = L_1 \cap \dots \cap L_k$. (For a finite poset P, its dimension is always finite since P is the intersection of all its linear extensions.) Similarly, we define the *dimension* of a comparability graph X, denoted by $\dim(X)$, as the dimension of any transitive orientation of X. (Every transitive orientation has the same dimension.) By k-DIM, we denote the subclass consisting of all comparability graphs X with $\dim(X) \le k$. We get the following infinite hierarchy of graph classes: 1-$\mathsf{DIM} \subsetneq$ 2-$\mathsf{DIM} \subsetneq$ 3-$\mathsf{DIM} \subsetneq$ 4-$\mathsf{DIM} \subsetneq \cdots \subsetneq \mathsf{COMP}$. For instance, the bipartite graph of the incidence between the vertices and the edges of a planar graph belongs to 3-DIM [19].

Function and Permutation Graphs. An intersection representation of a graph assigns one set to each vertex in such a way that two vertices are adjacent if and only if their sets intersect. Intersection-defined graph classes are obtained when these sets are restricted to a particular class of (geometric) objects. For instance, *function graphs* (FUN) are intersection graphs of continuous real-valued function on the interval $[0, 1]$. *Permutation graphs* (PERM) are function graphs representable by linear functions called *segments* [1]; see Fig. 1b and c. They are related to comparability graphs: $\mathsf{FUN} = \mathsf{co}\text{-}\mathsf{COMP}$ [9] and $\mathsf{PERM} = \mathsf{COMP} \cap \mathsf{co}\text{-}\mathsf{COMP} =$ 2-DIM [6], where co-COMP are the complements of comparability graphs.

Automorphism Groups of Graphs. The automorphism group $\mathrm{Aut}(X)$ of a graph X describes its symmetries. Every *automorphism* is a permutation of the vertices which preserves adjacencies and non-adjacencies. Frucht [7] proved that every finite group is isomorphic to $\mathrm{Aut}(X)$ of some graph X. Many graph theory results rely on highly symmetrical graphs.

Definition 1 *For a graph class $\mathcal{C}$, let $\mathrm{Aut}(\mathcal{C}) = \{G : X \in \mathcal{C}, G \cong \mathrm{Aut}(X)\}$. The class $\mathcal{C}$ is called* universal *if every abstract finite group is in $\mathrm{Aut}(\mathcal{C})$, and* non-universal *otherwise.*

In 1869, Jordan [12] characterized the automorphism groups of trees (TREE):

Theorem 2 (Jordan [12]) *The class $\mathrm{Aut}(\mathsf{TREE})$ is defined inductively as follows:*

(a) $\{1\} \in \mathrm{Aut}(\mathsf{TREE})$.
(b) *If $G_1, G_2 \in \mathrm{Aut}(\mathsf{TREE})$, then $G_1 \times G_2 \in \mathrm{Aut}(\mathsf{TREE})$.*
(c) *If $G \in \mathrm{Aut}(\mathsf{TREE})$, then $G \wr \mathbb{S}_n \in \mathrm{Aut}(\mathsf{TREE})$.*

Graph Isomorphism Problem. This famous problem asks whether two input graphs X and Y are the same up to a relabeling. It obviously belongs to NP, and it is not known to be polynomially-solvable or NP-complete. It is related to computing generators of an automorphism group: X and Y are isomorphic if and only if there exists an automorphism swapping them in $X \dot\cup Y$, and generators of $\mathrm{Aut}(X)$ can be computed using $\mathcal{O}(n^4)$ instances

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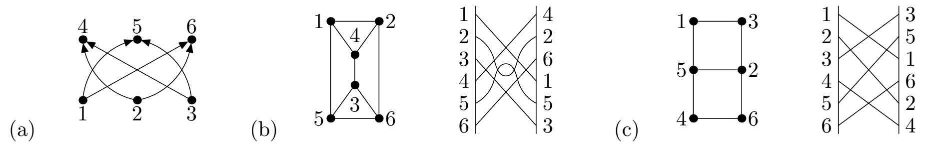

Figure 1: (a) A comparability graph with a transitive orientation. (b) A function graph and one of its representations. (c) A permutation graph and one of its representations.

of graph isomorphism [17]. By GI, we denote the complexity class of all problems that can be reduced to graph isomorphism in polynomial time.

As a rule of thumb, graph classes with very restricted automorphism groups have relatively easy polynomial-time algorithms solving their graph isomorphism; for instance, interval graphs [14, 15] and planar graphs [13, 10], circle graphs [14, 11] and permutation graphs [3]. On the other hand, when automorphism groups are very rich, testing graph isomorphism is GI-complete (e.g., chordal graphs [15], trapezoidal graphs [21]), or polynomial-time algorithms are quite involved (e.g., graphs of bounded degree [16]).

Our results. Since 1-DIM consists of all complete graphs, $\mathrm{Aut}(\textsf{1-DIM}) = \{\mathbb{S}_n : n \in \mathbb{N}\}$. The automorphism groups of 2-DIM = PERM are the following:

Theorem 3 *The class* $\mathrm{Aut}(\textsf{PERM})$ *is described inductively as follows:*

(a) $\{1\} \in \mathrm{Aut}(\textsf{PERM})$,
(b) *If* $G_1, G_2 \in \mathrm{Aut}(\textsf{PERM})$*, then* $G_1 \times G_2 \in \mathrm{Aut}(\textsf{PERM})$.
(c) *If* $G \in \mathrm{Aut}(\textsf{PERM})$*, then* $G \wr \mathbb{S}_n \in \mathrm{Aut}(\textsf{PERM})$.
(d) *If* $G_1, G_2, G_3 \in \mathrm{Aut}(\textsf{PERM})$*, then* $(G_1^4 \times G_2^2 \times G_3^2) \rtimes \mathbb{Z}_2^2 \in \mathrm{Aut}(\textsf{PERM})$.

In comparison to Theorem 2, there is the operation (d) which shows that $\mathrm{Aut}(\textsf{TREE}) \subsetneq \mathrm{Aut}(\textsf{PERM})$. Geometrically, the group $\mathbb{Z}_2^2$ in (d) corresponds to the horizontal and vertical reflections of a symmetric permutation representation.

Colbourn [3] described an $\mathcal{O}(n^3)$ algorithm for graph isomorphism of permutation graphs. This was improved by Spinrad [20] to $\mathcal{O}(n^2)$ by computing modular decompositions and testing tree isomorphism on them. The bottleneck of this algorithm is computing the modular decomposition, so by combining with [18], the running time is improved to $\mathcal{O}(n + m)$.

Combining this and the results of Mathon [17] one obtains a polynomial-time algorithm for computing automorphism groups of permutation graphs. We are not aware of any previous polynomial-time algorithm that computes automorphism groups of permutation graphs directly. We describe a linear-time algorithm, by combining the modular decomposition and computing automorphism groups of trees.

Corollary 4 *There exists a linear-time algorithm for computing automorphism groups of permutation graphs.*

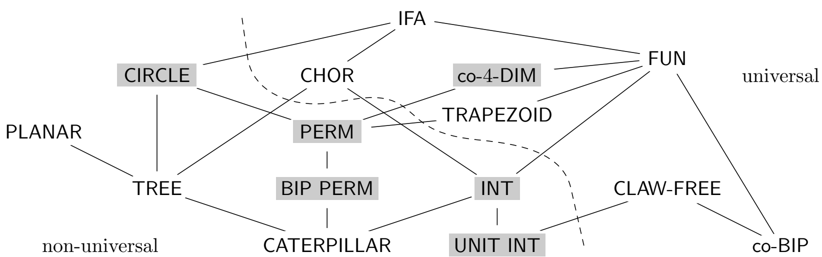

Figure 2: The inclusions between the considered graph classes. We characterize the automorphism groups of the classes in gray.

Further, using Theorem 3, our algorithm implicitely computes a structural decomposition of $\mathrm{Aut}(X)$, which is better and faster than outputting permutation generators of $\mathrm{Aut}(X)$. (These generators might be superlinear in size of X, but they can be computed compressed as in [4].) Also, our algorithm can be easily modified to solve graph isomorphism and canonization of permutation graphs, so it gives a more detailed description than [20].

Our characterization also easily gives the automorphism groups of *bipartite permutation graphs* (BIP PERM), in particular $\mathrm{Aut}(\mathsf{CATERPILLAR}) \subsetneq \mathrm{Aut}(\mathsf{BIP\ PERM}) \subsetneq \mathrm{Aut}(\mathsf{PERM})$. (For a description of Aut(CATERPILLAR), see [14].)

Corollary 5 *The class* Aut(connected BIP PERM) *consists of all abstract groups G_1, $G_1 \wr \mathbb{Z}_2 \times G_2 \times G_3$, and $(G_1^4 \times G_2^2) \rtimes \mathbb{Z}_2^2$, where G_1 is a direct product of symmetric groups, and G_2 and G_3 are symmetric groups.*

Comparability graphs are universal since they contain bipartite graphs; we can orient all edges from one part to the other. Since the automorphism group is preserved by complementation, FUN = co-COMP implies that also function graphs are universal. We explain the universality of FUN and COMP in more detail using the induced action on the set of all transitive orientations.

Bipartite graphs have arbitrarily large dimensions: the *crown graph* ($K_{n,n}$ without a matching) has the dimension n. We give a construction which encodes any graph X into a comparability graph Y with $\dim(Y) \leq 4$, while preserving the automorphism group.

Theorem 6 *For every $k \geq 4$, the class k-DIM is universal and its graph isomorphism is* GI*-complete. The same holds for posets of the dimension k.*

Yannakakis [23] proved that recognizing 3-DIM is NP-complete by a reduction from 3-coloring. For a graph X, a comparability graph Y is constructed with several vertices representing each element of $V(X) \cup E(X)$. It is proved that $\dim(Y) = 3$ if and only if X is 3-colorable. Unfortunately, the automorphisms of X are lost in Y since it depends on the labels of $V(X)$ and $E(X)$ and Y contains some additional edges according to these labels. We give a simple different construction which achieves only the dimension 4, but preserves the automorphism group: for a given graph X, we create Y by replacing each edge with a path of length eight. However, it is non-trivial to show that $Y \in 4$-DIM, and the constructed four linear orderings are inspired by [23]. A different construction follows from [2, 22].

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Partitioning graphs into induced subgraphs[1]

Dušan Knop

Department of Applied Mathematics (KAM), Charles University, Prague, Czech Republic

Extended Abstract

We begin with the definition of the Partition into H problem. We will then present the problem in the light of some well-known problems from computational complexity—for example Perfect Matching or Equitable Coloring—thus demonstrating it as a natural generalization of these and other problems. Finally, we give the summary of our results presented in this paper.

The Partition into H problem For graphs $G = (V, E), H = (W, F)$ with $|V| = |W| \cdot r$, we say that it is possible to *partition G into copies of H* if there exist disjoint sets $V_1, V_2, \ldots, V_r$ such that

- $\bigcup_{i=1}^{r} V_i = V$, and
- $G[V_i] \simeq H$ for every $i = 1, 2, \ldots, r$,

where by $G[V_i]$ we mean the subgraph of G induced by the set of vertices V_i. The corresponding decision problem is defined as follows.

Problem 1 (Partition into H)

FIXED:	*Pattern graph $H = (W, F)$.*
INPUT:	*Graph $G = (V, E)$ with $|V| = r \cdot |W|$ for an integer r.*
QUESTION:	*Is there a partition of G into copies of H?*

The complexity of the Partition into H problem has been studied by Hell and Kirkpatrick [7] and has been proven to be NP-complete for any fixed graph H with at least 3 vertices—they have studied the problem under a different name as the Generalized Matching problem. There are applications in the printed wiring board design [6] and code optimization [1].

Some variants of this problem are studied extensively in graph theory. For example when $H \simeq K_2$ the problem Partition into K_2 is the well known Perfect Matching problem, which can be solved in polynomial time due to Edmonds [3]—the algorithm works even for the optimization version, when one tries to maximize the number of copies of K_2 in G. The characterization theorem for $H \simeq K_2$, that is a characterization of graphs admitting a perfect matching is known due to Tutte [11].

Another frequently studied case of our problem is the Partition into K_3 problem—also known as the Triangle Partition problem. The Triangle Partition problem arises as a special case of the Set Partition problem (also known to be NP-complete [5]). Gajarský et al. [4] pointed out that the parameterized complexity of the Triangle Partition problem parameterized by the tree-width of the input graph was not resolved so far.

Parameterized complexity results. When dealing with an NP-hard problem it is usual to study the problem in the framework of parameterized complexity. While in the previous part we have introduced several problems of the classical complexity, here we give references to parameterized results for these problems.

A similar but more general problem (called the MSOL Partitioning problem) was studied by Rao [9]. Here the task is to partition the vertices of the graph G into several sets $A_1, A_2, \ldots, A_r$ such that $\varphi(A_i)$ holds for every $i = 1, 2, \ldots, r$, where $\varphi(\cdot)$ is an MSO_1 formula with one free set variable. If the number r and the clique-width $\mathrm{cw}(G)$ are fixed then the algorithm runs in polynomial time and hence the problem belongs to an XP class with parameterization by the clique-width.

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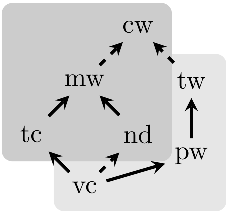

Figure 1: A map of assumed parameters. A full arrow stands for a linear upper bound, while a dashed arrow stands for an exponential upper bound. For example if a graph G has $\mathrm{vc}(G) \le k$ then $\mathrm{nd}(G) \le 2^k + k$.

Graph parameters and width

Definition 2 (Vertex cover) *For a graph $G = (V, E)$ the set $U \subset V$ is called a* vertex cover *of G if for every edge $e \in E$ it holds that $e \cap U \neq \emptyset$.*

The vertex cover number *of a graph, denoted as* $\mathrm{vc}(G)$, *is the least integer k for which there exists a vertex cover of size k.*

As the vertex cover number is (usually) too restrictive, many authors focused on defining other structural parameters. Three most well-known parameters of this kind are the path-width, the tree-width and the clique-width. Classes of graphs with bounded tree-width (respectively path-width) are contained in the so called sparse graph classes.

There are (more recent) structural graph parameters which also generalize the vertex cover number but in contrary to the tree-width these parameters focus on dense graphs. First, up to our knowledge, of these parameters is the *neighborhood diversity* defined by Lampis [8]. We denote the neighborhood diversity of a graph $G = (V, E)$ as $\mathrm{nd}(G)$.

We say that two (distinct) vertices u, v are of the same *neighborhood type* if they share their respective neighborhoods, that is when $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

Definition 3 (Neighborhood diversity [8]) *A graph $G = (V, E)$ has* neighborhood diversity *at most w (*$\mathrm{nd}(G) \le w$*), if there exists a partition of V into at most w sets (we call these sets* types*) such that all the vertices in each set have the same neighborhood type.*

Both previous approaches are generalized by a modular-width, defined by Gajarský et al. [4]. Here we deal with graphs created by an algebraic expression that uses four following operations:

1. create an isolated vertex,
2. the disjoint union of two graphs, that is from graphs $G = (V, E), H = (W, F)$ create a graph $(V \cup W, E \cup F)$,
3. the complete join of two graphs, that is from graphs $G = (V, E), H = (W, F)$ create a graph $(V \cup W, E \cup F \cup \{\{v, w\} \colon v \in V, w \in W\})$, note that the edge set of the resulting graph can be also written as $E \cup F \cup V \times W$.
4. The substitution operation with respect to a template graph T (for an example see Figure 2) with vertex set $\{v_1, v_2, \dots, v_k\}$ and graphs $G_1, G_2, \dots, G_k$ created by algebraic expression. The substitution operation, denoted by $T(G_1, G_2, \dots, G_k)$, results in the graph on vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$ and edge set
$$E = E_1 \cup E_2 \cup \cdots \cup E_k \cup \bigcup_{\{v_i, v_j\} \in E(T)} \{\{u, v\} \colon u \in V_i, v \in V_j\},$$
where $G_i = (V_i, E_i)$ for all $i = 1, 2, \dots, k$.

Definition 4 (Modular-width [4]) *Let A be an algebraic expression that uses only operations* 1–4. *The* width of expression *A is the maximum number of operands used by any occurrence of operation* 4 *in A.*

The modular-width *of a graph G, denoted as* $\mathrm{mw}(G)$, *is the least positive integer k such that G can be obtained from such an algebraic expression of width at most k.*

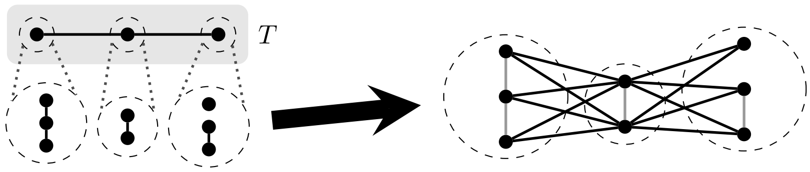

Figure 2: An illustration of the modular-width decomposition of a graph. A schema of a decomposition is depicted in the left part of the picture. In the right part of the picture there is the resulting graph—gray edges represent edges from the previous step of the decomposition (with template graph T). The resulting graph has nd $= 5$ (split the leftmost and the right most parts of the graph into two to obtain neighborhood diversity decomposition) and by the decomposition mw $= 3$.

When a graph H is constructed by the fourth operation, that is $G = T(G_1, G_2, \ldots, G_k)$, we call the graph T the *template graph*. An algebraic expression of width $\mathrm{mw}(G)$ can be computed in linear time [10].

Our contribution

Our first algorithm is based on the celebrated theorem of Courcelle [2]—an usual starting point for parameterized algorithm design. We would like to point out, that even though the result follows easily, the application is not straightforward.

Theorem 5 *For any fixed connected graph H the* Partition into H *problem is expressible by an MSO_2 formula.*

Thus, we have solved an open question raised by Gajarský et al. [4].

As the first algorithm is for graphs with bounded tree-width and thus a sparse class of graphs, we also analyze some variants of the Partition into H problem for a particular class of dense graphs. Many parameters are suitable for dense graph classes, such as neighborhood diversity and modular-width. We say that a graph H is a *prime graph* if no two vertices of H have the same neighborhood (the set of adjacent vertices). The class of prime graphs is thoroughly studied in the context of modular decompositions. Equivalently a graph H is prime graph if its neighborhood diversity equals the number of its vertices. In order to present our result we have to introduce the following strengthening of prime graphs.

Definition 6 (Strict prime graph) *We say that a graph $G = (V, E)$ is* strict prime graph *if for every subset of vertices $U \subsetneq V$ with at least two vertices there exist $v \in V \setminus U$ such that v is connected to at least one vertex in U and v is not connected to at least one vertex in U.*

Indeed, if G is a strict prime graph, then G is a prime graph. To see this assume that a prime graph contains two vertices, say u, v, with same neighborhood. If we take $U = \{u, v\}$ we have contradiction with Definition 6. On the other hand take any strict prime graph $G = (U, E)$ and add to G a new vertex v and attach v to all vertices of G. It is not hard to see that the resulting graph G' is prime graph. However, G' is not strict prime graph as witnessed by the set U.

Theorem 7 *For any fixed strict prime graph H and a graph G the* Partition into H *problem belongs to the* FPT *class when parameterized by modular width of graph G.*

It is worth to mention that even though the condition on strict prime graphs may seem very restrictive this class of graphs contains for example paths P_k on $k \geq 4$ vertices and cycles C_k on $k \geq 5$ vertices. Applications of the Partition into H problem with H being a path may be found in code optimization [1].

When it is shown that there is an FPT-algorithm for some problem, it is natural to ask, whether the problem admits a polynomial kernel—that is a preprocessing routine running in polynomial time which outputs an equivalent instance of size polynomially bounded in the assumed parameter. We prove that the Partition into H problem does not have polynomial kernel parameterized by modular-width for any reasonable graph H, that is when H has at least 3 vertices and thus the Partition into H problem is NP-hard. More precisely, we prove the following.

Theorem 8 *For any fixed graph H with at least 3 vertices. There is no polynomial kernel routine for the* Partition into *H problem when parameterized by modular width of graph G unless* $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

Theorem 9 *For any fixed graph H there is an* FPT*-algorithm for the* Partition into *H problem parameterized by neighborhood diversity of graph G.*

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On a conjecture on k-Thue sequences

Borut Lužar [1], *Martina Mockovčiaková* [2], *Pascal Ochem* [3], *Alexandre Pinlou* [3], and *Roman Soták* [4]

[1] Faculty of Information Studies, Novo mesto, Slovenia
[2] University of West Bohemia, Pilsen, Czech Republic
[3] Université Montpellier 2, Montpellier, France
[4] P. J. Šafárik University in Košice, Slovakia

Extended Abstract

A *repetition* in a sequence S is a subsequence $\xi_1 \dots \xi_t \xi_{t+1} \dots \xi_{2t}$ of consecutive terms of S such that $\xi_i = \xi_{t+i}$ for every $i = 1, \dots, t$. A sequence is called *nonrepetitive* or *Thue* if it does not contain a repetition of any length.

In 1906 Thue [6] showed that using only three symbols one can construct an arbitrarily long sequence without a repetition. His famous work attracted a considerable attention and later many applications have been found in various fields of science.

Let us consider the following generalization of Thue sequences introduced by Currie and Simpson [1]: a (possibly infinite) sequence S is *k-Thue* (or *nonrepetitive up to mod k*) if every j-subsequence of S is Thue, for $1 \leq j \leq k$. Here, a *j-subsequence* of S is a subsequence $\xi_i \xi_{i+j} \xi_{i+2j} \dots$, for any i. Notice that a 1-Thue sequence is simply a Thue sequence. As an example, consider a sequence $a\,b\,d\,c\,b\,c$, which is Thue, but not 2-Thue, since the 2-subsequence $b\,c\,c$ is not Thue. On the other hand, $a\,b\,c\,a\,d\,b$ is 2-Thue, but not 3-Thue.

Currie and Simpson [1] introduced this notion in connection with *nonrepetitive tilings*, i.e. assignments of symbols to the lattice points of the plane such that all lines in prescribed directions are nonrepetitive.

A natural question arises what is the minimum number of symbols required to construct an arbitrarily long k-Thue sequence. In [1], the authors showed that four symbols are enough to create 2-Thue sequences and five symbols suffice for 3-Thue sequences of arbitrary lengths. The lower bound on the number of symbols needed to construct a k-Thue sequence of an arbitrary length is obvious; for a positive integer k, at least $k + 2$ symbols are required to construct such sequences.

In 2002 Grytczuk conjectured that, in fact, the upper bound is equal to the lower bound for any k.

Conjecture 1 (Grytczuk, 2002 [3]) *For any k, $k + 2$ symbols suffices to construct a k-Thue sequence.*

This conjecture is hence true for $k = 2, 3$, and as shown in [2] also for $k = 5$. However, the upper bound for general k is still far from the conjectured. The bound of $e^{33}k$ established in [3] was substantially improved to $2k + O(\sqrt{k})$ in [4]. The authors in [5] presented a construction of arbitrarily long k-Thue sequences using at most $2k$ symbols which improves the previous bounds and currently it is the best known upper bound.

Theorem 2 (Kranjc et al., 2015 [5]) *For an arbitrary $k \geq 3$, there is an arbitrarily long k-Thue sequence using at most $2k$ symbols.*

Notice that this proof is constructive and provides a k-Thue sequence of a given length.

Recently, we solved Conjecture 1 for the cases $k = 4$ and 6 in two ways.

Theorem 3 (Lužar, Mockovčiaková, Ochem, Pinlou, Soták, 2016) *There is an arbitrarily long k-Thue sequence using $k + 2$ symbols for $k = 4$ and 6.*

In this talk we present constructions of 4- and 6-Thue sequences using 6 and 8 symbols, respectively.

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Parameterized complexity of fair deletion problems[1]

Tomáš Masařík[2] and *Tomáš Toufar*

Computer Science Institute of Charles University in Prague, Czech Republic

Extended Abstract

The deletion problems are a standard reformulation of some classical combinatorial problems and were examined by Yannakakis [5, 6]. For a graph property π we can formulate an *edge-deletion problem* in the following way. Given a graph $G = (V, E)$, find a set of edges F such that the graph $G' = (V, E \setminus F)$ satisfies the property π. The *vertex deletion* problem can be formulated analogously.

Additionally, the goal is to find smallest such a set. Many classical combinatorial problems can be formulated in this way, e.g. Minimum vertex cover, Maximum matching, or Minimum feedback arc set. For example, Minimum vertex cover can be reformulated as a deletion problem in the following way: find smallest set of vertices such that the graph obtained by its deletion contains no edge.

The fair deletion problems change the goal – instead of finding the smallest set, we want to find a set that is smallest in a local sense. More precisely, in Fair edge-deletion problem we are given a graph $G = (V, E)$ and a property π and we want to find a set $F \subseteq E$ which minimizes the maximum degree of graph $G^* = (V, F)$ where graph $G' = (V, E \setminus F)$ has the property π. Fair vertex-deletion problem is defined in similar manner.

In our work we only deal with graph properties definable in monadic second order logic (MSO for short). We also focus on an extension of fair deletion problems that was studied by Kolman et al. [2]. In this extension, we are given a formula with one free set-variable representing the set of removed edges F instead of a sentence describing the resulting graph $G \setminus F$. This version is slightly more general, as we can impose constraints not only on the graph $G \setminus F$, but also on the set F itself.

The formal definitions of Fair vertex-deletion problem and Fair edge-deletion problem follow.

Fair MSO edge-deletion

Input:	An undirected graph G, an MSO formula ψ with one free edge-set variable, and a positive integer k.
Question:	Is there a set $F \subseteq E(G)$ such that $G \models \psi(F)$ and for every vertex v of G, the number of edges in F incident with v is at most k?

Fair MSO vertex-deletion

Input:	An undirected graph G, an MSO formula ψ with one free vertex-set variable, and a positive integer k.
Question:	Is there a set $W \subseteq V(G)$ such that $G \models \psi(W)$ and for every vertex v of G, it holds that $\lvert N(v) \cap W \rvert \leq k$?

The following notions are useful when discussing fair edge-deletion problems. Given a graph $G = (V, E)$, the *fair cost of a set* $F \subseteq E$ is defined as $\max_{v \in V} \lvert \{e \in F \mid v \in e\} \rvert$. The function that assigns each set F its fair cost is referred to as *fair objective function*. Same notions can be defined for fair vertex-deletion problems. The *fair cost of a set* $W \subseteq V$ is defined as $\max_{v \in V} \lvert N(v) \cap W \rvert$. The *fair objective function* is defined analogously. Fair deletion problems can be rephrased as finding a set satisfying given property that minimizes the fair objective function.

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Courcelle and Mosbah [1] introduced a semiring homomorphism framework that can be used to minimize various function over all sets satisfying a given MSO formula. It is therefore natural to ask if we can use this framework to come up with algorithms for fair deletion problems. However, direct use of semiring homomorphism framework does not lead to an FPT algorithm for fair deletion problems; there is a natural choice of semiring that captures the fair objective function, but the size of such semiring is $n^{\mathrm{tw}(G)}$, so we obtain only XP algorithm in this way. It is unlikely that we can obtain an FPT algorithm by a more clever choice of semirings, as our results exclude the existence of an FPT algorithm under reasonable assumptions.

Known results

The fair deletion problems were introduced by Lin an Sahni [4]. In this paper, they showed that fair deletion problems are NP-hard for certain graph properties.

The main result concerning the parameterized complexity of fair deletion problems is an XP algorithm for FAIR EDGE-DELETION PROBLEMS for parameterization by tree-width due to Kolman et al. [2].

Our Contribution

Before going through our results, let us list structural graph parameters used throughout the paper:

- the *tree-width* of a graph, denoted by $\mathrm{tw}(G)$,
- the *path-width* of a graph, denoted by $\mathrm{pw}(G)$,
- the *size of minimum feedback vertex set* of a graph, denoted by $\mathrm{fvs}(G)$,
- the *neighborhood diversity* of a graph, denoted by $\mathrm{nd}(G)$ (introduced by Lampis [3]), and
- the *size of minimum vertex cover*, denoted by $\mathrm{vc}(G)$.

In the paper, we show the following hardness results.

Theorem 1 *If there is an FPT algorithm for* FAIR MSO_2 EDGE-DELETION *parameterized by the size of the formula ψ, the path-width of G, and the size of smallest feedback vertex set of G combined, then* $\mathsf{FPT} = \mathsf{W}[1]$. *Moreover, let k denote* $\mathrm{pw}(G) + \mathrm{fvs}(G)$. *If there is an algorithm for* FAIR MSO_2 EDGE-DELETION *with running time $f(|\psi|, k)n^{o(\sqrt{k})}$, then Exponential Time Hypothesis fails.*

Theorem 2 *If there is an FPT algorithm for* FAIR MSO_1 VERTEX-DELETION *parameterized by the size of the formula ψ, the path-width of G, and the size of smallest feedback vertex set of G combined, then* $\mathsf{FPT} = \mathsf{W}[1]$. *Moreover, let k denote* $\mathrm{pw}(G) + \mathrm{fvs}(G)$. *If there is an algorithm for* FAIR MSO_1 VERTEX-DELETION *with running time $f(|\psi|, k)n^{o(\sqrt{k})}$, then Exponential Time Hypothesis fails.*

In other words, we rule out the existence of an FPT algorithm with respect to given parameters under the assumption $\mathsf{W}[1] \neq \mathsf{FPT}$. If we assume that the Exponential Time Hypothesis holds (this is stronger assumption than $\mathsf{W}[1] \neq \mathsf{FPT}$), then we rule out even existence of an XP algorithms with running time faster than $f(k)n^{o(\sqrt{k})}$.

Note that since tree-width can be bounded by either path-width or the size of minimum feedback vertex set, we obtain corresponding result for parameterization by tree-width.

Corollary 3 *If there is an FPT algorithm for* Fair MSO_2 edge-deletion *parameterized by the size of the formula* ψ *and the tree-width of* G *combined, then* $\mathsf{FPT} = \mathsf{W}[1]$. *Furthermore, if there is an algorithm for* Fair MSO_2 edge-deletion *with running time* $f(|\psi|, \mathrm{tw}(G))n^{o(\sqrt{\mathrm{tw}(G)})}$, *then the Exponential Time Hypothesis fails.*

Corollary 4 *If there is an FPT algorithm for* Fair MSO vertex-deletion *parameterized by the size of the formula* ψ *and the tree-width of* G *combined, then* $\mathsf{FPT} = \mathsf{W}[1]$. *Furthermore, if there is an algorithm for* Fair MSO vertex-deletion *with running time* $f(|\psi|, \mathrm{tw}(G))n^{o(\sqrt{\mathrm{tw}(G)})}$, *then the Exponential Time Hypothesis fails.*

In addition to these hardness results, we also show that there are FPT algorithms for certain parameterizations.

Theorem 5 Fair MSO_1 vertex-deletion *is in* FPT *with respect to the neighborhood diversity* $\mathrm{nd}(G)$ *and the size of the formula* ψ.

Theorem 6 Fair MSO_2 edge-deletion *is in* FPT *with respect to the minimum size of vertex cover* $\mathrm{vc}(G)$ *and the size of the formula* ψ.

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Covering signed eulerian graphs with signed circuits

Edita Máčajová [1], *Edita Rollová* [2] and *Martin Škoviera* [1]

[1] Faculty of mathematics, physics and informatics, Comenius University, Bratislava, Slovakia
[2] NTIS- New Technologies for Information Society, Faculty of applied sciences, University of West Bohemia, Czech republic

Extended Abstract

A *circuit cover* of a graph G is a collection of circuits of G such that every edge of G belongs to at least one of them. The *length* of a circuit cover is the sum of lengths of its circuits. It is a natural problem to look for a circuit cover of minimum length. In 1985, Alon and Tarsi [1] independently conjectured that every bridgeless graph G admits a circuit cover of length at most $7/5 \cdot |E(G)|$. The best approximation of this conjecture to this date was given in 1983 by Bermond, Jackson and Jaeger [2] who established the ratio 5/3 instead of 7/5. Note that the requirement that G be bridgeless is necessary – a bridge of a graph does not belong to any circuit.

The problem of finding a short circuit cover has its natural counterpart in the theory of signed graphs. Somewhat surprisingly, a systematic study of signed circuit covers has been initiated only very recently by Máčajová, Raspaud, Rollová and Škoviera [4]. Recall that a *signed graph* is a graph G together with a mapping, the *signature* of G, which assigns $+1$ or -1 to each edge. The sign of each edge will usually be known from the immediate context, therefore no special notation for the signature will be used. A circuit C of a signed graph G is said to be *balanced* if it has an even number of negative edges; otherwise C is *unbalanced.* The signature should be regarded only as a means of introducing the concept of balance: two signatures that give rise to the same list of balanced circuits are therefore considered as *equivalent.* A signed graph G is said to be *balanced* if each circuit in G is balanced; otherwise G is unbalanced. It is obvious that any balanced signed graph is equivalent to the one with each edge positive.

A signed circuit cover is defined similarly as a circuit cover, however, we need to define a concept of a signed circuit, which is a natural signed analogue of a circuit. As argued in [4], a *signed circuit* is one of three types of signed graphs: (1) a balanced circuit, (2) a union of two disjoint unbalanced circuits connected by a path which intersects the circuits only at its end-vertices, and (3) a union of two unbalanced circuits that share exactly one vertex. Signed circuits of type (2) and (3) are often referred to as *barbells.* Finally, a *signed circuit cover* of a signed graph G is a collection of signed circuits such that each edge of G belongs to at least one of them. The *length* of a signed circuit cover is the sum of lengths of its signed circuits. Note that a signed circuit cover of an all-positive signed graph coincides with a circuit cover of the underlying unsigned graph. Thus the problem of finding a shortest circuit cover of a graph is a special case of the problem of finding a shortest signed circuit cover of a signed graph.

Theory of signed circuit covers is closely related to the theory of nowhere-zero integer flows on signed graphs, see [4]. In particular, a signed graph admits a signed circuit cover if and only if it admits a nowhere-zero integer flow; such a signed graph is called *flow-admissible.* The flow condition for the existence of a signed circuit cover is completely analogous to the unsigned case, with one significant difference: flow-admissible signed graphs may have more complicated structure. For example, there exist bridgeless signed graphs that fail to have a nowhere-zero integer flow.

Máčajová et al. [4] proved that a flow-admissible signed graph admits a signed circuit cover of length at most $11 \cdot |E(G)|$. This bound has been recently improved by Cheng, Lu, Luo, and Zhang [3] to approximately $4 \cdot |E(G)|$, which is, perhaps, still far from the optimal bound expected to be smaller than $2 \cdot |E(G)|$. The signed Petersen graph whose negative edges form a 5-circuit is known to require a signed circuit cover of length $5/3 \cdot 15$ (see [4]), the fraction 5/3 being the largest covering ratio of a signed graph known so far.

As the proofs of Máčajová et al. [4] and Zhang et al. [3] suggest, for finding a short signed circuit cover it is crucial to cover signed eulerian graphs efficiently. Covering an unsigned eulerian graph by circuits is trivial because every eulerian graph can be decomposed into circuits. However, this is not the case for signed graphs and, as our main result shows, the situation for signed eulerian graphs is much more complicated.

Theorem 1 *Let G be a signed eulerian graph. Then*

(i) *G does not admit a signed circuit cover if and only if it is unbalanced and contains an edge e such that $G - e$ is balanced;*

(ii) *If G has an even number of negative edges, then it is flow-admissible and admits a signed circuit cover of length at most $4/3 \cdot |E(G)|$ – this bound is tight;*

(iii) *If G has an odd number of negative edges and is flow-admissible, then it admits a signed circuit cover of length at most $3/2 \cdot |E(G)|$, and this bound is tight.*

The bound of (ii) is tight for at least two different infinite families of signed graphs. Here we provide an example of one of them (see Figure 1; dashed lines in the figures represent negative edges).

Example 2 Consider the signed graph G_n resulting from the star $K_{1,n}$, with $n \geq 2$, by replacing each edge with a pair of positive parallel edges and attaching a negative loop to every vertex which was pendant in $K_{1,n}$ (see Figure 1). Clearly, G_n has $3n$ edges. Let $\mathcal{C}$ be a shortest signed circuit cover of G_n. We show that at least n edges of G_n must be covered twice. There are two ways how a balanced circuit C of G_n can be covered by $\mathcal{C}$. Either C itself belongs to $\mathcal{C}$ or else each edge of C is contained in a barbell from $\mathcal{C}$. If $C \in \mathcal{C}$, then a barbell that covers the loop incident with C contains an edge of C; this edge is covered twice. If $C \notin \mathcal{C}$, then there must be two distinct barbells that cover C, and both have to contain the loop incident with C; in this case the loop is covered twice. Summing up, each balanced circuit of G_n yields at least one doubly covered edge, and distinct balanced circuits give rise distinct doubly covered edges. As there are n balanced circuits in G_n, there are at least n edges of G_n covered by $\mathcal{C}$ twice. The length of $\mathcal{C}$ is therefore at least $4n$, which in turn equals $4/3 \cdot |E(G_n)|$.

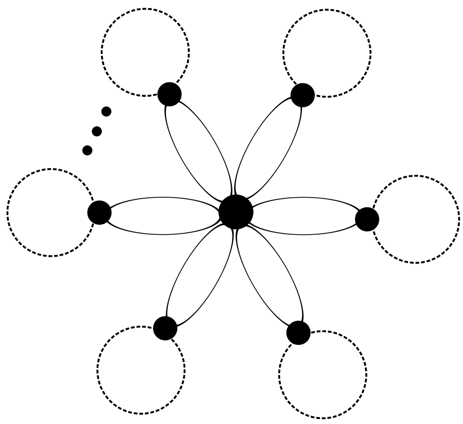

Figure 1: Example of a signed graph with shortest signed circuit cover of length $4/3 \cdot |E|$

The bound of (iii) is tight for the signed graph G of Figure 2 which consists of two vertices joined by a pair of edges with different signs with one negative loop added at each vertex. Note that G contains exactly four distinct signed circuits, each of them being a barbell of

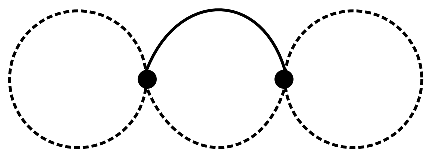

Figure 2: Example of a signed graph with shortest signed circuit cover of length $3/2 \cdot |E|$

length 3. Every signed circuit cover of G consists of at least two of the barbells, therefore the length of the shortest signed circuit cover is at least $6 = 3/2 \cdot 4$.

Sketch of the proof of Theorem 1.

(i) The edge e for which $G - e$ is balanced is contained in every unbalanced circuit of G, which immediately implies that there is no signed circuit containing e.

(ii) Without loss of generality we may suppose that G is connected. Recall that a *cactus* is a connected graph in which any two circuits have at most one vertex in common. It follows that an eulerian graph is a cactus if and only if it has a uniquely determined circuit decomposition. For a circuit decomposition $\mathcal{D}$, we define an *incidence multigraph* of $\mathcal{D}$ to be a graph $J(\mathcal{D})$ whose vertices are the circuits from $\mathcal{D}$ and whose edges are all triples of the form $\{C, v, C'\}$ where C and C' are distinct elements of $\mathcal{D}$ and v is a vertex from $C \cap C'$. The incidence relation in $J(\mathcal{D})$ is defined simply by set membership. If $\mathcal{D}$ has a pair of circuits with more than one common vertex, then $J(\mathcal{D})$ will clearly have parallel edges between the corresponding vertices. Note that the incidence multigraph of any cactus is acyclic.

The proof consists of two steps. At first we show that the result holds whenever G is a cactus, and then finish the proof by induction with cacti forming the induction basis. We now sketch the proof for cacti.

Let $\mathcal{D}$ be the circuit decomposition of G; clearly, the incidence multigraph $J(\mathcal{D})$ is a tree. A circuit of G will be called a *pendant circuit* if the corresponding vertex of $J(\mathcal{D})$ is of degree 1. We may exclude the case when G is balanced, because G admits a signed circuit cover of length $|E(G)|$, and thus G is unbalanced. We may assume that G has no balanced pendant circuits, because we can cover them separately, and the rest of the graph satisfies the conclusion of the theorem whenever G does. Since G has even number of negative edges, it has an even number of unbalanced circuits. Thus $J(\mathcal{D})$ has at least two vertices. If $J(\mathcal{D})$ has exactly two vertices, then G is a barbell and admits a signed circuit cover of length $|E(G)|$, and the theorem follows. Thus we assume that $J(\mathcal{D})$ has at least three vertices.

Our aim is to construct three signed circuit covers $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ such that each edge of G is covered at most twice by any $\mathcal{C}_i$ and at most four times in total by $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. If we find such covers, by pigeon hole principle one of the covers must have length at most $\frac{4}{3} \cdot |E(G)|$.

Choose an arbitrary eulerian trail T in G and pick an initial vertex v_0 for the traversal of T. Since any two intersecting circuits in G share exactly one vertex, for any two distinct circuits K and K' there exists a *unique* subtrail $T[K, K']$ of T starting at a vertex v of K and terminating at a vertex v' of K' such that $T[K, K']$ intersects $K \cup K'$ only at v and v'; it may happen that $v = v'$. Let us order the unbalanced circuits of G in the order as we proceed along T starting from v_0: an unbalanced circuit U is recorded into a list whenever T uses an edge of U for the first time. Let $U_0, U_1, \dots, U_{t-1}$ be the resulting list, with indices taken modulo t. Since G has even number of negative edges, t is even.

Consider two distinct unbalanced circuits U_x and U_y. Assume that either $y = x + 1$ or that U_x is pendant and U_y is the first pendant circuit in the list following U_x (taken in cyclic order); in the latter case we say that U_x and U_y are two *consecutive pendant circuits*. We claim that in both cases $U_x \cup T[U_x, U_y] \cup U_y$ is a barbell, and we call it the barbell *between* U_x and U_y. The sought covers $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are the following:

- $\mathcal{C}_1$ contains all barbells between two consecutive pendant circuits.
- $\mathcal{C}_2$ contains all balanced circuits of G and all barbells of the form $U_i \cup T[U_i, U_{i+1}] \cup U_{i+1}$, where i is an even index from $\{0, 1, \ldots, t-1\}$.
- $\mathcal{C}_3$ contains all balanced circuits of G and all barbells of the form $U_i \cup T[U_i, U_{i+1}] \cup U_{i+1}$, where i is an odd index from $\{0, 1, \ldots, t-1\}$.

Note that each of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ is a signed circuit cover, and since $J(\mathcal{D})$ has at least three vertices, they are all distinct. It is not difficult to see that every edge of G is covered twice by at most one of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, as required.
(iii) We use induction on the maximum number of unbalanced circuits in a circuit decomposition of G. The base case is when G contains three unbalanced circuits. In the induction step, we find two flow-admissible subgraphs of G, each of them containing fewer unbalanced circuits than G. For one of them, we apply (ii), because it has even number of negative edges, for the other one, we apply induction hypothesis.

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Decomposition of eulerian graphs into odd closed trails

Edita Máčajová and *Martin Škoviera*

Department of Computer Science, Comenius University, Bratislava, Slovakia

Extended Abstract

A *decomposition* of a graph G is a set $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of subgraphs of G whose edge sets partition the edge set of G. Here we are concerned with the case where G is an eulerian graph (a connected graph with all vertices of even degree) and all the subgraphs G_i constituting the decomposition are also eulerian. Throughout, all graphs may have multiple edges and loops.

Decomposing eulerian graphs into eulerian subgraphs, or equivalently, into closed trails, belongs to typical problems in eulerian graph theory [3]. For example, the classical result of Veblen [8] states that the existence of a circuit decomposition in a connected graph is equivalent to the existence of an eulerian trail in it. Several papers have been devoted to decompositions into closed trails of given lengths. These decompositions are difficult to find and in most cases the graphs where such decompositions are known are complete multipartite graphs, thus having a very explicit structure [1, 2, 5].

In contrast, our focus is on the existence of closed trail decompositions in fairly general graphs, with the only restriction that each member of the decomposition have an odd length. The motivation for the study of such decompositions comes from nowhere-zero flows on signed graphs: our main result, Theorem 4, is crucial for determining the flow number of a signed eulerian graph, see [6].

1. Unrooted decompositions

A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a graph G is an *odd closed trail decomposition* if each G_i is a closed trail of odd length; equivalently, if each G_i is an eulerian subgraph with an odd number of edges. The decomposition $\mathcal{D}$ will be called *rooted* if there exists a vertex r of G, called a *root* of $\mathcal{D}$, such that each G_i contains r.

The first question about odd closed trail decompositions that suggests itself is obvious: *Given an integer $k \geq 2$, which graphs admit a decomposition into k odd closed trails, rooted or not?*

Clearly, every connected graph G that admits a decomposition into k odd closed trails must be eulerian with $|E(G)| \equiv k \pmod 2$ and has to contain at least k pairwise edge-disjoint odd circuits, one for each closed trail. We show that this necessary condition is also sufficient.

Theorem 1 *An eulerian graph G admits a decomposition into k closed trails of odd length if and only if it contains at least k edge-disjoint odd circuits and $|E(G)| \equiv k \pmod 2$.*

Proof. We sketch the proof of sufficiency. Let $\mathcal{C}$ be a set of k pairwise edge-disjoint circuits of G and let $\mathcal{K}$ be a circuit decomposition of G such that $\mathcal{C} \subseteq \mathcal{K}$. We construct the graph $J(\mathcal{K})$ whose vertices are the elements of $\mathcal{K}$ and edges join pairs of elements that have a vertex of G in common. Clearly, every connected subgraph of $J(\mathcal{K})$, with vertex set a subset $\mathcal{L} \subseteq \mathcal{K}$, uniquely determines an eulerian subgraph of G. The latter subgraph will have an odd number of edges whenever $\mathcal{L}$ contains an odd number of odd circuits. Thus it suffices to show that $\mathcal{K}$ can be partitioned into k subsets, each containing an odd number of odd circuits and each inducing a connected subgraph of $J(\mathcal{K})$. In fact, we may assume that $J(\mathcal{K})$ is a tree as the general case follows with the partition of $\mathcal{K}$ obtained from a spanning tree T of $J(\mathcal{K})$. Thus it remains to prove the following claim.

Claim 1. *Let T be a tree, $k \geq 1$ an integer, and $B \subseteq V(T)$ a subset with $|B| \geq k$ vertices. If $|B| \equiv k \pmod 2$, then $V(T)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that each V_i contains an odd number of vertices from B and induces a subtree of T.*

The proof of this claim can be performed by induction on k, and for every fixed k by induction on the order of T. □

Corollary 2 *Let G be a connected $2d$-regular graph of odd order with $d \geq 1$. Then G admits a decomposition into k closed trails of odd length for each $k \in \{1, 2, \ldots, d\}$ such that $k \equiv d \pmod 2$.*

Proof. By Petersen's 2-factor theorem, G can be decomposed into k pairwise edge-disjoint 2-factors, each containing an odd circuit. The conclusion now follows from Theorem 1. □

2. Rooted decompositions

Our primary interest lies in odd closed trail decompositions which have a root. It is easy to see that a graph G has a rooted decomposition into k odd closed trails if and only if it contains an eulerian trail T and a vertex r such that r divides T into k closed subtrails $T_1, T_2, \ldots, T_k$ based at r with an odd number of edges each. If a graph G is to have such a decomposition, then G has to satisfy the conditions of Theorem 1 (so that G has an unrooted decomposition into k odd closed trails) and the root v must have degree at least $2d$. By Corollary 2, both conditions are satisfied in $2d$-regular graphs of odd order for $k = d$. Thus it is a natural question to ask whether for $2d$-regular graphs these necessary conditions are also sufficient. For $d = 1$ the answer is trivially positive. The next two theorems show that the answer is positive also for $d = 2$ and $d = 3$.

Theorem 3 *Every connected 4-regular graph of odd order has a rooted decomposition into two closed trails of odd length.*

Proof. Take a maximal odd closed subtrail of the graph and its complement. □

The following theorem is our main result. Its proof is significantly more involved and will be sketched in the next section. Details will appear in [7].

Theorem 4 *Every connected 6-regular graph of odd order has a rooted decomposition into three closed trails of odd length.*

We believe that these two theorems can be generalised to an arbitrary positive integer and propose the following conjecture.

Conjecture 5 *Every connected $2d$-regular graph of odd order, with $d \geq 1$, has a rooted decomposition into d closed trails of odd length.*

3. Sketch of proof of Theorem 4

Let G be a connected 6-regular graph of odd order. The proof is by induction on the order of G. The result is trivial if G has a single vertex. For the induction step, let G have an odd order greater than 1, and assume that the result is true for all smaller 6-regular graphs of odd order. Suppose to the contrary that G fails to have a rooted 3-odd decomposition; in other words, let G be a smallest counterexample. We first show that G must be 6-edge-connected.

For convenience, a subgraph H of G will be called *even* if it has an even number of edges, and it will be called *odd* if it has an odd number of edges. Further, we will say that H is *u-v-eulerian* if all vertices except possibly u and v have an even degree.

Proposition 6 *A smallest counterexample to Theorem 4 must be 6-edge-connected.*

Proof. It is straightforward to prove that the smallest counterexample G has no 2-edge-cut. Suppose that G has a 4-edge-cut S. Since G has an odd number of edges, $G - S$ has two components H and K, where H is even and K is odd. Let $S = \{a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$ with $\{a_1, a_2, a_3, a_4\} \subseteq V(H)$ and $\{b_1, b_2, b_3, b_4\} \subseteq V(K)$. We first show that H contains an odd a_i-a_j-eulerian subgraph $H_{i,j}$ and an odd a_k-a_l-eulerian subgraph $H_{k,l}$ such that $\{H_{i,j}, H_{k,l}\}$ is a decomposition of H and $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Details are quite involved and therefore omitted here. We only remark that the proof splits into two main cases depending on whether

H is or is not bipartite. Next, we take the graph K' formed from K by adding the edges b_ib_j and b_kb_l. Since K' is 6-regular, has odd order, and is smaller than G, it has a decomposition $\mathcal{D} = \{K_1, K_2, K_3\}$ into three odd closed trails with root at some vertex r of K. If we replace in the subgraphs from $\mathcal{D}$ the edge b_ib_j with $H_{i,j}$ and the edge b_kb_l with $H_{k,l}$, we obtain a decomposition of G into three closed trails of odd length rooted at r – a contradiction. □

To complete the proof of Theorem 4 it remains to show that every 6-edge-connected 6-regular graph of odd order has a rooted decomposition into three closed trails of odd length. The proof of this fact requires a special tool – signed graphs.

Recall that a *signed graph* is a graph G equipped with a mapping, called the *signature* of G, which assigns $+1$ or -1 to each edge. An edge receiving value $+1$ is said to be *positive* while one with value -1 is said to be *negative*. The sign of each edge will be known from the immediate context, therefore no special notation for the signature will be required.

The signature of a signed graph is a means of introducing the concept of balance, which is more important than the signature itself. A circuit of a signed graph is said to be *balanced* if it contains an even number of negative edges, and is *unbalanced* otherwise. A signed graph in which all circuits are balanced is itself called *balanced*; an *unbalanced* signed graph is one that contains at least one unbalanced circuit. In general, the essence of any signed graph is constituted by the list of all balanced circuits: two signed graphs with the same underlying graph are therefore considered to be *identical* if their lists of balanced circuits coincide. The corresponding signatures are called *equivalent*. It is well known that any two equivalent signatures can be turned into each other by a series of vertex-switchings [10]. To *switch* the signature of a signed graph G at some vertex v means to change the signs of all edges incident with v except for loops. In particular, by applying vertex switching sufficiently many times one can turn any balanced signed graph into the all-positive one.

Let G be a signed eulerian graph. It is easy to see that switching does not change the parity of the number of negative edges in G. Thus all signed eulerian graphs fall into two natural classes – *even* signed eulerian graphs and *odd* eulerian graphs depending on whether the number of negative edges is even or odd, respectively. From this point of view, a natural signed analogue of an odd closed trail is a trail with an odd number of negative edges.

By using signed graphs we can successfully deal with certain parity problems that would occur in the unsigned case and we can prove the following result.

Theorem 7 *Let G be a 6-edge-connected 6-regular signed graph with an odd number of negative edges. If G contains at least three pairwise edge-disjoint unbalanced circuits, then G has a rooted decomposition into three closed trails with an odd number of negative edges each.*

Proof. The proof is by induction on the number of vertices of G. Again, the result is trivial if G has a single vertex. If G has $n \geq 2$ vertices, we construct a 6-regular signed graph G' of order $n-1$ by removing an arbitrary vertex of G and by reconnecting its neighbours with three new signed edges in a suitable manner. A result of Frank [4, Theorem A$'$] implies that this can always be done in such a way that G' is again 6-edge-connected. By considering a number of cases for G, we either find the required decomposition directly in G, or construct one in G' and transform it to a decomposition of G. □

Now we can return to unsigned graphs.

Corollary 8 *Every 6-edge-connected 6-regular graph of odd order has a rooted decomposition into three closed trails of odd length.*

Proof. Let G be a 6-edge-connected 6-regular graph of odd order. Let us endow G with the all-negative signature. Since G has an odd order, the number of negative edges is odd. By Petersen's 2-factor theorem G can be decomposed into three pairwise disjoint 2-factors. Each of them contains an odd cycle, so with respect to the all-negative signature G has at least three edge-disjoint unbalanced circuits. Thus we can apply Theorem 7 to conclude that G

has a rooted decomposition into three closed trails with an odd number of negative edges each. Forgetting the signs, this decomposition has three closed trails of odd length rooted at some vertex of G. $\square$

4. Final remarks

1. We have shown that every connected 6-regular graph G of odd order has a rooted decomposition into three odd closed trails. This poses a natural question about the distribution of roots within the graph. We conjecture that if G is 6-edge-connected, then any vertex of G occurs as a root of some odd trail decomposition. However, there are examples with connectivity 2 and 4 where a root cannon be chosen arbitrarily.

2. As already mentioned, rooted decompositions into odd closed trails are closely related to nowhere-zero integer flows on signed eulerian graphs. It was proved by Xu and Zhang [9] that a signed eulerian G graph admits a nowhere-zero 2-flow if and only if it has an even number of negative edges. In [6] we prove that a signed eulerian graph has a nowhere-zero 3-flow if and only if it admits a rooted decomposition into three closed trails with an odd number of negative edges each. The crucial argument of the proof uses Theorem 4.

3. In contrast to Theorem 4, the assumption of 6-edge-connectivity in Theorem 7 cannot be relaxed. Examples are not difficult to find.

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Construction of permutation snarks of order 2 (mod 8)

Edita Máčajová and *Martin Škoviera*

Department of Computer Science, Comenius University, Bratislava, Slovakia

Extended Abstract

A *cycle permutation graph* is a cubic graph consisting of two disjoint circuits of equal length and a perfect matching between them; equivalently, it is a cubic graph which has a 2-factor consisting of two chordless circuits. A *permutation snark* is a cycle permutation graph that has no 3-edge-colouring.

Cycle permutation graphs were introduced in 1967 by Chartrand and Harary in [4] as a generalisation of the Petersen graph and were subsequently studied by various authors, see e. g. [7, 10, 11]. Since the Petersen graph is a snark, permutation snarks are a natural family to investigate. In spite of that, until very recently permutation snarks have attracted only very little attention, with a notable exception of Zhang's book [13].

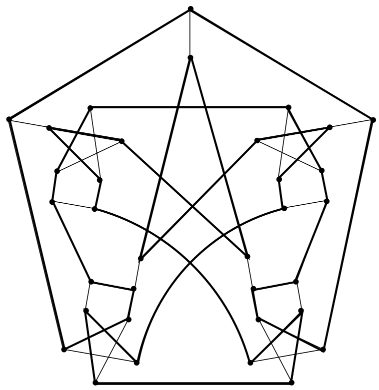

Figure 1: A cyclically 5-edge-connected permutation snark of order 34

It is easy to see that every permutation snark has cyclic connectivity at least 4 and girth at least 5. In [13], Zhang made a conjecture that the only cyclically 5-edge-connected permutation snark is the Petersen graph. However, in 2013, Brinkmann et al. [2] disproved this conjecture by exhibiting a cyclically 5-edge-connected permutation snark on 34 vertices found by an exhaustive computer search. This snark is displayed in Figure 1 in a form rather different from that in [2]; the defining 2-factor is shown in bold lines. Recently, using an *ad hoc* construction, Hägglund and Hoffmann-Ostenhof [5] extended this example to an infinite series of cyclically 5-edge-connected permutation snarks of order $24n + 10$ for each integer $n \geq 1$.

The defining 2-factor of every permutation snark consists of two odd circuits. It follows that the number of vertices of every permutation snark must be 2 (mod 4). However, all currently known permutation snarks have order 2 (mod 8) while orders 6 (mod 8) are completely missing. The purpose of this paper is to construct cyclically 4-edge-connected permutation snarks of every possible order 2 (mod 8) and cyclically 5-edge-connected connected snarks of order n for every integer $n \equiv 2 \pmod 8$ with $n \geq 34$. In contrast to [5], our constructions are very simple as they only use classical operations on snarks – the dot product of snarks (4-product) and a similar operation (5-product) introduced by Cameron, Chetwynd, and Watkins [3], which is capable of producing cyclically 5-edge-connected snarks.

1. Permutation snarks with cyclic connectivity 4

The most common method of constructing snarks with cyclic connectivity 4 is the dot product of snarks. Given two cubic graphs G and H, a *dot product* $G\,.\,H$ is a cubic graph defined as follows. Choose two independent edges $e_1 = a_1b_1$ and $e_2 = a_2b_2$ in G and an edge $e = uv$ in H. Let a_1', b_1', and v be the neighbours of u, and let a_2', b_2', and u be the neighbours of v. Remove the edges e_1 and e_2 from G and the vertices u and v from H. Finally, connect a_1 to a_1', b_1 to b_1', a_2 to a_2', and b_2 to b_2'. Although the notation $G\,.\,H$ is common, the result depends on the choice of the edges e_1 and e_2 in G and the edge $e = uv$ in H as well as on the chosen labelling of the neighbours of u and v in H and their counterparts in G. If we need to be more specific, we will write $G[e_1, e_2]\,.\,[e]H$ instead of $G\,.\,H$.

The operation of dot product was introduced by Adelson-Velskii and Titov [1] in 1973 and independently by Isaacs [6] in 1975. In [1] it was shown that if both G and H are cyclically 4-edge-connected, then so is $G\,.\,H$. Moreover, in both [1] and [6] it was proved that $G\,.\,H$ is non-3-edge-colourable provided that each of G and H is.

Now assume that G and H are permutation snarks. Let $P = \{P_1, P_2\}$ be a *permutation 2-factor* of G, that is, a 2-factor that consists of two chordless circuits P_1 and P_2. Also let $Q = \{Q_1, Q_2\}$ be a permutation 2-factor of H. Let us perform the dot product $G[e_1, e_2]\,.\,[e]H$ in such a way that the edges e_1 and e_2 belong to different circuits of P and that the edge e is a *spoke* of Q, that is, an edge of the 1-factor complementary to Q. Since the end-vertices vertices u and v of e belong to different circuits of Q, there is an index $i \in \{1, 2\}$ such that the dot product operation welds the circuit P_1 of P with the circuit Q_i of Q and the circuit P_2 of P with Q_{3-i} of Q. In this manner a 2-factor of $G\,.\,H$ consisting of two chordless circuits is produced. Thus $G\,.\,H$ is a permutation snark.

It is well known that the dot product operation is essentially reversible (see for example Cameron et al. [3]). This means that if a snark G has a cycle-separating edge-cut S of size 4, then there exist snarks G_1 and G_2 such that G is isomorphic to $G_1\,.\,G_2$. If G is a permutation snark, then the snarks G_1 and G_2 are uniquely determined by S and both can be shown to be again permutation snarks. Thus the following result is true.

Theorem 1 *Let G and H be permutation snarks. A dot-product $G[e_1, e_2] \cdot [e]H$ is a permutation snark if and only if the edges e_1 and e_2 belong to different circuits of a permutation 2-factor of G and e belongs to the 1-factor complementary to a permutation 2-factor of H. Furthermore, every permutation snark with cyclic connectivity 4 arises in this way.*

This theorem can be used to produce huge amounts of permutation snarks with cyclic connectivity 4. For example, starting from the Petersen graph and applying Theorem 1 repeatedly we can we can construct cyclically 4-edge-connected snarks of every possible order $n \equiv 2 \pmod 8$.

Corollary 2 *For every integer $n \equiv 2 \pmod 8$ with $n \geq 10$ there exists a permutation snark of order n.*

By performing dot products more specifically we can show that several well-known infinite families of snarks are permutation snarks. For example, this is true for generalised Blanuša snarks of Type 1 and Type 2 (see [12] for their definitions).

Theorem 1 also explains an explosion of permutation snarks observed by Brinkmann et al. in [2]: there is one permutation snark of order 10, two of order 18, 64 of order 26, and 10771 of order 34. Exluding the Petersen graph, only twelve of all these snarks are cyclically 5-edge-connected, all of order 34. This naturally directs our interest to cyclically 5-edge-connected permutation snarks.

2. Permutation snarks with cyclic connectivity 5

The dot product has a lesser-known cyclically 5-connected analogue which we call *star product*. This operation was first mentioned by Cameron et al. in [3] and can be described as follows. Consider two cubic graphs G and H and containing 5-cycles $C = v_0v_1v_2v_3v_4 \subseteq G$ and $D = w_0w_1w_2w_3w_4 \subseteq H$. For each vertex x on either of these circuits let x' denote the corresponding neighbour not lying on the circuit. Define $G \star H$ to be the cubic graph obtained by removing C from G and D from H and by connecting each vertex v'_i to the vertex w'_{2i} with the indices reduced modulo 5. Observe that the result is not uniquely determined as it depends on the chosen labelling of vertices in the 5-cycles C and D. Furthermore, if Ps denotes the Petersen graph, then $G \star Ps \cong G$ and $Ps \star H \cong H$, which means that the Petersen graph serves both as the right and as the left identity of this operation.

It can be shown that $G \star H$ is non-3-edge-colourable provided that G and H are (see [3]). The following result provides another important property of the star product.

Theorem 3 *If G and H are cyclically 5-edge-connected cubic graphs, then so is $G \star H$.*

Proof. If $G \star H$ contains a cycle separating edge-cut S with $|S| < 5$, then S must intersect the 5-edge-cut $T = \{v'_i w'_{2i};\ i \in \{0, 1, 2, 3, 4\}\}$ arising from the star product operation. This situation gives rise to a number of cases according to the mutual position of S and T, and each case leads to the conclusion that either G or H is not cyclically 5-edge-connected, which is a contradiction. □

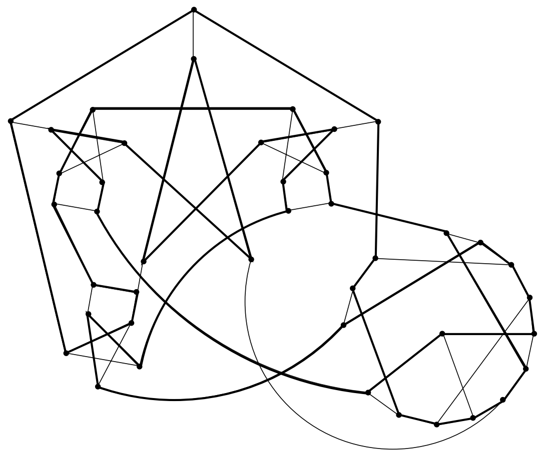

Figure 2: A cyclically 5-edge-connected permutation snark of order 42

Let G be a snark with a permutation 2-factor $P = \{P_1, P_2\}$ and let C be an arbitrary 5-cycle in G. Since each circuit of P is chordless, there exists an index $i \in \{1, 2\}$ such that C has two common edges with P_i and one common edge with P_{3-i}. Hence, precisely one of the edges connecting C to the rest of G is a spoke.

Now let G and H be permutation snarks with 5-cycles $C = v_0v_1v_2v_3v_4 \subseteq G$ and $D = w_0w_1w_2w_3w_4 \subseteq H$. Let us choose the labelling of vertices of the 5-cycles C and D in such a way that $v_0v'_0$ and $w_0w'_0$ are spokes of the respective permutation 2-factors. We will say that the star product performed with respect to such a labelling is *rooted*.

The following holds.

Theorem 4 *A star product of two permutation snarks is a permutation snark if and only if it is rooted.*

We now apply Theorem 3 and Theorem 4 to construct cyclically 5-edge-connected permutation snarks of every order $n \equiv 2 \pmod 8$ with $n \geq 34$. For this purpose it is sufficient to display cyclically 5-edge-connected permutation snarks of order 34, 42, and 50, each containing at least two disjoint 5-cycles, and apply rooted star product repeatedly. A pair of disjoint 5-cycles guarantees that the star product can indeed be iterated, because one of the 5-circuits is used for the product and the other one survives in each factor of the star product. For the snark of order 34 we can use the one in Figure 1 since it clearly contains two disjoint 5-cycles. Next we construct a cyclically 5-edge-connected permutation snark of order 42 from the one of order 34 by substituting a 7-vertex subgraph isomorphic to $Ps - P_2$ where P_2 is a path of length 2 with a 15-vertex subgraph $B_2 - R$ obtained from the second Blanuša snark B_2 (also called Blanuša double according to [9]) by removing a path R of length 2 that intersects its unique cycle-separating 4-edge-cut. The resulting snark is shown in Figure 2. A cyclically 5-edge-connected permutation snark of order 50 can be obtained from the one of order 42 by another similar substitution. Thus we have the following result.

Theorem 5 *There exists a cyclically* 5*-edge-connected permutation snark of order* n *for each* $n \equiv 2 \pmod 8$ *with* $n \geq 34$.

Full proofs of the presented results and additional results about permutation snarks will appear in [8].

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Neighbour distinguishing graph colourings - distant generalizations

Jakub Przybyło

AGH University of Science and Technology
al. A. Mickiewicza 30, 30-059 Krakow, Poland

Extended Abstract

In [28] Zhang et al. posed the following problem. Consider a graph $G = (V, E)$ containing no isolated edges and its proper edge colouring $c : E \to \{1, 2, \dots, k\}$. For any $v \in V$ denote by $S_c(v)$ the set of colours incident with v, i.e.,

$$S_c(v) := \{c(uv) : u \in N(v)\},$$

where $N(v)$ is the set of neighbours of v. We shall also denote $S_c(v)$ simply by $S(v)$ when this causes no ambiguities, and refer to it as the *colour pallet* of v. We call vertices u and v *distinguished* if $S(u) \neq S(v)$. The least number of colours in a proper edge colouring c which distinguishes the ends of all edges of G, i.e. such that $S_c(u) \neq S_c(v)$ for every $uv \in E$, is called the *neighbour set distinguishing index* or the *adjacent strong chromatic index* and denoted by $\chi'_a(G)$, see [3, 4, 8, 9, 28] (also for other notations used).

Conjecture 1 ([28]) *For every connected graph G, $\chi'_a(G) \leq \Delta(G) + 2$, unless G is isomorphic to K_2 or C_5.*

This problem remains open despite many articles studying it. In particular, it has been showed that $\chi'_a(G) \leq 3\Delta(G)$, [3], and $\chi'_a(G) \leq \Delta(G) + O(\log \chi(G))$, [4]. Moreover, the conjecture was verified to hold for special families of graphs, like e.g. bipartite graphs or graphs of maximum degree 3, see [4]. The following thus far best general upper bound is due to Hatami.

Theorem 2 ([9]) *If G is a graph with no isolated edges and maximum degree $\Delta > 10^{20}$, then $\chi'_a(G) \leq \Delta + 300$.*

Note that this implies that $\chi'_a(G) \leq \Delta(G) + C$ for every graph G containing no isolated edges, where C is some constant.

Let r be any positive integer. Vertices u, v of G shall be called *r-neighbours* (or *r-adjacent*) if $1 \leq d(u, v) \leq r$, where $d(u, v)$ denotes the distance of u and v in G. In [21], similarly as e.g. within the concept of distant chromatic numbers (see [14] for a survey of this topic), we proposed an extension of the study above towards distinguishing not only neighbours, but also vertices at some limited distance (from each other). The least number of colours in a proper edge colouring c of G such that $S_c(u) \neq S_c(v)$ for every pair of vertices $u, v \in V$ with $1 \leq d(u, v) \leq r$, so-called *r-distant set distinguishing colouring* (or *r-adjacent strong edge colouring*), shall be called the *r-distant set distinguishing index* or *r-adjacent strong chromatic index*, and denoted by $\chi'_{a,r}(G)$. This graph invariant has already been considered in [13] and [16] under the name of *d-strong chromatic index* (see [3, 26, 27] for other notations), mainly for paths, cycles and circulant graphs, aiming towards providing a series of counterexamples to a conjecture from [27]. Some aspects of this concept were also investigated in [3] with respect to trees and small values of r. As for general upper bounds, in [26] it was proved that $\chi'_{a,2}(G) \leq 32(\Delta(G))^2$ if $\Delta(G) \geq 4$, $\chi'_{a,3}(G) \leq 8(\Delta(G))^{\frac{5}{2}}$ if $\Delta(G) \geq 6$ and $\chi'_{a,r}(G) \leq 2\sqrt{2(r-1)}(\Delta(G))^{\frac{r+2}{2}}$ for $r \geq 4$ if $\Delta(G) \geq 4$. Improvements of these bounds shall be presented

The cornerstone of the general field of vertex distinguishing colourings, which is rich in many interesting open problems and conjectures, is the graph invariant called the irregularity strength. Consider a (not necessarily proper) edge colouring $c : E \to \{1, 2, \dots, k\}$ of a graph $G = (V, E)$ containing no isolated edges. Denote by

$$s_c(v) = \sum_{u \in N(v)} c(uv)$$

the sum of colours incident with any vertex $v \in V$. The *irregularity strength* of G, $s(G)$, is then the least integer k admitting such c with $s_c(u) \neq s_c(v)$ for all $u, v \in V$, $u \neq v$, see [7]. Note that equivalently, it is equal to the minimal integer k so that we are able to construct an irregular multigraph (a multigraph with pairwise distinct degrees of all the vertices) of G by multiplying some of its edges, each at most k times. Out of the extensive bibliography devoted to this parameter, it is in particular worth mentioning [2] and [17], where the sharp upper bound $s(G) \leq n - 1$ (with $n = |V|$) was settled, and [10], where a better upper bound $s(G) \leq 6\lceil n/\delta \rceil$ for graphs with sufficiently large minimum degree δ is proved. See [15] for an interesting survey and open problems in this topic as well. This problem is also related with the study of irregular graphs by Chartrand, Erdős and Oellermann, see [6], and gave rise to many other intriguing graph invariants. In particular the following concept was closely related to the later local version of irregularity strength – the well known problem commonly referred to as 1–2–3–Conjecture, see [12] by Karoński, Łuczak and Thomason (and [11] for the best result concerning this). Let K be the least integer K so that for every graph $G = (V, E)$ without isolated edges there exists a (not necessarily proper) edge colouring $c : E \to \{1, 2, \ldots, K\}$ such that for each edge $uv \in E$, the multisets of colours incident with u and v are distinct. Note that this problem is a natural correspondent of the concept of $\chi'_a(G)$, as the only difference is the requirement concerning the properness of the colourings investigated. However, by [1] it is known that $K \leq 4$, or even $K \leq 3$ suffices for graphs with minimum degree $\delta \geq 10^3$, while χ'_a is bounded from below by Δ. In fact one of our main motivations for studying the parameters $\chi'_{a,r}$ is a desire to expose the leading impact of the required properness of colourings on the number of colours needed to distinguish vertices by their incident (multi)sets, and that usually not many more colours are needed if we wish to distinguish not only neighbours, but also vertices at distance 2, 3,..., $r < \infty$. We believe, and prove in many cases that for every fixed r, if only $\delta(G)$ is 'not very small', then $\chi'_{a,r}(G) \leq \Delta + C$ for each graph G without an isolated edge, where C is some constant dependent on r. We pose the following general conjecture, where we believe that δ_0 should be roughly equal to r (up to some small additive constant).

Conjecture 3 ([21]) *For each positive integer r there exist constants δ_0 and C such that*

$$\chi'_{a,r}(G) \leq \Delta(G) + C$$

for every graph without an isolated edge and with $\delta(G) \geq \delta_0$.

We also prove that our assumption on $\delta(G)$ is unavoidable, and support this conjecture by showing the following.

Theorem 4 ([21]) *For every $r \geq 2$ and every graph G of maximum degree Δ with $\delta(G) \geq r + 2$ and without isolated edges,*

$$\chi'_{a,r}(G) \leq (1 + o(1))\Delta.$$

Theorem 5 ([21]) *For every positive $\varepsilon \leq 1$ and a positive integer r, there exist Δ_0 and a constant $C = C(\varepsilon, r)$ such that:*

$$\chi'_{a,r}(G) \leq \Delta(G) + C$$

for every graph G without an isolated edge and with $\delta(G) \geq \varepsilon\Delta(G)$, $\Delta(G) \geq \Delta_0$. In particular, $C \leq \varepsilon^{-2}(7r + 200) + r + 6$.

Moreover, or maybe even more importantly, by our research, we wish to reveal a difference between the two main concepts of distinguishing vertices (at a bounded distance) of the field, i.e., this with respect to sets and this based on sums. Intriguingly this expected difference was elusive while distinguishing only neighbours. The least integer k so that a proper edge colouring $c : E \to \{1, 2, \ldots, k\}$ exists with $s_c(u) \neq s_c(v)$ for every edge $uv \in E$ is denoted by $\chi'_\Sigma(G)$. Note that $\chi'_a(G) \leq \chi'_\Sigma(G)$ for every graph G without isolated edges. Though the requirement $s_c(u) \neq s_c(v)$ is much stronger than $S_c(u) \neq S_c(v)$, it was conjectured in [8] that $\chi'_\Sigma(G) \leq \Delta(G)+2$ for every connected graph non-isomorphic to K_2 nor C_5 (similarly as for χ'_a in [28], cf. Conjecture 1), what was also asymptotically confirmed in [18], where it was proved that $\chi'_\Sigma(G) \leq (1+o(1))\Delta$. In fact (almost) all exact values of the both parameters, settled for some special families of graphs coincide, see [8] for further comments. See also [5, 8, 24, 25] for other results concerning χ'_Σ. On the other hand, if we consider a distant version of the problem in sum environment, even setting aside the properness of colourings, then the number of colours required grows rapidly with r. For any positive integer r, let $s_r(G)$ be the least integer k so that an edge colouring $c : E \to \{1, 2, \dots, k\}$ exists with $s_c(u) \neq s_c(v)$ for every pair of r-neighbours u, v. Then it is known that $s_r(G) \leq 6\Delta^{r-1}$ for every graph G without isolated edges, see [19]. On the other hand, it can be proved that there are graphs for which the parameter $s_r(G)$ cannot be much smaller than Δ^{r-1} for arbitrarily large Δ, also in the family of regular graphs, contrary to the parameter $\chi'_{a,r}$ – cf. Conjecture 3 and Theorems 4 and 5 above. Using the probabilistic method we provide the following strengthening of the upper bound above, which also work similarly in the case of proper edge colourings.

Theorem 6 ([20]) *For every integer $r \geq 2$ there exists a constant Δ_0 such that for each graph G with maximum degree $\Delta \geq \Delta_0$ and minimum degree $\delta \geq \ln^8 \Delta$,*

$$s_r(G) < 4\Delta^{r-1}\left(1 + \frac{1}{\ln \Delta}\right) + 12,$$

hence $s_r(G) \leq (4 + o(1))\Delta^{r-1}$ for all graphs with $\delta \geq \ln^8 \Delta$ and without isolated edges.

Similar results hold also in total versions of the problem discussed, see [22, 23].

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Strong Rainbow Connection in Digraphs

Elżbieta Sidorowicz [1], and *Éric Sopena* [2,3]

[1] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Poland
[2] Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France
[3] CNRS, LaBRI, UMR5800, F-33400 Talence, France

Extended Abstract

In an edge-coloured graph G, a path is said to be rainbow if it does not use two edges with the same colour. Then the graph G is said to be rainbow-connected if any two vertices are connected by a rainbow path. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [1]. Since then the rainbow connection number of various graph classes has been determined. Also, different other parameters similar to rainbow connection were introduced, such as strong rainbow connectivity, rainbow k-connectivity, k-rainbow index and rainbow vertex connection. See [7] for a survey about these different parameters.

The notions of rainbow connection and strong rainbow connection readily extend to digraphs, using arc-colouring instead of edge-colourings and directed paths instead of paths. The study of rainbow connection in oriented graphs was initiated by Dorbec et al. in [6] and then studied by Alva-Samos and Montellano-Ballesteros in [2, 3, 4]. Rainbow connectivity in digraphs was considered by Ananth, Nasre and Sarpatwar in [5] from the computational point of view.

A digraph G is *strongly connected* (*strong* for short) if there exists a uv-path in G for every two vertices u and v. The digraph G is *minimally strongly connected* if G is strong and, for every arc xy in G, the digraph $G - xy$ (obtained from G by removing the arc xy) is not strong.

Let G be a digraph. A *k-arc-colouring* of G, $k \geq 1$, is a mapping $\varphi : A(G) \to \{1, \dots, k\}$. Note that adjacent arcs may receive the same colour. An *arc-coloured* digraph is then a pair (G, φ) where G is a digraph and φ an arc-colouring of G. A path P in (G, φ) is *rainbow* if no two arcs of P are coloured with the same colour. An arc-coloured digraph (G, φ) is *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G) if any two vertices in G are connected by a rainbow path.

For any two vertices u, v of G, a *rainbow uv-geodesic* is a rainbow uv-path of length $\mathrm{dist}_G(u, v)$. An arc-coloured digraph (G, φ) is *strongly rainbow connected* (or, equivalently, φ is a *strong rainbow arc-colouring* of G) if there exists a rainbow uv-geodesic for any two vertices u and v in G.

Note that in order to admit a rainbow arc-colouring, or a strong rainbow arc-colouring, a digraph must be strong. The *rainbow connection number* of a strong digraph G, denoted by $\vec{rc}(G)$, is the smallest number k such that G admits a rainbow k-arc-colouring. The *strong rainbow connection number* of a strong digraph G, denoted $s\vec{rc}(G)$, is the smallest number k such that G admits a strong rainbow k-arc-colouring. Observe that for any strong digraph G we have $\vec{rc}(G) \leq s\vec{rc}(G)$.

Another easy observation is that $s\vec{rc}(G) \leq n$ for every digraph G of order n. Let $V(G) = \{x_1, \dots, x_n\}$. We define an n-arc-colouring φ of G by setting $\varphi(x_i x_j) = j$ for every arc $x_i x_j$ in $A(G)$. Obviously, every elementary path in G is rainbow and, therefore, φ is a strong rainbow arc-colouring of G, which gives $s\vec{rc}(G) \leq n$.

Moreover, we clearly have $\mathrm{diam}(G) \leq \vec{rc}(G)$, since any rainbow uv-path such that (u, v) is an antipodal pair of vertices must use at least $\mathrm{diam}(G)$ colours. We thus have the following proposition.

Proposition 1 *If G is a strong digraph of order n with diameter d, then $d \leq \vec{rc}(G) \leq s\vec{rc}(G) \leq n$.*

We prove that the value of $s\vec{rc}(G) - \vec{rc}(G)$ can be arbitrarily large:

Theorem 2 *For every integer k there exists a strong digraph G_k with $s\vec{rc}(G) - \vec{rc}(G) = k$.*

It follows from the definitions that if H is a strong spanning subdigraph of G (that is, $V(H) = V(G)$), then $\vec{rc}(G) \leq \vec{rc}(H)$. However, such an equality does not hold for the strong rainbow connection number of digraphs. Alva-Samos and Montellano-Ballesteros [2] showed that there exists digraphs G and H, such that H is a spanning subdigraph of G and $s\vec{rc}(G) > s\vec{rc}(H)$. The digraph G constructed in [2] contains pairs of opposite arcs. We can prove that this property also holds for oriented graphs:

Proposition 3 *There exist strong oriented graphs G and H, such that H is a spanning subdigraph of G and $s\vec{rc}(G) > s\vec{rc}(H)$.*

The upper bound in Proposition 1 is tight since $\vec{rc}(C_n) = s\vec{rc}(C_n) = n$ for the cycle C_n on n vertices. Moreover, we prove the following theorem.

Theorem 4 *Let G be a minimally strongly connected digraph of order n. Then $s\vec{rc}(G) = n$ if and only if G is a cycle.*

Since $\vec{rc}(G) \leq \vec{rc}(H)$ whenever H is a spanning subdigraph of an oriented graph G, it follows from Theorem 4 that $\vec{rc}(G) \leq n - 1$ for every non-Hamiltonian oriented graph G of order n (this was already observed in [6]). Although the inequality $\vec{rc}(G) \leq \vec{rc}(H)$ does not hold in general for digraphs (see Proposition 3), we can prove a similar result for non-Hamiltonian digraphs, with an additional assumption on the diameter. However, we conjecture that this diameter condition can be dropped.

Theorem 5 *Let G be a strong digraph of order n. If G is non-Hamiltonian and $\mathrm{diam}(G) = n - 1$, then $s\vec{rc}(G) \leq n - 1$.*

Conjecture 6 *Let G be a strong digraph of order n. If G is non-Hamiltonian then $s\vec{rc}(G) \leq n - 1$.*

Furthermore, we study the strong rainbow connection number of strong tournaments. We first prove the following result.

Theorem 7 *Let T be a strong tournament of order $n \geq 4$. Then $3 \leq s\vec{rc}(T) \leq n - 1$.*

Both bounds in Theorem 7 are tight and, moreover, $s\vec{rc}(T)$ can take any value between 3 and $n - 1$, as shown by the following result.

Theorem 8 *For every two integers n and k, $3 \leq k \leq n-1$, there exists a strong tournament $T_{n,k}$ of order n with $s\vec{rc}(T_{n,k}) = k$.*

In [6], it has been proved that the rainbow connection number of every tournament is bounded by a function of its diameter, namely $\vec{rc}(T) \leq \mathrm{diam}(T) + 2$. However, such a result does not hold for the strong rainbow connection number, as shown by the next theorem.

Theorem 9 *For every two integers d and k, $3 \leq d \leq k$, there exists a strong tournament $F_{d,k}$ with $\mathrm{diam}(F_{d,k}) = d$ and $s\vec{rc}(F_{d,k}) = k$.*

Theorem 9 says that the strong rainbow connection number of a tournament with diameter at least 3 can be arbitrarily large. We believe that this is no longer true for tournaments with diameter 2 and thus propose the following:

Conjecture 10 *There exists a constant t such that for every strong digraph G with $\mathrm{diam}(G) = 2$, $s\vec{rc}(G) \leq t$.*

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On Group Divisible Designs $\mathsf{GDD}(m, n; 2, \lambda)$

Panon Sinsap and *Chariya Uiyyasathian*

Department of Mathematics and Computer Science,
Chulalongkorn University, Bangkok, Thailand

Extended Abstract

A *group divisible design* $\mathsf{GDD}(v = v_1 + v_2 + \cdots + v_g, g, k; \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is a v-set of symbols and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V satisfying the following properties: the v-set is divided into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$.

Group divisible designs can be described graphically as follows. Let λK_v denote the graph on v vertices in which each pair of distinct vertices is joined by λ edges. Let G_1 and G_2 be vertex disjoint graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. Let G and H be graphs with G is a subgraph of H. A *G-decomposition* of a graph H is a partition of the edge set of H such that each element of the partition induces a copy of G. Thus an existence of a $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to an existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$.

The problem of the existence of group divisible design has been interested for a long time. In 1952, Bose and Shimamoto published a work that classified the designs [2]. The case when $g = 2$ and $k = 3$ is of highly interest recently (Specifically, we will use the notation $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ to means $\mathsf{GDD}(v = m + n, 2, 3; \lambda_1, \lambda_2)$). Note that the existence of $\mathrm{GDD}(m, n; \lambda_1, \lambda_2)$ is equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. In particular, the case where $\lambda_1 = \lambda_2 = \lambda$ is equivalent to K_3-decomposition of λK_{m+n}, such a result is known as 2-fold triple system and appears in many standard textbooks (See [7]). Series of research articles had collectively solved the problem of existence of group divisible design with parameter $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ where $\lambda_1 > \lambda_2$. (eg. [4],[6],[8],[11].) The case where $\lambda_1 < \lambda_2$ is considered more difficult. The case where $\lambda_1 = 1, \lambda_2 = 2$ and $m < \frac{n}{2}$ is established in 2012 [10]. When $\lambda_1 = 1, \lambda_2 = 3$, the problem was partially solved recently in 2015 [12]. A $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ is called *gregarious* if each block in the design contains elements from both groups. El-Zanati et al. found all (m, n) where gregarious $\mathsf{GDD}(m, n; 1, 2)$ exists [5]. Up to date, no other result where $\lambda_1 < \lambda_2$ has been known.

In this paper we continue along this line of work. In particular, we solve the existence of a $\mathsf{GDD}(m, n; 2, \lambda)$ where $\lambda > 2$.

An existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ yields the following necessary conditions of the existence of our designs.

Theorem 1 *Let $(m, n) \in \mathbb{N}^2$, if $\mathsf{GDD}(m, n; 2, \lambda)$ exists then*

1. $3 \mid m(m-1) + n(n-1) + \lambda mn$,
2. $2 \mid \lambda m$ *and* $2 \mid \lambda n$, *and*
3. $\frac{\lambda}{2} \leq \frac{m-1}{n} + \frac{n-1}{m}$.

For sufficiency, we have finished the constructions of our designs for almost all the cases. This paper uses various graph decomposition techniques, for instant, 1-factors, 2-factors, Hamiltonian cycles and resolvable classes. One of the major tools is the result from the Alspach's problem [1]. He asked whether it is possible to decompose the complete graph on n vertices, denoted K_n, into t cycles of specified lengths $m_1, \ldots, m_t$ given the obvious necessary conditions are satisfied. After series of research articles, Bryant, Horsley and Pettersen finally solved the problem in [3] as the result of the following theorm.

Theorem 2 [3]

1. *There is a decomposition $G_1, G_2, ..., G_t$ of K_n in which G_i is an m_i-cycle for $i = 1, 2, ..., t$ if and only if n is odd, $3 \le m_i \le n$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = \frac{n(n-1)}{2}$.*

2. *There is a decomposition $G_1, G_2, ..., G_t, I$ of K_n in which G_i is an m_i-cycle for $i = 1, 2, ..., t$ and I is a one-factor if and only if n is even, $3 \le m_i \le n$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + ... + m_t = \frac{n(n-2)}{2}$.*

Let $\mathcal{S}$ be the set of the residue modulo 6 of all ordered triples, (m, n, λ), where $m > n$, that satisfies necessary conditions (1) and (2) in Theorem 1. First, we divide $\mathcal{S}$ into seven groups for the sake of their constructions as followings:

1. $\mathcal{S}_1 = \{(m, n, \lambda) \in \mathcal{S} \mid m \not\equiv 2 \pmod 3 \wedge n \not\equiv 2 \pmod 3 \wedge \lambda \equiv 0 \pmod 2\}$
2. $\mathcal{S}_2 = \{(m, n, \lambda) \in \mathcal{S} \mid m \equiv 2 \pmod 3 \wedge n \not\equiv 2 \pmod 3 \wedge \lambda \equiv 0 \pmod 2\}$
3. $\mathcal{S}_3 = \{(m, n, \lambda) \in \mathcal{S} \mid m \not\equiv 2 \pmod 3 \wedge n \not\equiv 2 \pmod 3 \wedge \lambda \equiv 1 \pmod 2\}$
4. $\mathcal{S}_4 = \{(m, n, \lambda) \in \mathcal{S} \mid m \equiv 2 \pmod 3 \wedge n \not\equiv 2 \pmod 3 \wedge \lambda \equiv 1 \pmod 2\}$
5. $\mathcal{S}_5 = \{(m, n, \lambda) \in \mathcal{S} \mid n \equiv 2 \pmod 3 \wedge \lambda \equiv 0 \pmod 2\}$
6. $\mathcal{S}_6 = \{(m, n, \lambda) \in \mathcal{S} \mid m \equiv 2 \pmod 3 \wedge n \equiv 2 \pmod 3 \wedge \lambda \not\equiv 0 \pmod 2\}$
7. $\mathcal{S}_7 = \{(m, n, \lambda) \in \mathcal{S} \mid m \not\equiv 2 \pmod 3 \wedge n \equiv 2 \pmod 3 \wedge \lambda \not\equiv 0 \pmod 2\}$

Our result yields the following theorems.

Theorem 3 *Let $(m, n, \lambda) \in \mathbb{N}^3$ and $\frac{m-1}{n} \ge \frac{\lambda}{2}$. If the residue modulo 6 of (m, n, λ) is in $\mathcal{S}_1$ or $\mathcal{S}_2$ then $\mathsf{GDD}(m, n; 2, \lambda)$ exists.*

Theorem 4 *Let $(m, n, \lambda) \in \mathbb{N}^3$ and $\frac{m-1}{n} \ge \frac{\lambda}{2}$. If the residue modulo 6 of (m, n, λ) is in $\mathcal{S}_3$ or $\mathcal{S}_4$ then $\mathsf{GDD}(m, n; 2, \lambda)$ exists.*

Theorem 5 *Let $(m, n, \lambda) \in \mathbb{N}^3$ and $\frac{m-1}{n} - \frac{2}{n} \ge \frac{\lambda}{2}$ and $n - 1 \ge \frac{\lambda}{2}$. If the residue modulo 6 of (m, n, λ) is in $\mathcal{S}_5$ then $\mathsf{GDD}(m, n; 2, \lambda)$ exists.*

Theorem 6 *Let $(m, n, \lambda) \in \mathbb{N}^3$ and $\frac{m-1}{n} - \frac{2}{n} \ge \frac{\lambda}{2}$ and $n(n-1) \ge \lambda n - 2$. If the residue modulo 6 of (m, n, λ) is in $\mathcal{S}_6$ then $\mathsf{GDD}(m, n; 2, \lambda)$ exists.*

Theorem 7 *Let $(m, n, \lambda) \in \mathbb{N}^3$ and $\frac{m-1}{n} - \frac{8}{n} \ge \frac{\lambda}{2}$ and $\frac{n-1}{2} \ge \lambda$. If the residue modulo 6 of (m, n, λ) is in $\mathcal{S}_7$ then $\mathsf{GDD}(m, n; 2, \lambda)$ exists.*

We would note further that our techniques in Theorems 3-7 to construct all GDDs under certain restrictions, for example the assumption $\frac{m-1}{n} \ge \frac{\lambda}{2}$ in Theorem 3. However, according to the necessary condition (3) in Theorem 1, there should exist the designs for all $(m, n, \lambda) \in \mathcal{S}$ satisfying $\frac{\lambda}{2} \le \frac{m-1}{n} + \frac{n-1}{m}$. This means that our construction still could not provide the complete solution of the problem. But, from further analysis, we essentially have proved that our construction works most of the cases. The following theorems are our results.

Theorem 8 *If $n \not\equiv 2$ (mod 3) then the necessary condition for the existence of $\mathsf{GDD}(m, n; 2, \lambda)$ is also sufficient except for possibly the largest λ possible.*

Theorem 9 *If $n \equiv 2 \pmod 3$, $\lambda \equiv 0 \pmod 2$ and $n > \sqrt{m-1}$ then the necessary condition for the existence of $\mathsf{GDD}(m, n; 2, \lambda)$ is also sufficient except for possibly the largest λ possible.*

Theorem 10 *If $m \equiv 2 \pmod 3$, $n \equiv 2 \pmod 3$, $\lambda \not\equiv 0 \pmod 2$ and $n > \sqrt{m-1}$ then the necessary condition for the existence of $\mathsf{GDD}(m, n; 2, \lambda)$ is also sufficient except for possibly the largest λ possible.*

Theorem 11 *If $m \not\equiv 2 \pmod 3$, $n \equiv 2 \pmod 3$, $\lambda \not\equiv 0 \pmod 2$ and $n > 2\sqrt{m-1}$ then the necessary condition for the existence of $\mathsf{GDD}(m,n;2,\lambda)$ is also sufficient except for possibly the largest λ possible.*

Open Problem

Consider a fixed (m,n) and $(m,n,\lambda) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. By the necessary condition (3) in Theorem 1, there will be the largest λ which satisfies $\frac{m-1}{n} + \frac{n-1}{m} \geq \frac{\lambda}{2}$. We call such λ, $\lambda_{max}(m,n)$. Furthermore, there will be the largest λ which satisfies $\frac{m-1}{n} \geq \frac{\lambda}{2}$, and we call $\lambda_t(m,n)$. Now since $m > n$ and $\lambda > 2$, we have $\lambda_{max}(m,n) - \lambda_t(m,n) < 2$.

Here are the cases which the problem is still open. We list them in the following table.

$m\backslash n$	0	1	2	3	4	5
0	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$		$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	
1	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	$*$	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	$*$
2		$\lambda_{max}(m,n)$	$*$		$\lambda_{max}(m,n)$	$*$
3	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$		$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	
4	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	†,††	$\lambda_{max}(m,n)$	$\lambda_{max}(m,n)$	$*$
5		$\lambda_{max}(m,n)$	$*$		$\lambda_{max}(m,n)$	$*$

$*$ means If $n > \sqrt{m-1}$, $\lambda_{max}(m,n)$. If $n \leq \sqrt{m-1}$, all λ.
† means If $n > \sqrt{m-1}$, $\lambda_{max}(m,n)$. If $n \leq \sqrt{m-1}$, all λ. (If $\lambda_t(m,n) \equiv 2 \pmod 6$)
†† means If $n > 2\sqrt{m-1}$, $\lambda_{max}(m,n)$. If $n \leq 2\sqrt{m-1}$, all λ. (If $\lambda_t(m,n) \equiv 5 \pmod 6$)

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Tournament limits: score functions and degree distributions

Erik Thörnblad

Department of Mathematics, Uppsala University, Sweden

Extended Abstract

The score sequence of a tournament (a directed complete graph) is the sequence of outdegrees of its vertices. Landau's theorem characterises those sequences that can be realised as the score sequence of some tournament.

Theorem 1 ([3]) *A sequence $(d_i)_{i=1}^n$ of non–negative integers is the score sequence of some tournament if and only if*

$$\sum_{i\in I} d_i \geq \binom{|I|}{2}$$

for all $I \subseteq \{1, 2, \ldots, n\}$, with equality for $I = \{1, 2, \ldots, n\}$.

We will deal with the theory of *tournament limits* and *tournament kernels*, with the ultimate aim of proving a limit and kernel analogue of Theorem 1. The theory of tournament limits is developed in [2, 4] and is based on the homomorphism numbers of test graphs. Given a digraph F on k vertices and a tournament G on n vertices, define

$$t(F, G) = \frac{\hom(F, G)}{n^k},$$

where $\hom(F, G)$ is the number of homomorphisms of F into G. Roughly speaking, $t(F, G)$ is the normalised number of "copies" of F inside G. If $(G_n)_{n=1}^\infty$ is a sequence of tournaments such that $t(F, G_n)$ converges for each digraph F, then we say that the sequence $(G_n)_{n=1}^\infty$ is *convergent*. Corresponding to each convergent tournament sequence is a tournament limit (which can be constructed by taking the completion of the set of tournaments embedded in a suitable metric space). If G_n converges to the tournament limit Γ, then it holds by definition that

$$t(F, G_n) \to t(F, \Gamma)$$

for each digraph F. The tournament limits are very much abstract limit objects, defined completely in terms of their homomorphism densities, but each tournament limit Γ can be represented analytically by so–called a tournament kernels, which are measurable functions $W : [0, 1]^2 \to [0, 1]$ satisfying $W(x, y) + W(y, x) = 1$. The functions of this type are precisely the functional versions of the adjacency matrices of finite tournaments. Put shortly, one can define homomorphism numbers of kernels via the relation

$$t(F, W) := \int_{[0,1]^{|V(F)|}} \prod_{ij\in E(F)} W(x_i, x_j) \prod_{i=1}^{|V(F)|} dx_i$$

for any digraph F. The kernels W which represent a given tournament limit Γ are precisely those which satisfy $t(F, W) = t(F, \Gamma)$ for each digraph F. This dual world offers both some complications (especially in terms of determining which tournament kernels represent the same limit) but also a great deal of flexibility, since one can often choose to work with limits or with kernels.

To extend Theorem 1 to tournament kernels and tournament limits, we need to extend the notion of score sequence to these two settings. For tournament limits, we will use the degree distribution. The outdegree distribution of a finite tournament G is found by simply counting the relative frequence of the numbers $\{0, 1, 2, \ldots, |G| - 1\}$ in the score sequence of

G. This is the same as counting the number of occurences of copies of S_k, $k = 0, 1, \ldots, |G|$, in G, where S_k denotes the digraph with one vertex joined by outgoing edges to k other vertices. This defines a distribution on $[0, 1]$, the moments of which are the numbers $t(\mathsf{S}_k, G)$. Since tournament limits are defined in terms of homomorphism densities, we may define the *outdegree distribution*, denoted $\nu^+(\Gamma)$ of a tournament limit Γ as the (unique) probability measure on $[0, 1]$ with k:th moment $t(\mathsf{S}_k, \Gamma)$. (One should, of course, check that this is well–defined.)

For tournament kernels we employ the analogy the adjacency matrix. If W is a tournament kernel, the function $f(x) = \int_0^1 W(x, y)dy$ is called the *score function* of W. (Integrating across is the analogue of summing the entries of a row in the adjacency matrix, which gives the score of that vertex.) Since $W(x, y) + W(y, x) = 1$ almost everywhere, it follows that

$$\int_A f(x)dx = \int_A \int_0^1 W(x, y)dxdy \geq \int_{A\times A} W(x, y)dxdy = \frac{\lambda(A)^2}{2},$$

for any measurable $A \subseteq [0, 1]$, where λ denotes Lebesgue measure on $[0, 1]$. It turns out that this necessary conditions is also sufficient, in the sense of the following thoerem. (Note that we identify distributions with the corresponding measures.)

Theorem 2 ([5]) *Let $f : [0, 1] \to [0, 1]$ be a function. Then the following are equivalent.*

1. *There exists a tournament limit Γ such that $\nu^+(\Gamma) = \lambda(f^{-1}(\cdot))$.*
2. *There exists a tournament kernel W with score function f (almost everywhere).*
3. *For all measurable $A \subseteq [0, 1]$,*

$$\int_A f(x)dx \geq \frac{\lambda(A)^2}{2}.$$

with equality if $\lambda(A) = 1$.

The main ingredients in the proof of Theorem 2 are discretisations, some results on weak convergence, and the Hardy–Littlewood inequality.

Avery [1] posed and resolved the question which score sequences are realised by a unique tournament (up to isomorphism). A natural question to ask is therefore which score functions or degree distributions are realised by a unique kernel or limit. Since the question of uniqueness for kernels is rather involved, we state the limit version of the result. It transpires that the presence of 3–cycles is of importance. Denote by C_3 the digraph with vertex set $\{1, 2, 3\}$ and edge set $\{12, 23, 31\}$.

Theorem 3 ([5]) *Let Γ some tournament limit. Then there exists some tournament limit $\Gamma' \neq \Gamma$ with $\nu^+(\Gamma) = \nu^+(\Gamma')$ if and only if $t(\mathsf{C}_3, \Gamma) > 0$.*

It should be mentioned that the number $t(\mathsf{C}_3, \Gamma)$ is determined by the outdegree distribution; alternatively by the numbers $t(\mathsf{S}_k, \Gamma)$, $k \geq 1$. To see why the presence of 3–cycles should be critical, one should perhaps consider the situation for finite tournaments. Reversing the arcs in a 3–cycle in a tournament does not change the score sequence of the tournament. However, these arc reversals would typically (but not always) result in changing the isomorphism type of the tournament. A similar situation occurs in the limit setting. Indeed, the proof of Theorem 3 goes via a more–or–less explicit construction of Γ' by "reversing arcs" of 3–cycles in Γ, leading eventually to $t(\mathsf{C}_4, \Gamma) \neq t(\mathsf{C}_4, \Gamma')$, which implies $\Gamma \neq \Gamma'$.

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Square-free graphs are multiplicative

Marcin Wrochna

Institute of Informatics, University of Warsaw, Poland

Extended Abstract

Graph homomorphism is an ubiquitous notion in graph theory with a variety of applications, and a first step to understanding more general constraints given by relational structures, see e.g. the monograph of Hell and Nešetřil [5]. We write $\mu : G \to H$ if μ is a homomorphism from the graph G to H, or simply $G \to H$ if such a homomorphism exists. For example, a graph G is k-colorable iff it has a homomorphisms to the complete graph on k vertices, $G \to K_k$.

The tensor product $G \times H$ is a natural operation arising in this context, defined as having a vertex for every pair $(g, h) \in V(G) \times V(H)$, and an edge between (g, h) and (g', h') whenever $gg' \in E(G)$ and $hh' \in E(H)$. It coincides with the so called categorical product in the category of graphs, with homomorphisms as arrows. In particular, for graphs K, G, H we have that $K \to G \times H$ if and only if $K \to G$ and $K \to H$.

Hedetniemi [4] conjectured the following for the chromatic number of a product of graphs:

Conjecture 1 *For any two graphs G, H,*

$$\chi(G \times H) = \min(\chi(G), \chi(H))$$

This seemingly innocent statement withstood 50 years of research, and is only known to hold in very special cases. Seeing colorings as homomorphism, we get the following equivalent statement: for any graphs G, H and any integer k, $G \times H \to K_k \Leftrightarrow G \to K_k$ or $H \to K_k$.

The same motivations lead to the more general question of which graphs K satisfy that $G \times H \to K \Leftrightarrow G \to K$ or $H \to K$ (for all graphs G, H). Such graphs are called **multiplicative**. Hedetniemi's conjecture is then the statement that all cliques K_k are multiplicative.

There are many equivalent definitions. To give one more example, the tensor product and disjoint union define a distributive lattice, in which multiplicative graphs are precisely the *meet-irreducible* elements, crucial for understanding this lattice structure. We refer to the surveys of Zhu [11], Sauer [8], and Tardif [10] for a variety of results on Hedetniemi's conjecture and multiplicative graphs.

Results

Previously, the only graphs shown to be multiplicative were: K_1 and K_2 (trivially), K_3 by El-Zahar and Sauer [2], a generalization to odd cycles C_{2i+1} by Häggkvist et al. [3], and, much later, a further generalization to all circular cliques $K_{p/q}$ with $\frac{p}{q} < 4$ by Tardif [9].

We use topology to show the following, giving in particular the first examples of multiplicative graphs with arbitrarily high chromatic number.

Theorem 2 *Every square-free graph is multiplicative.*

Square-free graphs, that is, graphs without the 4-cycle C_4 as a subgraph (whether induced or not), were previously considered by Delhommé and Sauer [1], who showed them to be multiplicative if we limit ourselves to products of graphs that both contain a triangle. Our proofs, however, share little with those in [1], instead formalizing and extending the topological intuitions implicitly used in [2] and [3]. We use the same methods to give a different proof for circular cliques, which subsumes all previously known cases.

Theorem 3 *Every circular clique $K_{p/q}$ with $\frac{p}{q} < 4$ is multiplicative.*

Our general approach relies on assigning a topological space to every graph and using topological invariants to show or improve properties of a given coloring of a product of graphs. While topology underlies most of the ideas, the formal proofs only need to use relatively simple, discrete, algebraic descriptions of the invariants.

The topological spaces we use could be equivalently defined with the so called *box complex* – one of the basic objects studied in *topological combinatorics*; we refer to Matoušek's book [7] for a gentle, but in-depth introduction to this field. The box complex is usually used to give lower bounds on the chromatic number of a graph, as in Lovász' famous proof of Kneser's conjecture [6]; however, we use it for square-free graphs, which is precisely the case where the topological lower bounds are almost trivial. This suggests there are still unexplored connections of topology with combinatorics and hints at many avenues for generalizations.

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First order and monadic second order logic of very sparse random graphs[1]

Maksim Zhukovskii

Moscow Institute of Physics and Technology, Laboratory of Advanced Combinatorics and Network Applications
Dolgoprudny, Moscow region, Russia

Extended Abstract

1 Logic of the random graph

We study asymptotical probabilities of first order and monadic second order properties of Erdős–Rényi random graphs $G(n, p(n))$ [1]–[4]. Recall that edges in this graph on the set of vertices $V_n = \{1, \ldots, n\}$ appear independently with probability $p = p(n)$ (i.e., for any undirected graph $H = (V_n, E)$ without loops and multiple edges the equality $\mathsf{P}(G(n,p) = H) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$ holds).

Formulae in the first order language of graphs (first-order formulae) [3]–[6] are constructed using relational symbols $\sim$ (the symbol of adjacency) and $=$; logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$; variables $x, y, x_1, \ldots$ (that express vertices of a graph); and quantifiers $\forall, \exists$. Monadic second order formulae [7, 8] are built of the above symbols of the first order language and variables $X, Y, X_1, \ldots$ that express unary predicates. Following [3]–[6], we call a number of nested quantifiers in the longest chain of nested quantifiers of a formula ϕ *the quantifier depth* $q(\phi)$. For example, the formula

$$(\forall X\ ([\exists x_1 \exists x_2\ (X(x_1) \wedge (\neg(X(x_2))))] \Rightarrow [\exists y \exists z\ (X(y) \wedge (\neg(X(z))) \wedge (y \sim z))]))$$

has quantifier depth 3 and expresses the property of being connected. It is known that this property is not expressed by a first order formula (see, e.g., [3]).

We say that $G(n,p)$ *obeys FO zero-one law* (*MSO zero-one law*) if for any first order formula (monadic second-order formula) it is either true asymptotically almost surely (a.a.s.) or false a.a.s. (as $n \to \infty$). In 1988, S. Shelah and J. Spencer [9] proved the following zero-one law for the random graph $G(n, n^{-\alpha})$.

Theorem 1 *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey FO zero-one law if and only either $\alpha \in (0,1] \cap \mathbb{Q}$ or $\alpha = 1 + 1/l$ for some integer l.*

Obviously, there is no MSO zero-one law when even FO zero-one law does not hold. In 1993, J. Tyszkiewicz [7] proved that $G(n, n^{-\alpha})$ does not obey MSO zero-one law for irrational $\alpha \in (0,1)$ also. When $\alpha > 1$ and does not equal to any of $1+1/l$ MSO zero-one law holds. The last statement simply follows from standart arguments from the theory of logical equivalence.

Theorem 2 *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey MSO zero-one law if and only if either $\alpha \in (0,1]$ or $\alpha = 1 + 1/l$ for some integer l.*

For a formula ϕ, consider the set $S(\phi)$ of α such that $G(n, n^{-\alpha})$ does not obey the zero-one law for the fixed formula ϕ. Both theorems do not give any explanation of how the set $S(\phi)$ depends on ϕ (or even on a quantifier depth of this formula). However, better insight into an asymptotical behavior of probabilities of the properties expressed by first order and monadic second order formulae is given by zero-one k-laws (see Section 3), which are well studied only for the first order language and $\alpha \leq 1$ (see, e.g., [3]). We obtain new zero-one k-laws (both for first order and monadic second order languages) when $\alpha > 1$ and give their statements in Section 3. Proofs of these results are based on the existed study of first order equivalence classes and our study of monadic second order equivalence classes (see Section 2). The respective results are of interest by themselves.

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2 Logical equivalence

For two graphs G and H and any positive integer k, the notation $G \equiv_k^{\mathrm{FO;\,graphs}} H$ denotes that any first order formula ϕ with $q(\phi) \leq k$ is true on both G and H or false on both G and H. The notation $G \equiv_k^{\mathrm{MSO;\,graphs}} H$ is defined similarly. Obviously, $\equiv_k^{\mathrm{FO;\,graphs}}$ and $\equiv_k^{\mathrm{MSO;\,graphs}}$ are both equivalence relations on the set of all graphs. Moreover, for every k there are only finitely many equivalence classes (see, e.g., [6]) and an upper bounds for the cardinality $r_k^{\mathrm{FO;\,graphs}}$ of the set of all $\equiv_k^{\mathrm{FO;\,graphs}}$-equivalence classes $\mathcal{R}_k^{\mathrm{FO;\,graphs}}$ is known [10] and given below. Let $T(s)$ be the tower function: $T(s) = 2^{T(s-1)}$, $T(1) = 2$. Let $\log^*(k) = \min\{i : T(i) \geq k\}$.

Theorem 3

$$r_k^{\mathrm{FO;\,graphs}} \leq T(k + 2 + \log^*(k)) + O(1).$$

Similarly, $r_k^{\mathrm{MSO;\,graphs}}$ and $\mathcal{R}_k^{\mathrm{MSO;\,graphs}}$ are defined. In this paper, we prove a similar result for $\equiv_k^{\mathrm{MSO;\,graphs}}$-equivalence.

Theorem 4 *For any positive integer k,*

$$r_k^{\mathrm{MSO;\,graphs}} \leq T(k + 2 + \log^*(k)).$$

Obviously, from our result it follows that $O(1)$ can me removed in Theorem 3.

In order to prove the results on zero-one laws from Section 3, we also need an extension of the above theory to the case of rooted trees. Recall that *a rooted tree T_R* is a tree with one distinguished vertex R, which is called *the root*. If $R, \ldots, x, y$ is a simple path in T_R, then x is called a parent of y and y is called a child of x. The first order language for rooted trees has a constant symbol R (for the root) and the parent–child relation $P(x, y)$. For two rooted trees T_R and $T'_{R'}$ and any positive integer k, the notations $T_R \equiv_k^{\mathrm{FO;\,trees}} T'_{R'}$, $T_R \equiv_k^{\mathrm{FO;\,trees}} T'_{R'}$, $r_k^{\mathrm{FO;\,trees}}$, $r_k^{\mathrm{MSO;\,trees}}$, $\mathcal{R}_k^{\mathrm{FO;\,trees}}$, $\mathcal{R}_k^{\mathrm{MSO;\,trees}}$ are defined in the same way as for graphs. The following result is proved in [10].

Theorem 5

$$r_k^{\mathrm{FO;\,trees}} \leq T(k + 2 + \log^*(k)) + O(1).$$

For any $A \in \mathcal{R}_k^{\mathrm{FO;\,trees}}$, the following inequality holds:

$$\min_{T_R \in A} |V(T_R)| \leq T(k + 4 + \log^*(k)) + O(1).$$

We prove a similar result for $\equiv_k^{\mathrm{MSO;\,trees}}$-equivalence.

Theorem 6 *Let $k \geq 4$ be an integer. Then*

$$r_k^{\mathrm{MSO;\,trees}} \leq T(k + 2 + \log^*(k)).$$

For any $A \in \mathcal{R}_k^{\mathrm{MSO;\,trees}}$, the following inequality holds:

$$\min_{T_R \in A} |V(T_R)| \leq T(k + 4 + \log^*(k)).$$

Note that $O(1)$ can be removed in Theorem 5 for all $k \geq 4$.

3 Zero-one k-laws

By Theorem 1, for any rational $\alpha \in (0,1]$ there is a first order formula ϕ which is true on $G(n,p)$ with probability whose asymptotics either does not exist or does not equal to 0 or 1. Obviously, this statement need not be true for formulae with bounded quantifier depth. We say that $G(n,p)$ *obeys FO zero-one k-law* (*MSO zero-one k-law*) if for any first order formula (monadic second-order formula) having quantifier depth at most k it is either true a.a.s. or false a.a.s. (as $n \to \infty$). In [11]–[13], the following zero-one k-laws are proved.

Theorem 7 *For any $k \geq 3$ and any*

$$\alpha \in (0, 1/(k-2))$$

the random graph $G(n,n^{-\alpha})$ obeys FO zero-one k-law. If $\alpha = 1/(k-2)$, then $G(n,n^{-\alpha})$ does not obey FO zero-one k-law.

For any $k \geq 4$ and any

$$\alpha \in (1 - 1/(2^k - 2), 1)$$

the random graph $G(n,n^{-\alpha})$ obeys FO zero-one k-law. If $\alpha = 1 - 1/(2^k-2)$, then $G(n,n^{-\alpha})$ does not obey FO zero-one k-law.

In this paper, we consider the very sparse case $\alpha > 1$. From Theorems 1, 2, $G(n,n^{-\alpha})$ obeys both zero-one k-laws if $\alpha \neq 1 + 1/l$ for any positive integer l. We get bounds on the smallest l such that $G(n,n^{-1-1/l})$ obeys FO zero-one k-law and the largest l such that $G(n,n^{-1-1/l})$ does not obey MSO zero-one k-law. Using the statements from Section 2, we get the following zero-one laws.

Theorem 8 *Let l be positive integer, $\alpha = 1 + 1/l$.*

- *Let $k \geq 7$ be an arbitrary integer. If $l \leq 2T(k-4)$, then the random graph $G(n,n^{-\alpha})$ does not obey FO zero-one k-law.*
- *Let $k \geq 4$ be an arbitrary integer. If $l \geq T(k + \log^*(k) + 4)$, then the random graph $G(n,n^{-\alpha})$ obeys MSO zero-one k-law.*

The second (positive) statement of Theorem 8 easily follows from Theorem 6.

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Authors